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BETHE ANSATZ AND QUANTUM GROUPS

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ABSTRACT

The formulation and resolution of integrable lattice statistical models in a quantum group covariant way is the subject of this review. The Bethe Ansatz turns to be remarkably useful to implement quantum group symmetries and to provide quantum group representations even when q is a root of unity. We start by solving the six-vertex model with fixed boundary conditions (FBC) that guarantee exact $SU(2)_q$ invariance on the lattice. The algebra of the Yang-Baxter (YB) and $SU(2)_q$ generators turns to close and the transfer matrix is $SU(2)_q$ invariant for FBC. In addition, the infinite spectral parameter limit of the YB generators yields **cleanly** the $SU(2)_q$ generators. The Bethe Ansatz states constructed for FBC are shown to be **highest weights** of $SU(2)_q$. The light-cone evolution operator for FBC is introduced and shown to follow from the row-to-row FBC transfer matrix with alternating inhomogeneities. This operator is shown to describe the SOS model after an appropiate gauge choice. Using this FBC light-cone approach, the scaling limit of both six-vertex and SOS models easily follows. Finally, the higher level Bethe Ansatz equations (describing the physical excitations) are explicitly derived for FBC. We then solve the RSOS(p) models on the light-cone lattice with fixed boundary conditions by disentangling the type II representations of $SU(2)_q$, at $q = e^{i\pi/p}$, from the full SOS spectrum obtained through Algebraic Bethe Ansatz. The rule which realizes the quantum group reduction to the RSOS states is that there must not be **singular** roots in the solutions of the Bethe Ansatz equations describing the states with quantum spin $J < (p-1)/2$. By studying how this rule is active on the particle states, we are able to give a microscopic derivation of the lattice $S-$matrix of the massive kinks. The correspondence between the light-cone Six-Vertex model and the Sine-Gordon field theory implies that the continuum limit of the RSOS($p+1$) model is to be identified with the $p-$restricted Sine-Gordon field theory.

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1. Introduction

As is by now well known integrability is a consequence of the Yang-Baxter equation (YBE) in two-dimensional lattice models and two-dimensional quantum field theory (QFT), (for recent reviews see for example ref.[1]) . More precisely, a statistical model is integrable when the local weights are solutions of the YBE. Analogously, for two dimensional integrable QFT the two-body S-matrix fulfils the YBE.

Quantum groups are closely related to Yang-Baxter algebras[1,2]. However, quantum group invariance holds for an integrable lattice model only for specific choices of the boundary conditions. As we showed in refs.[3-4] (see also refs.[5,6,7]), choosing fixed boundary conditions (FBC), the transfer matrix commutes with the quantum group generators.

The purpose of this paper is to review the work in collaboration with Claudio Destri in refs.[3-4] on integrable lattice models and their scaling limit using a fully quantum group covariant Bethe Ansatz (BA) framework. Let us recall that the YB algebra of monodromy operators acting on the space of physical states, is the main tool to construct the transfer matrix eigenvectors by the (algebraic) Bethe Ansatz. To do that we choose FBC (Dirichlet type) and use the Sklyanin-Cherednik[8,9] construction of the Yang-Baxter algebra. In this framework, besides the R-matrix defining the local statistical weights, there are two matrices $K^{\pm}(\theta)$ that define the boundary conditions. $K^{\pm}(\theta)$ must fulfil a set of equations [eqs.(2.12) and (2.15)] in order to respect integrability. (FBC is one special case out of a continuous family of boundary conditions compatible with integrability).

The appropiate monodromy operators $U_{ab}(\lambda,\tilde{\omega})$ for a N-sites line take now the form depicted in fig.1 where arbitrary inhomogeneities ω_i , $(1 \leq i \leq N)$, are allowed at each site. We compute the large θ limit of the monodromy operators $U_{ab}(\lambda,\tilde{\omega})$ and find that they are just the quantum group generators. We do that explicitly for the six-vertex model where the quantum group is $SU(2)_q$. We find in this way an explicit representation of the $SU(2)_q$ generators acting on the space of states.

In addition, the $\theta \to \infty$ limit of the Yang-Baxter algebra for the $U_{ab}(\lambda,\tilde{\omega})$ shows that the transfer matrix $t(\lambda,\tilde{\omega})$ commutes with the $SU(2)_q$ generators and that the algebra of the $U_{ab}(\lambda,\tilde{\omega})$ $(1 \leq a, b \leq 2)$ with the quantum group generators ($J_{\pm}$ and q^{J_z}) closes [see eqs.(2.33)-(2.37)]. Moreover, $t(\lambda,\tilde{\omega})$ in the $\theta \to \infty$ limit yields the q-Casimir operator $\mathcal{C}_q$ through

$$t(\infty,\tilde{\omega}) = q + q^{-1} + (q - q^{-1})^2\, \mathcal{C}_q \tag{1.1}$$

It must be recalled that for periodic boundary conditions (PBC) the $\theta \to \infty$ limit of the Yang-Baxter algebra also gives the quantum group generators but that the algebra with the PBC monodromy operators does not close [1] .

Then, we investigate the BA construction in this quantum group covariant framework. It is natural to define for FBC a creation operator of pseudoparticles $\hat{\mathcal{B}}(\theta)$ (proportional to $U_{12}(\lambda,\tilde{\omega})$) which is odd in θ [eq.(3.3)]. The exact eigenvectors of the transfer matrix $t(\lambda,\tilde{\omega})$ are then given by

$$\Psi(\vec{v}) = \hat{\mathcal{B}}(v_1)\hat{\mathcal{B}}(v_2)....\hat{\mathcal{B}}(v_r)\Omega \tag{1.2}$$

where $v_1, v_2, \ldots, v_r$, fulfil the BA equations (3.4) (BAE) and Ω is the ferromagnetic ground state (3.2). We show that only BAE roots with strictly positive real part must be consider ($\mathrm{Re}\, v_j > 0$). In particular, $v_j = 0$ and purely imaginary roots must be discarded. We map the BAE (3.4) for FBC onto BAE for PBC in 2N sites [see eq.(5.2)]. We find the usual PBC BAE plus an extra source and two important constraints :

a) the total number of roots is even and they are symmetrically distributed with respect to the origin,

b) a root at the origin as well as purely imaginary roots are excluded.

Therefore, the antiferroelectric ground state (and the excitations on the top of it) contain always a hole at the origin. This hole combined with the extra source accounts for the surface energy (see for example ref.[12]).

Starting from the general BA state (1.2) we prove that they are highest weights for $SU(2)_q$. That is,

$$J_+\Psi(\vec{v}) = 0 \tag{1.3}$$

provided the BAE (3.4) hold for $v_1, v_2, \ldots, v_r$. Therefore, the eigenvalues of $t(\theta,\tilde{\omega})$ are degenerate with respect to the quantum group and the eigenvectors :

$$J_-\Psi(\vec{v}),\ (J_-)^2\Psi(\vec{v}), \ldots,\ (J_-)^{2J}\Psi(\vec{v}), \tag{1.4}$$

are linearly independent from $\Psi(\vec{v})$.

SOS and vertex models are related by the vertex-face correspondence. (The degrees of freedom lie on faces for SOS models whereas they lie on links for the vertex model). The correspondence between them amounts to an application of the q-analog of the Wigner-Eckart theorem. As we explain in sec. 4, the SOS space of states is **identical** to the set of maximal weight six vertex states in a quantum group invariant framework like ours. It then follows from eq.(1.3) that the Solid-On-Solid (SOS) states **are just** the f.b.c. BA states given by eq.(1.2). The six vertex Hilbert space includes the whole $SU(2)_q$ multiplets and follow then by repeatedly applying the lowering operator J_- to the highest weight BA states (1.2) as shown in eq.(1.4). In other words, we have found the BA solution of the SOS model since we derived the transfer matrix eigenstates. In ref.[10] an alternative but equivalent solution of the SOS model is derived using a BA in **face** language. In addition PSOS (periodic SOS models where the face states l and $l+p$ are identified) are solved in ref.[10].

The light-cone approach is a direct way to give a field theoretical interpretation to a lattice model and furthermore obtain its scaling limit as a massive QFT. The light-cone approach with periodic boundary conditions has been investigated in refs.[19-20] . Here we consider this approach in the case of fixed boundary conditions leading to a quantum group invariant framework.

In this context we show in sec. 4 that the diagonal-to-diagonal transfer matrix $U(\Theta)$ [fig.6] can be obtained from the row-to-row transfer matrix $t(\lambda, \tilde{\omega})$ choosing the inhomogeneities appropiately. This is analogous to the relation found in ref.[19,20] for PBC. From the transfer matrix $U(\Theta)$ we define the lattice hamiltonian through

$$H = \frac{i}{a} \log U(\Theta) \tag{1.5}$$

where a is the lattice spacing. In the $a \to 0, \Theta \to \infty$ limit this operator defines the continuous QFT hamiltonian provided the renormalized mass scale [see eq.(4.17)] is kept fixed. The evolution operator $U(\Theta)$ can be considered both in the vertex or in the face language. In face language, it has a simple expression provided we make an appropiate gauge transformation. That is, if one transforms each local R-matrix to a matrix $\tilde{R}$ such that the $SU(2)_q$ symmetry holds locally. In this way, we show that the associated light-cone evolution operator $\tilde{U}(\Theta)$ just describes the SOS ABF model[13] [eq.(4.27)].

As it is known, the physical states above the AF vacuum are described by the higher level BAE[14]. We obtain the higher level BAE for FBC in sec. 5 [eq.(5.14)]. They are the starting point to study the excitations in SOS and RSOS models . To conclude we show that the BAE for FBC admit solutions at infinity only when γ/π is a rational number. This is precisely the case when RSOS models can be defined and when the representations of $SU(2)_q$ algebras cease to be isomorphic to usual SU(2). Recall that when γ/π is rational, type I representations are reducible but indecomposable, type II are irreducible as in SU(2) (see for example ref.[5,6]).

It is known that the Six Vertex (6V) model, in the so-called light-cone formulation and with periodic boundary conditions (p.b.c.), yields the Sine-Gordon massive field theory in an appropriate scaling limit [19]. Hence the light-cone 6V model can be regarded as an exactly integrable lattice (minkowskian) regularization of the SG model.

Recently, the hidden invariance of the SG model under the quantum group $SU(2)_q$ was exhibited [21]. Our quantum group invariant light-cone formulation, provides a lattice formulation where such hidden invariance appears starting from first principles. That is, not on a bootstrap framework but deriving the field theory as a rigorous scaling limit of the six-vertex model.

We present in section 6 and 7 the derivation of the factorized $S-$matrices *on the lattice*, *i.e.* still in the presence of the UV cutoff. This derivation is based on the "renormalization" of the BA Equations, which consists in removing the infinitely

many roots describing the ground state. What is left is once again a f.b.c. BA structure involving the lattice rapidities of the physical excitations (the particles of the model) and the roots of the higher-level BAE obtained in paper I. The explicit form of two-body $S-$matrix for the 6V model and the SOS model can be extracted in a precise way (cft. eq. (6.11)) from this higher-level BA structure. In the massive scaling limit these lattice scattering amplitudes become the relativistic $S-$matrices of the SG model (or Massive Thirring model) and of the continuum SOS model. Let us remark that the SOS $S-$matrix, although closely related to the 6V and SG $S-$matrices from the analytical point of view, is conceptually different. It describes the scattering of *kinks* interpolating between *renormalized* local vacua labelled by integers. This kink $S-$matrix is most conveniently expressed in the so called face language (see eq.(7.3)).

In section 8 we investigate the f.b.c. BAE (eq.(2.4)) when the quantum group deformation parameter q is a root of unity, say $q^p = \pm 1$, with p some integer larger than 2 (the case $p = 2$ being trivial). In this case it is known that RSOS(p) model can be introduced by restricting to the finite set $(1, 2, \ldots, p-1)$ the allowed values of the local height variables of the SOS model [13]. At the same time when $q^p = \pm 1$, the representations of $SU(2)_q$ algebras cease to be isomorphic to usual SU(2). Recall that when γ/π is rational, type I representations are reducible but indecomposable, type II are irreducible as in SU(2) (see for example ref.[5,6]). The restriction leading to the RSOS model from the SOS model is equivalent to the projection of the full SOS Hilbert space (which is formed by the highest weight states of $SU(2)_q$) to the subspace spanned by the type II representations [5,6]. That is, those representations which remain irreducible when q becomes a root of unity. Our results on this matters, in the BA context, can be summarized as follows:

a) Only when q is a root of unity, the f.b.c. BAE admit singular roots (that is vanishing $z-$roots or diverging $v-$roots, in the notations of sec.1).

b) When q tend to a root of unity, say $q^p = \pm 1$, the BAE solutions can be divided into regular and singular solutions, having, respectively, no singular roots or some singular roots. Regular solutions correspond to irreducible type II representations. Singular solutions with r singular roots correspond to the reducible and generally indecomposable type I representations obtained by mixing two standard $SU(2)_q$ irreps of spin J and $J + r$ (we recall that $J = N/2 - M$, where N is the spatial size of the lattice and M the number of BA roots). Then necessarily $r < p$.

c) The r singular roots $z_1, z_2, \ldots, z_r$ vanish with fixed ratios

$$z_j = \omega^{j-1} z_1 \,, \quad \omega = e^{2\pi i/r} \,, \quad 1 \le j \le r \tag{1.6}$$

In terms of the more traditional hyperbolic parametrization, with $v-$roots related by $v_j = -\frac{1}{2} \log z_j$ to the $z-$roots, the singular roots form an asymptotic string-like configuration. They have a common diverging real part and are separated by π/r in the imaginary direction.

So we see that the RSOS eigenstates are easily singled out from the full set of BA eigenstates of the 6V or SOS transfer matrix. One must retain **all and only those BAE solutions with $\mathbf{M > (N - p + 1)/2}$** (which correspond to states with $J < (p-1)/2$) **and with all M z−roots different from zero.** This provides therefore an exact, explicit and quite simple solution for the RSOS model on the lattice (with suitable boundary conditions, as we shall later see). In particular the ground state of the 6V, SOS and RSOS models is the same f.b.c. BA state (in the thermodynamic limit $N \to \infty$ at fixed lattice spacing). It is the unique $SU(2)_q$ singlet with all real positive roots and no holes. The local height configuration which, loosely speaking, dominates this ground state can be depicted as a sequence of bare kinks jumping back and forth between neighboring bare vacua (see fig. 10). In the massive scaling limit proper of the light-cone approach [19], this ground state becomes the physical vacuum of the SG model as well as of and all the Restricted SG field theories.

In the discussion ending sec. 8, we argue that the kink S−matrix for the excitations of the RSOS models follows indeed by restriction from that of the SOS model. In the scaling limit it is to be identified with the relativistic S−matrix of the Restricted SG models. All these field-theoretical S−matrices are naturally related to the Boltzmann weights of the respective lattice models. This is because the higher-level BAE are identical in form to the "bare" BAE, apart from the renormalization of the anisotropy parameter γ (related to q by $q = e^{i\gamma}$)

$$\gamma \to \hat{\gamma} = \frac{\pi\gamma}{\pi - \gamma} \tag{1.7}$$

and the replacement of the rapidity cutoff $\pm\Theta$ with the suitably scaled rapidities θ_j of the physical excitations

$$\pm\Theta \to \frac{\gamma\theta_j}{\pi - \gamma} \tag{1.8}$$

In particular, since $\gamma = \pi/p$ for the RSOS(p) model, eq.(1.7) yields $p \to p-1$. This shows that the renormalized local vacua $\hat{\ell}$ run from 1 to $p-2$ when the bare local heights ℓ_n run from 1 to $p-1$. The higher-level BA structure of the light-cone 6V model (or lattice regularized SG model) thus provides a microscopic derivation of the bootstrap construction of ref. [21] and explains why the S−matrix of the p−restricted SG field theory has the same functional form of the Boltzmann weights of the lattice RSOS($p+1$) model. Moreover, the well-known correspondence between the critical RSOS(p) models and the Minimal CFT Models M_p imply the natural identification of the massive p−restricted SG model with a completely massive relevant perturbation of M_p. This is generally recognized as the perturbation induced by the primary operator $\phi_{1,3}$ with negative coupling.

Two detailed examples of the BA realization of the quantun group reduction to the RSOS models are presented in section 9. We considered the simplest cases

$p = 3$ and $p = 4$. The RSOS(3) model is a trivial statistical system with only one state, since all the local heigths are fixed once we choose, for example, the boundary condition $\ell_0 = 1$. In our quantum group covariant f.b.c. construction this corresponds to the existence of one and only one type II state when $\gamma = \pi/3$. We then obtain the following purely mathematical result: for any $N \geq 2$ and real $w = \exp(-2\Theta)$ the set of BAE

$$\left(\frac{z_j w - e^{\pi i/3}}{z_j w e^{\pi i/3} - 1}\,\frac{z_j - w e^{\pi i/3}}{z_j e^{\pi i/3} - w}\right)^N = \prod_{\substack{k=1\\k\neq j}}^{[N/2]} \frac{z_j - z_k e^{2\pi i/3}}{z_j e^{2\pi i/3} - z_k}\,\frac{z_j z_k - e^{2\pi i/3}}{z_j z_k e^{2\pi i/3} - 1}\,, \quad 1 \leq j \leq N \tag{1.9}$$

admit one and only one solution with non-zero roots within the unit disk $|z| < 1$. In addition, these roots are all real and positive.

The RSOS(4) model can be exactly mapped into an anisotropic Ising model as showed in eqs. (9.1–5). In our case the horizontal and vertical Ising couplings turn out to be Θ−dependent *complex* numbers. For even N, the Ising spins are fixed on both space boundaries. For odd N the spins are fixed on the left and free to vary on the right. We analyzed the BAE for the RSOS(4) model in some detail. In the thermodynamic limit the ground state, as already stated, is common to the 6V and SOS moodels. The elementary excitations correspond to the presence of holes in the sea of real roots charactering the ground state. Each holes describes a physical particle or kink and may be accompanied by complex roots. In sec. 9 we argue that in the RSOS(4) case a state with ν holes necessarily contains $[\nu/2]$ *two-strings* those position is entirely fixed by the holes. Notice that here the number ν of holes can be odd even for even N. This is not the case for the 6V or SOS models, where ν is always even for N even. What happens is that when $\gamma \to (\pi/4)^-$ the largest real v−root diverges in the $J = 1$ states of the 6V and SOS models. Therefore, these states get mixed with $J = 2$ states into type I representations and do not belong to the RSOS(4) Hilbert space. The RSOS(4) states with $J = 1$ are obtained by choosing the largest quantum integer $I_{N/2-1} = N/2$ (see the Appendix). There is no root associated to $N/2 + 1$. It follows that these states, from the 6V and SOS viewpoint, have a cutoff dependent term in the energy equal to π/a, where a is the lattice spacing (loosely speaking, one could say that "there is a hole at infinity"). They are removed from the physical SG spectrum in the continuum limit. We are thus led to propose as RSOS(4) hamiltonian, for even N

$$H_{RSOS(4)} = H_{SOS}(\gamma = \pi/4) - a^{-1}\pi J \tag{1.10}$$

where H_{SOS} is given by eqs. (4.14), (4.10), (4.21-27) and $J = 0$ or 1. In this way, the particle content of the light cone RSOS(4) and the corresponding S-matrix turn out to coincide with the results of the bootstrap-like approach of ref. [21]. Eq. (1.10) defines, in the scaling limit, the hamiltonian of the ($p = 3$)−restricted

SG model. Notice, in this respect, that the higher-level BAE (5.17) and (6.10) completely determine the physical states in terms of renormalized parameters.

To summarize, the picture we get from the BA solution of the f.b.c. 6V, SOS and RSOS(p) lattice models is as follows. Performing the scaling limit whithin the light-cone approach, these lattice models yields respectively: the SG model (or Massive Thirring Model), a truncated SG and the $(p-1)$restricted SG models. For the SG model we have essentially nothing to add to the existing literature, apart from the explicit unveiling, at the regularized microscopic level, of its $SU(2)_q$ invariance and for a better derivation of the $S-$matrix. The truncated SG follows by keeping only the highest weight states with respect to $SU(2)_q$, that is the kernel of the raising operator J_+. Finally we showed that the RSOS(p) lattice models with trigonometric weights yield in the scaling limit proper of the light-cone approach the $(p-1)$restricted SG field theories formulated at the bootstrap level in ref. [21].

2. Boundary conditions in lattice integrable models

Let us consider an integrable vertex model with R-matrix $R^{ab}_{cd}(\theta)$ (see fig.1). Each element $R^{ab}_{cd}(\theta)$ defines the statistical weight of this configuration. We assume $R(\theta)$ to fulfil the Yang-Baxter equations. The indices a, b, c, d, are assumed to run from 1 to q with $q \geq 2$.

$$\left[\mathbf{1} \otimes R(\theta-\theta')\right] \left[R(\theta) \otimes \mathbf{1}\right] \left[\mathbf{1} \otimes R(\theta')\right] = \left[R(\theta') \otimes \mathbf{1}\right] \left[\mathbf{1} \otimes R(\theta)\right] \left[R(\theta-\theta') \otimes \mathbf{1}\right] \tag{2.1}$$

We shall assume T and P invariance for $R(\theta)$

$$R^{ab}_{cd}(\theta) = R^{cd}_{ab}(\theta) = R^{ba}_{dc}(\theta) \tag{2.2}$$

In addition, we assume $R(\theta)$ to be regular, that is

$$R(0) = c\ \mathbf{1} \qquad \text{or} \qquad R^{ab}_{cd}(0) = c\ \delta^a_c \delta^b_d \tag{2.3}$$

where c is a numerical constant. Eqs. (2.1) and (2.3) imply the unitarity-related equation

$$R(\theta)\ R(-\theta) = \rho(\theta)\ \mathbf{1} \tag{2.4}$$

where $\rho(\theta)$ is an even c-number function. Furthermore, we assume crossed-unitarity. That is,

$$\hat{R}(\theta)\ \hat{R}(-\theta-2\eta) = \hat{\rho}(\theta)\ \mathbf{1} \tag{2.5}$$

where η is a constant and $\hat{R}^{ab}_{cd}(\theta) \equiv R^{ac}_{bd}(\theta)$.

Let us consider now a NxN' square lattice with periodic boundary conditions. Then the row-to-row transfer matrix is given by

$$\tau(\theta, \tilde{\omega}) = \sum_a T_{aa}(\theta, \tilde{\omega}) \tag{2.6}$$

where the operators $T_{ab}(\theta, \tilde{\omega})$ defined by (see fig.2)

$$T_{ab}(\theta, \tilde{\omega}) = \sum_{a_1, \ldots, a_{N-1}} t_{a_1 b}(\theta - \omega_1) \otimes t_{a_2 a_1}(\theta - \omega_2) \otimes \ldots\ldots \otimes t_{a a_{N-1}}(\theta - \omega_N) \tag{2.7}$$

act on the vertical space $\mathcal{V} = \bigotimes_{1 \leq i \leq N} V_i$, $V_i \equiv C^q$, and the simplest choice for the local vertices is $[t_{ab}(\theta)]_{cd} \equiv R^{bd}_{ca}(\theta)$. In eq.(2.7) $\omega_1, \omega_2, \ldots, \omega_N$ stand for arbitrary inhomogeneity parameters. The $T_{ab}(\theta, \tilde{\omega})$ fulfil the YB algebra

$$R(\lambda - \mu)\left[T(\lambda, \tilde{\omega}) \otimes T(\mu, \tilde{\omega})\right] = \left[T(\mu, \tilde{\omega}) \otimes T(\lambda, \tilde{\omega})\right] R(\lambda - \mu) \tag{2.8}$$

as follows from eq.(2.1). Thus, the $\tau(\theta, \tilde{\omega})$ are a commuting family

$$\left[\tau(\theta, \tilde{\omega}), \tau(\theta', \tilde{\omega})\right] = 0 \tag{2.9}$$

Let us now consider the generalization to other boundary conditions compatible with integrability[8]. Define (see fig.3)

$$U_{ab}(\theta, \tilde{\omega}) = \sum_{cd} T_{ad}(\theta, \tilde{\omega}) K^-_{dc}(\theta) T^{-1}_{cb}(-\theta, \tilde{\omega}) \tag{2.10}$$

where $T^{-1}(-\theta, \tilde{\omega})$ is the inverse in both the horizontal and vertical spaces :

$$\sum_b T_{ab}(\theta, \tilde{\omega}) T^{-1}_{bc}(-\theta, \tilde{\omega}) = \delta^a_c\ \mathcal{I} \tag{2.11}$$

and $\mathcal{I}$ is the identity on the vertical space $\mathcal{V}$. Summation over indices of the vertical space $\mathcal{V}$ are omitted both in eqs.(2.10)and (2.11)(cft. fig. 3). $K^-(\theta)$ in eq.(2.10) is a qxq matrix solely acting on the horizontal space. It must fulfil[8]

$$\begin{aligned} R(\lambda - \mu)\left[K^-(\lambda) \otimes \mathbf{1}\right] R(\lambda + \mu)\ \left[K^-(\mu) \otimes \mathbf{1}\right] = \\ \left[K^-(\mu) \otimes \mathbf{1}\right] R(\lambda + \mu) \left[K^-(\lambda) \otimes \mathbf{1}\right] R(\lambda - \mu) \end{aligned} \tag{2.12}$$

(Notice, that our R-matrix differs from ref.[8] in a permutation matrix $R \to PR$, $P^{ab}_{cd}(\theta) = \delta^a_d \delta^b_c$).

As is well known, $R^{ab}_{cd}(\theta-\theta')$ has the interpretation of scattering amplitude for a two-body collision where $a(d)$ and $b(c)$ label the initial (final) states of two particles with rapidities θ and θ' respectively. In this S-matrix context, $K^-_{ab}(\theta)$ is the scattering amplitude for one particle with a rigid wall on the left, a and b labelling the initial and final states and θ being its final rapidity (see fig.4).

Thanks to eqs.(2.8) and (2.12), $U(\theta,\tilde{\omega})$ fulfils the Yang-Baxter algebra

$$\begin{aligned} R(\lambda-\mu)\,[U(\lambda,\tilde{\omega})\otimes\mathcal{I}]R(\lambda+\mu)\,[U(\mu,\tilde{\omega})\otimes\mathcal{I}] = \\ [U(\mu,\tilde{\omega})\otimes\mathcal{I}]\,R(\lambda+\mu)\,[U(\lambda,\tilde{\omega})\otimes\mathcal{I}]\,R(\lambda-\mu) \end{aligned} \tag{2.13}$$

The transfer matrix is given now by

$$t(\lambda,\tilde{\omega}) = \sum_{ab} K^+_{ab}(\lambda+\eta)U_{ab}(\lambda,\tilde{\omega}) \tag{2.14}$$

Here $K^+(\lambda)$ describes the scattering with a rigid wall on the right (cft. fig.5). It is a solution of the equation :

$$\begin{aligned} R(\lambda-\mu)\,\left[\mathbf{1}\otimes K^+(\lambda)\right]R(\lambda+\mu)\,\left[\mathbf{1}\otimes K^+(\mu)\right] = \\ \left[\mathbf{1}\otimes K^+(\mu)\right]\,R(\lambda+\mu)\,\left[\mathbf{1}\otimes K^+(\lambda)\right]\,R(\lambda-\mu) \end{aligned} \tag{2.15}$$

It follows from eqs.(2.13)-(2.15) that $t(\lambda,\tilde{\omega})$ is a commuting family

$$\left[\,t(\theta,\tilde{\omega}),\ t(\theta',\tilde{\omega})\right] = 0 \tag{2.16}$$

As we can see from eqs. (2.10) and (2.14) the boundary conditions associated to $t(\theta,\tilde{\omega})$ follow from the form of $K^+(\lambda)$ and $K^-(\lambda)$ sitting on the right and left borders, respectively.

From now on we shall consider the case of the six-vertex model, where q = 2 and the R matrix reads, in terms of a generic spectral parameter θ,

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} b = b(\theta,\gamma) &= \frac{\sinh\theta}{\sinh(i\gamma-\theta)} \\ c = c(\theta,\gamma) &= \frac{\sinh i\gamma}{\sinh(i\gamma-\theta)} \end{aligned} \tag{2.17}$$

where the anisotropy parameter γ (we may assume $0\le\gamma\le\pi$) is related to the quantum group deformation q by $q=\exp(i\gamma)$. This R-matrix (2.17) is unitary for

real θ. Crossed unitarity (2.5) holds for $\eta = i\gamma$. It is also convenient to work with slightly modified local vertices

$$[t_{ab}(\theta)]_{cd} \equiv R^{bd}_{ca}(\theta - i\gamma/2) \tag{2.18}$$

in order to construct the row-to-row monodromy matrix $T_{ab}(\theta,\tilde{\omega})$. For this R-matrix the diagonal solutions $K^{\pm}(\theta)$ of eqs.(2.12) and (2.15) turn to be[8,9]

$$K^{\pm}(\theta) = K(\theta,\xi_{\pm})$$

where

$$K(\theta,\xi) = \frac{1}{\sinh\xi}\begin{pmatrix} \sinh(\xi+\theta) & 0 \\ 0 & \sinh(\xi-\theta) \end{pmatrix} \tag{2.19}$$

Here ξ_+ and ξ_- are arbitrary numbers that parametrize this boundary conditions compatible with integrability. (For the general solution of eqs.(2.12) and (2.15) in the six-vertex model see ref.[11].)

We are interested on boundary conditions yielding a quantum group covariant framework. [$SU(2)_q$ for the six-vertex model]. For periodic boundary conditions the quantum group transformation properties of $T_{ab}(\theta,\tilde{\omega})$ and $t(\theta,\tilde{\omega})$ are not simple [15]. A $SU(2)_q$ invariant XXZ hamiltonian requires fixed boundary conditions and special end-point terms[6].

The XXZ hamiltonian follows from $dt(\theta,\tilde{\omega}=0)/d\theta$ evaluated at $\theta=0$. One finds from eqs.(2.14),(2.17), and (2.19) [8]

$$\begin{aligned} H_{XXZ} &= -\frac{1}{4trK^+(0)}\frac{d}{d\theta}\left[t(\theta,\tilde{\omega}=0) - trK^+(\theta)\right] \\ &= -\frac{1}{2}\sum_{n=1}^{N-1}(\sigma^x_n\sigma^x_{n+1} + \sigma^y_n\sigma^y_{n+1} + \cos\gamma\,\sigma^z_n\sigma^z_{n+1}) \\ &\quad + \frac{1}{2}\sinh(i\gamma)(\sigma^z_1\coth\xi_- + \sigma^z_N\coth\xi_+). \end{aligned} \tag{2.20}$$

As one can check, the quantum group invariant case corresponds to $\xi_\pm = \pm\infty$. We choose therefore $\xi_\pm = \pm\infty$ in order to built a $SU(2)_q$ covariant framework for the six-vertex model and its scaling limit. This choice corresponds physically to boundary conditions of Dirichlet type. In that case

$$K^{\pm}(\theta) = \exp(\mp\theta\sigma_z) = \begin{pmatrix} e^{\mp\theta} & 0 \\ 0 & e^{\pm\theta} \end{pmatrix} \tag{2.21}$$

Now, in order to find the $SU(2)_q$ content of the YB algebra (2.14) and its associate Bethe Ansatz construction, we start by computing the $\theta\to\infty$ limit of the $U_{ab}(\theta,\tilde{\omega})$ operators.

The $\theta \to \pm\infty$ limit of the row-to-row monodromy matrix

$$T(\theta,\tilde{\omega}) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \tag{2.22}$$

yields $SU(2)_q$ generators [1,15]. We have

$$\begin{aligned}
A(\theta) &\overset{\theta\to\pm\infty}{=} e^{\mp iN\gamma/2}\ \exp(\pm i\gamma J_z)\ [1+O(e^{\mp 2\theta})]\ , \\
D(\theta) &\overset{\theta\to\pm\infty}{=} e^{\mp iN\gamma/2}\ \exp(\mp i\gamma J_z)\ [1+O(e^{\mp 2\theta})]\ , \\
B(\theta) &\overset{\theta\to\pm\infty}{=} \pm\, 2e^{\mp iN\gamma/2 \mp\theta}\ \sinh(i\gamma)\ J_-(\mp\tilde{\omega},\mp\gamma)\ [1+O(e^{\mp 2\theta})]\ , \\
C(\theta) &\overset{\theta\to\pm\infty}{=} \pm\, 2e^{\mp iN\gamma/2 \mp\theta}\ \sinh(i\gamma)\ J_+(\pm\tilde{\omega},\pm\gamma)\ [1+O(e^{\mp 2\theta})]\ ,
\end{aligned} \tag{2.23}$$

where

$$J_\pm(\tilde{\omega},\gamma) = \sum_{k=1}^{N}\prod_{j=1}^{k-1} \exp[-\frac{i\gamma}{2}(\sigma_j)_z] e^{\pm\omega_k}(\sigma_\pm)_k \prod_{l=k+1}^{N} \exp[\frac{i\gamma}{2}(\sigma_l)_z] \tag{2.24}$$

and

$$J_z = \frac{1}{2}\sum_{a=1}^{N}(\sigma_a)_z \tag{2.25}$$

Notice that $J_+ \equiv J_+(\tilde{\omega},\gamma)$, $J_- \equiv J_-(\tilde{\omega},\gamma)$ and J_z are $SU(2)_q$ generators with $q = e^{i\gamma}$, for they obey the commutation rules

$$\begin{aligned}
[J_+, J_-] &= \frac{\sin(2\gamma J_z)}{\sin\gamma} = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}} \\
[J_z, J_\pm] &= \pm\, J_\pm
\end{aligned} \tag{2.26}$$

For the boundary conditions $\xi_\pm = \pm\infty$, one sets

$$U(\theta,\tilde{\omega}) = \begin{pmatrix} \mathcal{A}(\theta) & \mathcal{B}(\theta) \\ \mathcal{C}(\theta) & \mathcal{D}(\theta) \end{pmatrix} \tag{2.27}$$

From eqs.(2.10) and (2.21) it follows that

$$\begin{aligned}
\mathcal{A}(\theta) &= e^{i\gamma/2-\theta}\ B(\theta)\ C(-\theta)\ -\ e^{-i\gamma/2+\theta}\ A(\theta)\ D(-\theta) \\
\mathcal{B}(\theta) &= e^{-i\gamma/2+\theta}\ A(\theta)\ B(-\theta)\ -\ e^{i\gamma/2-\theta}\ B(\theta)\ A(-\theta) \\
\mathcal{C}(\theta) &= e^{i\gamma/2-\theta}\ D(\theta)\ C(-\theta)\ -\ e^{-i\gamma/2+\theta}\ C(\theta)\ D(-\theta) \\
\mathcal{D}(\theta) &= e^{-i\gamma/2+\theta}\ C(\theta)\ B(-\theta)\ -\ e^{i\gamma/2-\theta}\ D(\theta)\ A(-\theta)
\end{aligned} \tag{2.28}$$

Let us compute the $\theta \to \infty$ limit of these operators. We find from eqs.(2.23) -(2.28)

$$\begin{aligned}
\mathcal{A}(\theta) &\overset{\theta\to\infty}{=} -\ e^{-i\gamma/2+\theta}\ \exp(2i\gamma J_z)\ [1+O(e^{-2\theta})]\ , \\
\mathcal{B}(\theta) &\overset{\theta\to\infty}{=} -\ e^{-i\gamma/2}\ \sinh(i\gamma)\ J_-\ [1+O(e^{-2\theta})]\ , \\
\mathcal{C}(\theta) &\overset{\theta\to\infty}{=} -\ e^{-i\gamma/2}\ \sinh(i\gamma)\ J_+\ e^{i\gamma J_z}\ [1+O(e^{-2\theta})]\ , \\
\mathcal{D}(\theta) &\overset{\theta\to\infty}{=} 2e^{-i\gamma/2-\theta}\sin[(J+J_z)\gamma]\sin[(J-J_z+1)\gamma] \\
&\quad -\frac{1}{2}e^{i\gamma/2}\ e^{-2i\gamma J_z-\theta}\ +O(e^{-3\theta})\ ,
\end{aligned} \tag{2.29}$$

The operator J is defined through the $SU(2)_q$ Casimir invariant $\mathcal{C}_q$

$$\mathcal{C}_q = \frac{1}{2}(J_+J_- + J_-J_+) + \cos\gamma\ \frac{\sin^2(\gamma J_z)}{\sin^2(\gamma)} = \frac{\sin\gamma(J+1)\sin\gamma J}{\sin^2\gamma} \tag{2.30}$$

As one can see from eqs.(2.29) the asymptotic form of $U_{ab}(\lambda,\tilde{\omega})$ is related with the $SU(2)_q$ generators in a very clean way.

Let us now consider the transfer matrix $t(\lambda,\tilde{\omega})$ [eq.(2.14)]. We find when $\xi_\pm = \pm\infty$

$$t(\theta,\tilde{\omega}) = e^{-i\gamma/2-\theta}\ \mathcal{A}(\theta) + e^{i\gamma/2+\theta}\ \mathcal{D}(\theta) \tag{2.31}$$

Now, when $\theta \to \infty$, inserting eq.(2.29) in (2.31) yields

$$t(\theta,\tilde{\omega}) \overset{\theta\to\infty}{=} 2\cos[\gamma(2J+1)] + O(e^{-2\theta}) \tag{2.32}$$

That is, the asymptotics of the transfer matrix expresses up to numerical constants in terms of the q-Casimir $\mathcal{C}_q$ [eq.(2.30)].

The operators $U_{ab}(\theta,\tilde{\omega})$ fulfil the Yang-Baxter algebra (2.13) with R-matrix (2.17) . Letting $\theta' \to \infty$ in eq.(2.13) we can compute the relevant commutation of $U_{ab}(\theta,\tilde{\omega})$ with the $SU(2)_q$ generators. We find

$$[\mathcal{A}(\theta), J_-] = -\mathcal{B}(\theta)\ e^{i\gamma J_z+\theta}\ ,\quad [\mathcal{D}(\theta), J_-] = \mathcal{B}(\theta)\ e^{i\gamma(J_z+1)+\theta} \tag{2.33}$$

It is well known that $B(\theta)$ and $C(\theta)$ act as lowering and raising operators for J_z , while $A(\theta)$ and $D(\theta)$ commute with it[1]. We find here the same properties for the

elements of $U(\theta,\tilde{\omega})$,

$$[\,\mathcal{B}(\theta),\ J_z] = \mathcal{B}(\theta)\ ,\ [\,\mathcal{C}(\theta),\ J_z] = -\mathcal{C}(\theta)\ ,$$
$$[\,\mathcal{A}(\theta),\ J_z] = [\,\mathcal{D}(\theta), J_z] = [\,t(\theta,\tilde{\omega}),\ J_z] = 0 \tag{2.34}$$

In addition, we find

$$\left[\,e^{-i\gamma J_z}\,\mathcal{B}(\theta),\ J_-\right] = 0\ \ ,\ \ \mathcal{B}(\theta)\ J_+ = e^{-i\gamma}\ J_+\ \mathcal{B}(\theta) + [\,e^{\theta-i\gamma}\,\mathcal{D}(\theta) - e^{-\theta}\,\mathcal{A}(\theta)]\ e^{i\gamma J_z} \tag{2.35}$$

Using now eq.(2.31) yields

$$[\,t(\theta,\tilde{\omega}),\ J_-] = 0 \tag{2.36}$$

One can analogously prove that

$$[\,t(\theta,\tilde{\omega}),\ J_+] = 0 \tag{2.37}$$

Therefore, the transfer matrix $t(\theta,\tilde{\omega})$ is $SU(2)_q$ invariant. As a corollary, we see that H_{XXZ} (see eq.(2.20)) is $SU(2)_q$-invariant, as shown in ref.[6] by direct calculation.

We investigate in the next section the Bethe Ansatz construction of eigenvectors of $t(\theta,\tilde{\omega})$. The first consequence of the the $SU(2)_q$ invariance of $t(\theta,\tilde{\omega})$ is the degeneracy of its eigenvalues with respect to the quantum group.

3. The Bethe Ansatz and the $SU(2)_q$ group

In section II we developped the Yang-Baxter framework with boundary conditions adapted to the $SU(2)_q$ invariance. We call this a quantum group covariant framework. Let us now investigate in such scheme the eigenvectors of the transfer matrix $t(\theta,\tilde{\omega})$ in the algebraic Bethe Ansatz[8]. These eigenvectors can be written as

$$\Psi(\vec{v}) = \hat{\mathcal{B}}(v_1)\hat{\mathcal{B}}(v_2)....\hat{\mathcal{B}}(v_r)\Omega \tag{3.1}$$

where Ω is the ferromagnetic ground state and $\vec{v} \equiv (v_1, v_2, \ldots, v_r)$.

$$\Omega = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes ..... \otimes \begin{pmatrix}1\\0\end{pmatrix} \tag{3.2}$$

and

$$\hat{\mathcal{B}}(\theta) \equiv \frac{\sinh 2\theta}{\sinh(2\theta - i\gamma)}\ e^{i\gamma/2}\ \mathcal{B}(\theta) = \ e^{-\theta}\ B(\theta)\ A(-\theta) - e^{\theta}\ B(-\theta)\ A(\theta) \tag{3.3}$$

Notice that use has been made of the Yang-Baxter algebra (2.8) to obtain the last form in eq.(3.3) from the expression (2.28) for $\mathcal{B}(\theta)$. The numbers $v_1, v_2, ..., v_r$, $(0 \leq$

$r \leq N/2)$ must be all distinct roots of the set of algebraic equations

$$\prod_{k=1}^{N} \frac{\sinh[v_m - \omega_k + i\gamma/2]}{\sinh[v_m - \omega_k - i\gamma/2]} \frac{\sinh[v_m + \omega_k + i\gamma/2]}{\sinh[v_m + \omega_k - i\gamma/2]} =$$
$$\prod_{k=1,k\neq m}^{r} \frac{\sinh[v_m - v_k + i\gamma]}{\sinh[v_m - v_k - i\gamma]} \frac{\sinh[v_m + v_k + i\gamma]}{\sinh[v_m + v_k - i\gamma]} \tag{3.4}$$
$$1 \leq m \leq r$$

These Bethe Ansatz Equations (BAE) guarantee the vanishing of the so-called unwanted terms. That is, vectors in $t(\theta,\tilde{\omega})\Psi(\vec{v})$ which are not proportional to $\Psi(\vec{v})$. They arise, together with the wanted terms, from the repeated commutation of $t(\theta,\tilde{\omega})$ with $\hat{\mathcal{B}}(v_1)\hat{\mathcal{B}}(v_2)\ldots$
$\hat{\mathcal{B}}(v_r)$ by means of the Yang-Baxter algebra (2.13) . Let us point out that the order of the factors $\hat{\mathcal{B}}(v_j)$ in eq.(3.1) is irrelevant since the Yang-Baxter algebra (2.13) implies

$$\left[\,\hat{\mathcal{B}}(\theta),\ \hat{\mathcal{B}}(\theta')\right] = 0 \tag{3.5}$$

From eq.(3.3) it is evident that $\hat{\mathcal{B}}(\theta)$ is an odd function of

$$\hat{\mathcal{B}}(-\theta) = -\hat{\mathcal{B}}(\theta) \tag{3.6}$$

Consequently, a direct check shows that the BAE (3.4) are invariant under negation of any single unknown v_j . This implies that it is enough to consider BAE roots with strictly positive real parts. In particular, purely imaginary pairs $(\pm i\eta)$ are ruled out. Moreover, one verifies by explicit calculation that $\hat{\mathcal{B}}(i\pi/2)\Omega = 0$. So that the selfconjugate root v = i/2 is also ruled out.

The eigenvalue of $t(\theta,\tilde{\omega})$ on $\Psi(\vec{v})$ is given by

$$\Lambda(\theta;\vec{v}) = \Lambda_+(\theta;\vec{v}) + \Lambda_-(\theta;\vec{v}) \tag{3.7}$$

$$\Lambda(\theta;\vec{v}) = \frac{\sinh(2\theta \pm i\gamma)}{\sinh(2\theta + i\gamma)} \prod_{k=1}^{N} \frac{\sinh[\theta - \omega_k + i\gamma/2]}{\sinh[\theta - \omega_k - i\gamma/2]} \frac{\sinh[\theta + \omega_k + i\gamma/2]}{\sinh[\theta + \omega_k - i\gamma/2]}$$
$$\prod_{m=1}^{r} \frac{\sinh[\theta - v_m \mp i\gamma]}{\sinh[\theta - v_m]} \frac{\sinh[\theta + v_m \mp i\gamma]}{\sinh[\theta + v_m]} \tag{3.8}$$

Taking logarithms, eq.(3.8) become

$$\begin{aligned}
&\sum_{k=1}^{N}[\phi(v_m-\omega_k,\gamma/2)+\phi(v_m+\omega_k,\gamma/2)] \\
&= 2\pi I_m + \sum_{k=1,k\neq m}^{r}[\phi(v_m-v_k,\gamma)+\phi(v_m+v_k,\gamma)] \\
&\quad 1\leq m\leq r
\end{aligned} \tag{3.9}$$

where the I_m are positive integers[16] and

$$\phi(\lambda,x)\equiv i\log\frac{\sinh(ix+\lambda)}{\sinh(ix-\lambda)} \qquad \text{with} \quad \phi(0,x)=0. \tag{3.10}$$

Before solving these equations in the $N\to\infty$ limit, let us study the $\theta\to\infty$ behavior of $\Lambda(\theta;\vec{v})$ in order to match with the asymptotics (2.29) . We find from eqs.(3.7)-(3.8)

$$\Lambda(\theta;\vec{v}) \overset{\theta\to\infty}{=} 2\cos[\gamma(N+1-2r)]\ [1+O(e^{-2\theta})] \tag{3.11}$$

Here all roots $v_1, v_2, ..., v_r$ are assumed to be finite. Actually this must be the case unless γ/π is rational (see sec.5). Comparison of eqs.(2.32) and (3.11) shows that there is only one positive (or zero) solution

$$J=\frac{N}{2}-r \tag{3.12}$$

except when γ/π is rational where other possibilities may happen. In addition, we have from eqs.(2.34) and (3.2) that

$$J_z\Psi(\vec{v})=\left(\frac{N}{2}-r\right)\Psi(\vec{v}) \tag{3.13}$$

Eqs.(3.12) and (3.13) show that $J=J_z$ for Bethe-Ansatz eigenvectors. That is, they are **maximal weight vectors for the quantum group** for non-rational values of γ/π . Thus,

$$J_+\Psi(\vec{v})=0 \tag{3.14}$$

In addition, the kernel of J_+ is stable under variations of γ . Therefore eq.(3.14) holds even when γ/π is rational (see below). Each eigenvalue $\Lambda(\theta;\vec{v})$ has (at least) a degeneracy (2J+1) for $t(\theta,\tilde{\omega})$, since the vectors

$$J_-\Psi(\vec{v}),\ (J_-)^2\Psi(\vec{v}),\ldots,\ (J_-)^{2J}\Psi(\vec{v}),$$

are all eigenvectors of $t(\theta,\tilde{\omega})$ with the same eigenvalue thanks to eqs.(2.36)-(2.37).

Of course, when γ/π is rational J **may not be equal** to J_z for some BA states $\Psi(\vec{v})$ as follows from eqs..(2.32),(3.11) and (3.13).

Let us give an algebraic proof of the highest weight condition (3.14) for the BA states. We shall apply J_+ to $\Psi(\vec{v})$ given by eq.(3.1) permuting J_+ through the $\hat{\mathcal{B}}(v_j),(1\leq j\leq r)$ with the help of eq.(2.35) . Finally we shall use that

$$J_+\Omega=0 \tag{3.15}$$

Actually it is more convenient to use the operator $\hat{\mathcal{D}}(\theta)$

$$\hat{\mathcal{D}}(\theta)=\frac{1}{\sinh(2\theta-i\gamma)}[\mathcal{D}(\theta)\sinh 2\theta-\mathcal{A}(\theta)\sinh(i\gamma)] \tag{3.16}$$

Eq.(2.35) can now be written as

$$J_+\,\hat{\mathcal{B}}(\theta)=e^{i\gamma}\,\hat{\mathcal{B}}(\theta)\,J_++[e^{-\theta}\,\mathcal{A}(\theta)-e^{\theta}\,\hat{\mathcal{D}}(\theta)] \tag{3.17}$$

We find from eq.(3.1) after using r times eq.(3.17) :

$$\begin{aligned}
J_+\Psi(\vec{v})=&\sum_{j=1}^{r}\exp[i\gamma(\frac{N+3}{2}-r+2j)]\,\hat{\mathcal{B}}(v_1)\hat{\mathcal{B}}(v_2).....\hat{\mathcal{B}}(v_{j-1}) \\
&\left[e^{-v_j}\mathcal{A}(v_j)-e^{v_j}\hat{\mathcal{D}}(v_j)\right]\hat{\mathcal{B}}(v_{j+1})....\hat{\mathcal{B}}(v_r)\Omega
\end{aligned} \tag{3.18}$$

where we also used eqs.(3.13) and (3.15) . Now we can push the operators $\mathcal{A}(v_j)$ and $\hat{\mathcal{D}}(v_j)$ in eq.(3.18) to the right using the Yang-Baxter algebra (2.13). That is,

$$\begin{aligned}
\mathcal{A}(\theta)\,\hat{\mathcal{B}}(\theta')=&\frac{\sinh(\theta-\theta'-i\gamma)\sinh(\theta+\theta'-i\gamma)}{\sinh(\theta-\theta')\sinh(\theta+\theta')}\,\hat{\mathcal{B}}(\theta')\,\mathcal{A}(\theta)+ \\
&\frac{\sinh(2\theta-i\gamma)\sinh(i\gamma)}{\sinh(2\theta)}\,\hat{\mathcal{B}}(\theta)\left[\frac{\mathcal{A}(\theta')}{\sinh(\theta-\theta')}-\frac{\hat{\mathcal{D}}(\theta')}{\sinh(\theta+\theta')}\right] \\
\hat{\mathcal{D}}(\theta)\,\hat{\mathcal{B}}(\theta')=&\frac{\sinh(\theta-\theta'+i\gamma)\sinh(\theta+\theta'+i\gamma)}{\sinh(\theta-\theta')\sinh(\theta+\theta')}\,\hat{\mathcal{B}}(\theta')\,\hat{\mathcal{D}}(\theta)+ \\
&\frac{\sinh(2\theta+i\gamma)\sinh(i\gamma)}{\sinh(2\theta)}\,\hat{\mathcal{B}}(\theta)\left[\frac{\mathcal{A}(\theta')}{\sinh(\theta+\theta')}-\frac{\hat{\mathcal{D}}(\theta')}{\sinh(\theta-\theta')}\right]
\end{aligned} \tag{3.19}$$

We find from eqs.(3.18)-(3.19) an expression of the form

$$J_+\Psi(\vec{v})=\sum_{j=1}^{r}c_j(\vec{v})\ \hat{\mathcal{B}}(v_1)\hat{\mathcal{B}}(v_2).....\hat{\mathcal{B}}(v_{j-1})\hat{\mathcal{B}}(v_{j+1})....\hat{\mathcal{B}}(v_r)\Omega \tag{3.20}$$

where the $c_j(\vec{v})$ are c-number coefficients. The easier is to first compute $c_1(\vec{v})$ and then deduce the rest by symmetry of the arguments (cfr. eq.(3.5)) . $c_1(\vec{v})$ follows by

permuting in the first term of eq.(3.18) the bracket $\left[e^{-v_j}\mathcal{A}(v_j) - e^{v_j}\hat{\mathcal{D}}(v_j)\right]$ through $\hat{\mathcal{B}}(v_2)....\hat{\mathcal{B}}(v_r)$ using eqs.(3.19) and keeping just the first terms in eqs.(3.19) . That is, those proportional to $\hat{\mathcal{B}}(v_j)\ \mathcal{A}(v_1)$ and to $\hat{\mathcal{B}}(v_j)\ \hat{\mathcal{D}}(v_1)$, $(2 \leq j \leq r)$. Collecting all factors, it is easy to see that $c_1(\vec{v})$ and the $c_j(\vec{v})$ $(1 \leq j \leq r)$ are proportional to the BAE (3.4) and hence identically zero. We have therefore proved that all BA vectors are highest weights [3,17,18].

4. The quantum group covariant ligth-cone approach.

The light-cone approach is a direct way to give a field theoretical interpretation to a lattice model and furthermore obtain its scaling limit as a massive QFT. We start from a diagonal lattice that we interpret as a discretized two-dimensional Minkowski spacetime. The sites in the light-cone lattice are considered as world events. Each site has then microscopic amplitudes associated to it which describe the different processes that may take place and must verify general properties like unitarity.

The light-cone approach with periodic boundary conditions has been investigated in refs.[19-20]. Here we consider this approach in the case of fixed boundary conditions leading to a quantum group invariant framework. For the six-vertex model, this can be done as follows. Consider the $SU(2)_q$ invariant transfer matrix $t(\theta, \tilde{\omega})$ constructed in sec.2, and identify the inhomogeneity parameters $\omega_1, \omega_2, ..., \omega_N$ with the cutoff rapidities of the diagonal lattice. Namely set

$$\omega_j = (-1)^j\ \Theta \tag{4.1}$$

Then define

$$U(\Theta) = \ t(\Theta + i\gamma/2, \omega_j = (-1)^j\ \Theta)\ \frac{\sinh(2\Theta + i\gamma)}{\sinh(2\Theta + 2i\gamma)} \tag{4.2}$$

We shall now show that $U(\Theta)$ has the natural interpretation of the unit time evolution operator on the diagonal lattice. To this end we introduce some convenient compact notation. We denote by 0 the auxiliary horizontal space and label by j, $(1 \leq j \leq N)$, the vertical spaces. Then

$$L_j(\theta) = R_{0j}(\theta - \omega_j)\ P_{0j} \tag{4.3}$$

is a local vertex acting only on the jth. vertical space. Here $R_{jk}(\theta)$ stand for the six-vertex R-matrix (2.17) acting on the spaces j and k , $(0 \leq j, k \leq N)$ and P_{jk} is the usual exchange operator. In practice, the ab matrix element of $L_j(\theta)$ in the 0th.

space act on the jth. space as the operators $t_{ab}(\theta - \omega_j)$ defined in sec.2. In terms of the $L_j(\theta)$ we can write the transfer matrix as

$$t(\theta + i\gamma/2, \tilde{\omega}) = Tr_0\{K_0^+(\theta + i\gamma)L_N(\theta).....L_1(\theta)K_0^-(\theta)L_1(-\theta)^{-1}.....L_N(-\theta)^{-1}\} \tag{4.4}$$

Since the R-matrix is regular, $R(0) = \mathbf{1}$, we have

$$L_j(\theta)\ |_{\theta=\omega_j} = P_{0j} \tag{4.5}$$

Hence, as soon as $\omega_j = (-1)^j\ \Theta$, one obtains

$$L_{2k}(\Theta) = P_{0,2k} \quad , \quad L_{2k+1}(\Theta) = S_{0,2k+1} \tag{4.6}$$

where $S_{jk} = R_{jk}(2\Theta)P_{jk}$. Inserting eq.(4.6) into eq.(4.4) yields for odd N

$$\begin{aligned}
& t(\theta + i\gamma/2, \omega_j = (-1)^j\ \Theta) \\
=& Tr_0\{K_0^+(\theta + i\gamma)\ S_{0,N}P_{0,N-1}S_{0,N-2}P_{0,N-3}.....S_{0,3}P_{0,2}S_{0,1}K_0^- \\
& P_{0,1}S_{0,2}P_{0,3}.....S_{0,N-1}P_{0,N}\} \\
=& Tr_0\{K_0^+(\theta + i\gamma)\ S_{0,N}S_{N-2,N-1}.....S_{1,2}P_{0,N-1}P_{0,N-3}.....P_{0,2}K_0^-(\Theta) \\
& P_{0,1}P_{0,3}.....P_{0,N}S_{2,3}.....S_{N-1,N}\} \\
=& Tr_0\{K_0^+(\theta + i\gamma)\ S_{0,N}P_{0,N}\}\ S_{12}.....S_{N-2,N-1}P_{0,N}P_{0,N-1}.....P_{0,2} \\
& P_{0,1}P_{0,3}....P_{0,N}K_1^-(\Theta)S_{23}.....S_{N-1,N} \\
=& Tr_0\{K_0^+(\theta + i\gamma)\ R_{0,N}(2\Theta)\}\ R_{12}(2\Theta).....R_{N-2,N-1}(2\Theta) \\
& K_1^-(\Theta)\ R_{23}(2\Theta).....R_{N-1,N}(2\Theta)
\end{aligned} \tag{4.7}$$

Actually, the computation up to now is completely general and would hold for any R-matrix and K-matrices. Let us exploit now the explicit form of $R(\theta)$ and $K^{\pm}(\theta)$ for the six-vertex model [eqs.(2.17)-(2.19)]. Specialising to the $SU(2)_q$ invariant case, we obtain

$$K_1^-(\Theta) = g_1(\Theta)^{-1} \quad , \quad Tr_0\{K_0^+(\theta + i\gamma)R_{0,N}(2\Theta)\} = g_N(\Theta)\ \frac{\sinh(2\Theta + i\gamma)}{\sinh(2\Theta + 2i\gamma)} \tag{4.8}$$

where

$$g(\theta) = \exp[-\theta\sigma_z] = \begin{pmatrix} e^{-\theta} & 0 \\ 0 & e^{+\theta} \end{pmatrix} \tag{4.9}$$

Thus

$$U(\Theta) = R_{12}R_{34}\ldots R_{N-1-\epsilon,N-\epsilon}g_N g_1^{-1}R_{23}R_{45}\ldots R_{N-2+\epsilon,N-1+\epsilon} \tag{4.10}$$

where $\epsilon = [1 - (-1)^N]/2$, g_j is the matrix $\exp[\Theta\sigma^z]$ acting nontrivially only on the two-dimensional vector space attached to the jth link, and R_{jk} is the 6V R–matrix

$R(2\Theta)$ acting nontrivially only on the tensor product of the jth and k^{th} vector spaces.

$U(\Theta)$ is clearly the unit time evolution operator on a diagonal lattice with "reflecting" type boundary conditions, $g(\theta)$ $[g(\theta)^{-1}]$ acting for collisions on the right [left] wall. Graphically $U(\Theta)$ can be depicted as in fig.6. By taking powers of $U(\Theta)$ one then obtains the evolution on the diagonal lattice for any discrete time (Fig.7). By construction, this time evolution is $SU(2)_q$ invariant.

The eigenvectors of $U(\Theta)$ can now be written as in sec.3, namely

$$\Psi(\vec{v}) = \hat{\mathcal{B}}(v_1)\hat{\mathcal{B}}(v_2)....\hat{\mathcal{B}}(v_r)\Omega \tag{4.11}$$

where $\omega_j = (-1)^j\ \Theta$ and the Bethe Ansatz equations (3.4) take the form

$$\begin{aligned}&\left[\frac{\sinh[v_m - \Theta + i\gamma/2]\sinh[v_m + \Theta + i\gamma/2]}{\sinh[v_m - \Theta - i\gamma/2]\sinh[v_m + \Theta - i\gamma/2]}\right]^N = \\ &\prod_{k=1,k\neq m}^{r} \frac{\sinh[v_m - v_k + i\gamma]\sinh[v_m + v_k + i\gamma]}{\sinh[v_m - v_k - i\gamma]\sinh[v_m + v_k - i\gamma]} \\ &1 \leq m \leq r\end{aligned} \tag{4.12}$$

The associated eigenvalue of $U(\Theta)$ is given by

$$\mathcal{U}(\Theta, v_1, v_2, \ldots, v_M) = \prod_{k=1}^{r} \frac{\sinh[\Theta - v_m - i\gamma/2]\sinh[\Theta + v_m - i\gamma/2]}{\sinh[\Theta - v_m + i\gamma/2]\sinh[\Theta + v_m + i\gamma/2]} \tag{4.13}$$

Let us now discuss the problem of unitarity for the time evolution defined by $U(\Theta)$. The R-matrix (2.17) is unitary for real θ. Hence the evolution operator U is unitary too, for real Θ, up to the boundary effects due to the term $g_N g_1^{-1}$ in eq.(4.10). As is evident from figs.(6-7) [eqs.(4.10)] there exist unitarity violating boundary effects (factors $g(\theta)_N$ and $g(\theta)_1^{-1}$). Then, the reflection of the bare particles on the boundaries is affected by "leaking". However, on average this leaking compesates since we have opposite factors $g(\theta)_N$ and $g(\theta)_1^{-1}$ on right and left sides respectively. Thus one can expect $U(\Theta)$ to still have unimodular eigenvalues. Indeed, this is the case from eq.(4.13) provided, as it is usually the case, the BA roots are either real, selfconjugate ($\mathrm{Im}\, v = \pi/2$) or organised in complex conjugated pairs. As seen below, the exact diagonalization of $U(\Theta)$, shows that all its eigenvalues are unimodular, so that there exists a similarity transformation mapping $U(\Theta)$ to an explicitly unitary operator. On the other hand, it is natural to expect that in the thermodynamic limit ($N \to \infty$) different boundary conditions become equivalent (the model has a finite mass gap). This suggest that the above mentioned similarity transformation

reduces to the identity as $N \to \infty$. We conclude therefore that the light-cone 6V model with fixed b.c. described by the evolution operator U of eq.(4.10) is another good, integrability-preserving regularization of the SG model. Its advantage over the more conventional setup with periodic b.c. is that $U(\Theta)$ is explicitly $SU(2)_q$ invariant even for finite N.

According to the general light-cone construction the physical lattice hamiltonian can be defined in terms of $U(\Theta)$ as

$$H = \frac{i}{a}\log U(\Theta) \tag{4.14}$$

where a is the lattice spacing. By a judicious choice of the logarithmic branches the energy eigenvalues can be written from eqs.(4.13)-(4.14)

$$E = a^{-1}\sum_{j=1} e_0(v_j) \tag{4.15}$$

where

$$e_0(v) = -\pi + 2\arctan\left(\frac{\cosh 2\Theta\cos\gamma - \cosh 2v}{\sinh 2\Theta\ \sin\gamma}\right) \tag{4.16}$$

Notice that $e_0(v)$ is smooth and negative definite for real v. The specific choice of the logarithmic branch in passing from eq.(4.13) to eqs.(4.15), (4.16) is dictated by the requirement that $e_0(v)$ should correspond to the negative energy branch of the spectrum of a single spin wave over the ferromagnetic state Ω.

The physical ground state and the particle-like excitations are obtained by filling the interacting sea of negative energy states. This sea is described by a set of real v-roots of the BAE with no "holes" (in the Appendix we give a detailed exposition on the treatment of the BAE, also to clarify some rather subtle matters). Excitations correspond to solutions of the BAE which necessarily contain holes and possibly complex v-roots. The crucial point is that in the limit $N \to \infty$ only the number of real roots of the sea grows like N, while the number of holes and complex roots stays finite to guarantee a finite energy above the ground state energy. Hence these solutions of the BAE can be described by densities $\rho(v)$ of real roots plus a finite number of parameter associated to the positions $x_1, x_2, \ldots x_\nu$ of the ν holes and to the location of the complex v-roots.

The continuous theory is defined by the double limit $a \to 0, \Theta \to \infty$ taken in such a way that a finite mass gap m emerges. The explicit result in the periodic case which will be shown to hold also here is [1,19,20]

$$m = \lim_{a\to 0\ \Theta\to\infty}\left[\frac{4}{a}e^{-\frac{\pi}{a}\Theta}\right] \tag{4.17}$$

[The double limit is taken such that the l.h.s. is finite and non-zero]. The invariance of $U(\Theta)$ under the quantum group $SU(2)_q$ allows to perform a transformation of

the original vertex model where states are assigned to the links and interactions as 'bare' scattering amplitudes, to the vertices, into a diagonal lattice face model, where states are defined on the plaquettes. The plaquettes are the sites of the dual lattice and the interaction involves the four plaquettes around each vertex, or equivalentely, the four sites around each face of the dual lattice.

On the faces of the diagonal lattice one introduces positive integer-valued local height variables $\ell_n(t)$, $t \in \mathbb{Z}$, according to (see fig. 2)

$$\begin{aligned} \ell_0(t) &= 1\,;\ \ell_1(t) = 2 \\ \ell_{n+1}(t) &= \ell_n(t) \pm 1\,,\ n = 1, 2, \ldots N-1 \end{aligned} \tag{4.18}$$

The configurations $\{\ell_1(t), \ldots \ell_N(t)\}$ at any fixed discrete time t are in one-to-one correspondence with the $SU(2)_q$ multiplets of the 6V model of dimension $\ell_n(t) = 2J+1$ (notice that $\ell_N(t)$ can be chosen to be time-independent thanks to $SU(2)_q$ invariance). Now the matrix elements of U between these highest weight states define the unit time evolution in the face language.

This vertex-IRF correspondence is well known : in practice, it amounts to an application of the q-analog of the Wigner-Eckart theorem. Indeed, by $SU(2)_q$ invariance, the matrix elements of $U(\Theta)$ in the subspace with definite total q-spin $J = j$, for γ/π not a rational, can be written

$$< jm; \tau | U(\Theta) | j'm'; \tau' >= \delta_{jj'}\, \delta_{mm'} \ < \tau || U(\Theta) || \tau' >_j \tag{4.19}$$

where $m = -j, -j+1, ..., +j$ labels the J_z eigenvalues and τ the degeneracy of the $J = j$ subspace which has dimension

$$\begin{pmatrix} N \\ N/2 - j \end{pmatrix} - \begin{pmatrix} N \\ N/2 - j - 1 \end{pmatrix} \equiv \Gamma_j^{(N)} \tag{4.20}$$

$< \tau || U(\Theta) || \tau' >_j$ thus denotes the reduced matrix elements, and a specific choice of basis is required to give explicit expressions for them. The most natural basis, which is the basis useful for the vertex-IRF correspondence, is that obtained by the successive composition of the q-spin 1/2 basic constituents assigned to the links. Consider a 'time zero' line cutting the diagonal lattice as in fig.8. By intersecting N links, it identifies the Hilbert space of the vertex model as $(\otimes[1/2])^N$, where $[j]$ denotes an irreducible representation of weight j . On the other hand, the line passes also through a well defined set of plaquettes, including the half-plaquette at the extreme left and right. Then, a set of configurations of a local height variable l , assigned to each plaquette, can be constructed as follows. Assign $l = l_0 = 1$ to the half plaquette on the left, the 0th. plaquette. Then pass to the next one on the right , the first plaquette, following the time zero line. This cuts a link

carrying a spin 1/2 representation. So that, at this stage the Hilbert space is just the representation $[1/2]$ itself, i.e. $J = 1/2$. We then set $l_1 = 2J+1 = 2$. Following the line, we now cut another spin 1/2 link and arrive at the second plaquette. This Hilbert space is now the direct sum $[J=0] \oplus [J=1]$ and the local height l can take two values, $l_2 = 2j+1 = 1$ or 3 . By repeating this procedure n times (with $n \leq N$), we land in the nth. plaquette after having cut n spin 1/2 links : the Hilbert space is $(\otimes[1/2])^n$ and contains irreps with J running from 0 or 1/2 up to $n/2$. Hence the local variables l_n take the values

$$l_n = 2J+1 = \begin{cases} 1, 3, 5, \ldots, n+1, & n \text{ even} \\ 2, 4, 6, \ldots\ldots, n+1, & n \text{ odd} \end{cases}$$

By construction, $|l_n - l_{n-1}| = 1$, since the l_n follow the composition laws of $SU(2)_q$ representations

$$[J] \otimes [1/2] = [J+1/2] \oplus [J-1/2]$$

which for $q = \exp(i\gamma)$ not a root of unity, are just those of the usual SU(2).

When $n = N$ we arrive at the half-plaquette on the right. Here the values of l_n are just the dimensions of the irreducible representations in which the full Hilbert space $(\otimes[1/2])^N$ can be decomposed. All together, a specific choice of $l_2, l_3, \ldots, l_N$ (l_0 and l_1 are fixed to $l_0 = 1$, $l_1 = 2$ by construction) defines one and only one of the $\Gamma_j(N)$ irreps with $J = j$. , if $l_N = 2j+1$. In other words, we have constructed a map from the N + 1 plaquettes at 'time zero' and the Bratteli diagrams giving the composition rules for the tensor product of N spin 1/2 reprersentations (see fig.9). We are now in position to identify the degeneracy label τ with a specific path in the Bratelli diagram, i.e. $\tau = (l_0, l_1, \ldots, l_N)$, with $l_0 = 1$ and $l_N = 2j+1$.

It remains to evaluate the reduced matrix elements

$$< \tau || U(\Theta) || \tau' >_j = < l_0, l_1, \ldots, l_N | U(\Theta) | l'_0, l'_1, \ldots, l_N > \tag{4.21}$$

where the height variables $l'_0, l'_1, \ldots, l_N$ are associated to the plaquettes crossed by the 'time one' line (see fig.8). Notice that $l_0 = l'_0 = 1$ by definition, while $l_N = l'_N = 2j+1$ by $SU(2)_q$ invariance. However, in spite of the factorised form of $U(\Theta)$ [cft. eqs.(4.10)], these matrix elements cannot be written in a factorised form, with each factor depending locally on the variables $l_0, l_1, \ldots l_N$ and $l'_0, l'_1, \ldots, l_N$. This is so because the full unit time evolution operator $U(\Theta)$ is $SU(2)_q$ invariant, but each single $R_{k,k+1}(2\Theta)$ which enters is not. Only the peculiar boundary terms g_1 and g_N enforce $SU(2)_q$ invariance. Nevertheless, it is easy to pass to a different vertex representation where the $SU(2)_q$ symmetry holds in a local sense. We define

a similarity transformation on $U(\Theta)$

$$\tilde{U}(\Theta) = GU(\Theta)G^{-1} \tag{4.22}$$

where

$$G = g_1^{1/2}\, g_2^{-1/2}\, g_3^{1/2} \dots g_N^{\epsilon-1/2} \tag{4.23}$$

$\epsilon = \frac{1-(-1)^N}{2}$ and g_j is $\exp[-\Theta\sigma_z]$ acting on the j-th. spin 1/2 space. Now comparing eqs.(4.10) and (4.22)-(4.23) it is easy to verify that $\tilde{U}(\Theta)$ can be written, e. g. for even N

$$\tilde{U}(\Theta) = \tilde{R}_{12}\tilde{R}_{34}\dots\tilde{R}_{N-1,N}\tilde{R}_{23}\tilde{R}_{45}\dots\tilde{R}_{N-2,N-1} \tag{4.24}$$

where

$$\tilde{R}_{k,k+1} = g_k^{-1/2} g_{k+1}^{1/2} R_{k,k+1}(2\Theta) g_k^{1/2} g_{k+1}^{-1/2} \tag{4.25}$$

One can explicitly verify that $\tilde{R}_{k,k+1}$ is $SU(2)_q$ invariant; that is

$$\begin{aligned} [\tilde{R}_{k,k+1},\ (\sigma_z)_m\ +\ (\sigma_z)_{m+1}] &= 0 \quad , \\ [\tilde{R}_{k,k+1},\ (q^{-\sigma_z/2})_m\,(\sigma_\pm)_{m+1}\ +\ (\sigma_\pm)_m\,(q^{\sigma_z/2})] &= 0 \end{aligned} \tag{4.26}$$

The reduced matrix elements of $\tilde{U}(\Theta)$ can now be expressed in a factorised form ,

$$< l_0, l_1, \dots, l_N|U(\Theta)|l'_0, l'_1, \dots, l_N > = W'_1 W'_3 W'_5 \dots W'_{N-1+\epsilon} W_2 W_4 \dots W_{N-\epsilon} \ ,$$

where

$$W_m = W(l_{m-1}, l_{m+1}|l_m, l'_m; \Theta) \ , \quad W'_m = W(l'_{m-1}, l'_{m+1}|l_m, l'_m; \Theta) \tag{4.27}$$

and

$$W(x, y|u, v; \Theta) = \delta_{uv} - \frac{\sinh 2\Theta}{\sinh(2\Theta + i\gamma)} \frac{\sqrt{[u]_q[v]_q}}{[x]_q}\, \delta_{xy}. \tag{4.28}$$

with the standard notation $[x]_q = (q^x - q^{-x})/(q - q^{-1}) = \sin(\gamma x)/\sin\gamma$. This completes the transformation from vertex to faces description. The global time evolution in the faces (or heights) language is obtained by taking matrix products of

$$< l_0, l_1, \dots, l_N|U(\Theta)|l'_0, l'_1, \dots, l_N >$$

The boundary conditions are l_0 = constant = 1 on the left, while l_N = constant = $2j+1$ on the right, if the reduction is performed onto the $\Gamma_j^{(N)}$- dimensional space of irreps of q-spin J = j . With the natural constraint that $|l - l'| = 1$, when l and l' sit on neighboring faces, the weights given by eqs.(4.27)-(4.28) define the ABF-SOS model in the trigonometric regime[13].

5. Analysis of the Bethe Ansatz equations

We investigate in this section the solution of the BAE (4.12) associated to the quantum group covariant BA in the light-cone approach. It is convenient to relate them with the BAE for periodic boundary conditions (see for example [1]). Define 2r variables λ_j as

$$\lambda_j = v_j \quad , \quad \lambda_{j+r} = -v_{r-j+1} \ , \quad 1 \le j \le r \tag{5.1}$$

Then eqs.(4.12) can be rewritten as

$$\begin{aligned} &\left[\frac{\sinh[\lambda_m - \Theta + i\gamma/2]\sinh[\lambda_m + \Theta + i\gamma/2]}{\sinh[\lambda_m - \Theta - i\gamma/2]\sinh[\lambda_m + \Theta - i\gamma/2]}\right]^N \frac{\sinh(2\lambda_m + i\gamma)}{\sinh(2\lambda_m - i\gamma)} = \\ &-\prod_{k=1}^{2r} \frac{\sinh[\lambda_m - \lambda_k + i\gamma]}{\sinh[\lambda_m - \lambda_k - i\gamma]} \ , \qquad 1 \le m \le 2r. \end{aligned} \tag{5.2}$$

These equations are like the BAE for periodic boundary conditions on a 2N sites line and with an additional source factor

$$\frac{\sinh(2\lambda_m + i\gamma)}{\sinh(2\lambda_m - i\gamma)} = -\frac{\sinh(\lambda_m + i\gamma/2)}{\sinh(\lambda_m - i\gamma/2)} \frac{\sinh(\lambda_m - i\frac{\pi-\gamma}{2})}{\sinh(\lambda_m + i\frac{\pi-\gamma}{2})} \tag{5.3}$$

More important, we have the following constraints on the roots λ_m :

a) the total number of roots is even (2r) and they are symmetrically distributed with respect to the origin according to eq.(5.1).

b) $\lambda_m = 0$ and purely imaginary λ_m are excluded as roots.

Let us start by considering the antiferroelectric ground state. It is formed by roots with fixed imaginary part equal to $\pi/2$. Therefore, we shift $\lambda_m \to \lambda_m + i\pi/2$ and take λ_m real for the ground state. Now $\lambda_m = 0$ is excluded whereas it may be present for periodic boundary conditions (PBC). Therefore, the ground state for eq.(4.12) is a one-hole solution of eq.(5.2) with the hole at $\lambda_m = 0$.We also redefine γ into $\pi - \gamma$ in order to agree with the conventions of refs.[1,19,20]. With this choice, γ is related to the sine-Gordon coupling constant β by $\beta^2/(8\pi) = 1 - \gamma/\pi$.

Let us now consider the $N \to \infty$ limit where a continuous density of BAE real roots can be introduced

$$\rho(\lambda_m) = \lim_{N\to\infty} \frac{1}{N(\lambda_{m+1} - \lambda_m)} \tag{5.4}$$

This function must always be even in our fixed boundary conditions case. Taking logarithms in both sides of eq.(5.2) and using eqs.(5.3) and (5.4) the BAE yield

respect to the $\mathrm{Re}\,\lambda = 0$ axis. Hence, restricting to positive real

$$\frac{\sinh\alpha[\chi_j - \theta_h + i\gamma/2]}{\sinh\alpha[\chi_j - \theta_h - i\gamma/2]}\frac{\sinh\alpha[\chi_j + \theta_h + i\gamma/2]}{\sinh\alpha[\chi_j + \theta_h - i\gamma/2]} =$$

$$\prod_{k\neq j}^{/2} \frac{\sinh\alpha[\chi_j - \chi_k + i\gamma]}{\sinh\alpha[\chi_j - \chi_k - i\gamma]}\frac{\sinh\alpha[\chi_j + \chi_k + i\gamma]}{\sinh\alpha[\chi_j + \chi_k - i\gamma]} \tag{5.14}$$

$$1 \leq j \leq M_p/2.$$

come back to the BAE (4.12). They can be rewritten in a form by introducing the variables $z_j \equiv \exp(-2v_j)$, $(1 \leq j \leq r)$.

$$\left(\frac{w - q}{wq - 1}\frac{z_j - wq}{z_j q - w}\right)^N = \prod_{\substack{k=1\\k\neq j}} \frac{z_j - z_k q^2}{z_j q^2 - z_k}\frac{z_j z_k - q^2}{z_j z_k q^2 - 1} \tag{5.15}$$

) and $q = \exp(i\gamma)$ is the quantum group deformation parameter. neral discussion following eq.(3.6), we can restrict the search for) strictly within the unit circle $|z_j| < 1$. Therefore, to any stinct numbers fulfilling eq.(5.15), $z_1, \ldots, z_r$, with $|z_j| < 1$, ne BA eigenstate of the transfer matrix (4.4) and hence of the n the light cone-lattice. Special attention should be paid to the or more roots z_j lay exactly at the origin. This corresponds to refore to the reduction of the corresponding $\hat{\mathcal{B}}(v_j)$ to a multiple ator J_- of $SU(2)_q$ [cft. eq.(2.29)].

$$\hat{\mathcal{B}}(\infty) = (1 - q^2)J_- \tag{5.16}$$

at infinity constitutes the only exception to the requirement $, v_r$ be all distinct. However, this is a very special possibility. t a given root in eq.(5.15) , which we can always identify with n $z_1 = 0$. We immediatly obtain from eq.(5.15) :

$$q^{2N} = q^{4(r-1)} \Longrightarrow q^{4(J_z+1)} = 1 \tag{5.17}$$

root of unit and γ a rational multiple of π . This correspond when certain irreducible representations of $SU(2)_q$ mix into indecomposable representations. This interesting phenomena 7-9 within the Bethe Ansatz framework in connection with the

Eqs.(5.14) can also be written in algebraic form:

$$\prod_{h=1}^{\nu} \frac{\xi_j w_h - \hat{q}}{\xi_j w_h \hat{q} - 1}\frac{\xi_j - w_h\hat{q}}{\xi_j\hat{q} - w_h} = \prod_{\substack{k=1\\k\neq j}}^{\hat{M}} \frac{\xi_j - \xi_k\hat{q}^2}{\xi_j\hat{q}^2 - \xi_k}\frac{\xi_j\xi_k - \hat{q}^2}{\xi_j\xi_k\hat{q}^2 - 1} \tag{5.18}$$

where

$$\xi_j = e^{-2\alpha\chi_j}\ ,\quad w_h = e^{-2\alpha x_h}\ ,\quad \hat{q} \equiv e^{i\hat{\gamma}} = e^{i\alpha\gamma}\ ,\quad \alpha = \frac{\pi}{\pi - \gamma} \tag{5.19}$$

These new BAE involve only a finite number of parameters and are formally identical to the "bare" BAE (4.12)-(5.15), with the holes acting as sources in the place of the alternating rapidities $\pm\Theta$. Moreover, from (5.18) one reads out the renormalization of the quantum group deformation parameter

$$q \to \hat{q} \quad i.e. \quad \gamma \to \hat{\gamma} = \frac{\pi\gamma}{\pi - \gamma} \tag{5.20}$$

As we shall see later on, this renormalization has a nice physical interpretation for the SOS and RSOS models related to the 6V model by the vertex face correspondence.

6. Higher–level Bethe Ansatz and S–matrix

The holes in the sea of real BAE roots are the particles of the light–cone 6V model. They are $SU(2)_q$ doublets and can be present in even (odd) number for N even (odd). Consider first the even N sector, setting $N = 2N'$. The energy of a BA state with an even number ν of holes located at $x_1, x_2, \ldots x_\nu$ can be calculated to be, in the $N' \to \infty$ limit,

$$E = E_0 + a^{-1}\sum_{h=1}^{\nu} \mathrm{e}(x_h) + O(a^{-1}N^{-1})\ ,\quad \mathrm{e}(x) = 2\arctan\left(\frac{\cosh\pi x/\gamma}{\sinh\pi\Theta/\gamma}\right) \tag{6.1}$$

where E_0 is the energy of the ground state, that is the state with no holes. E_0 is of order N and is explicitly given in the appendix along with some detail on the derivation of eq.(6.1). Now suppose N odd, with $N = 2N' - 1$. To compare this situation with the previous one, we need to slightly dilate the lattice spacing a to $a' = 2N'a/(2N' - 1)$, in order to keep constant the physical size $L = Na$ of the system. Then eq.(6.1) remains perfectly valid, as $N' \to \infty$, also for the case of N odd, with the same ground state energy E_0. The only difference is that now ν is odd. Hence, alltogether, we obtain that the number of holes can be arbitrary (unlike in the

treatment with periodic b.c.) and that the total energy, relative to the ground state and in the $L \to \infty$ limit, is the uncorrelated sum of the energy of each single hole, independently of the complex pair structure of the corresponding BAE solution. The BA state with one hole has $J = N/2 - M = (2N' - 1)/2 - (N' - 1) = 1/2$ and is therefore a $SU(2)_q$ doublet. The 2^ν polarizations of a state with ν holes are obtained by considering all solutions of the higher level BAE (5.18) with $0 \le \hat{M} \le \nu/2$. We see in this way that the holes can be consistently interpreted as particles.

As a lattice system the light–cone 6V model is not critical. The (dimensionless) mass gap is the minimum value of the positive definite $e(x)$, the energy of a single hole, that is

$$e(0) = 2\arctan\left(\frac{1}{\sinh \pi\Theta/\gamma}\right) \tag{6.2}$$

This gap vanishes in the limit $\Theta \to \infty$. Hence the continuum limit $a \to 0$ can be reached provided at the same time $\Theta \to \infty$ in such a way that the physical mass

$$m = a^{-1}e(0) \simeq 4a^{-1}e^{-\pi\Theta/\gamma} \tag{6.3}$$

stays constant [19]. In the same limit we obtain the relativistic expression

$$a^{-1}e(x) \to m\cosh \pi x/\gamma \tag{6.4}$$

so that $\pi x/\gamma$ is naturally interpreted as the physical rapidity θ of the hole. This is consistent also in our fixed b.c. framework, provided the limit $L \to \infty$ is taken before the continuum limit. Indeed the total momentum becomes a conserved quantity in the infinite volume limit and its eigenvalues can be expressed in terms of the BA v–roots exactly as in the periodic b.c. formulation. Then one finds that the momentum of a single particle takes the required form $m\sinh\theta$ in the continuum limit.

The above analysis shows that a relativistic particle spectrum appears in the $a \to 0$ limit above the antiferromagnetic ground state. For $\gamma > \pi/2$ one can show also that bound states appear, associated to appropriate strings of complex roots, just as in the periodic b.c setup. In the sequel we shall anyway restrict our analysis to the repulsive $\gamma < \pi/2$ region, where the only particles are the holes, to be identified with the solitons of the SG model. Of course at this moment, since the infinite volume limit is already implicit, the particles are in their asymptotic, free states: the rapidities $\theta_h = \pi x_h/\gamma$ may assume arbitrary continuus values and the total excitation energy is the sum of each particle energy and does not depend on the internal state of the particles. The situation changes if we consider L very large but finite, since in this case the hole parameters $x_1, \ldots, x_\nu$ are still quantized

through the "bare" BAE (4.12). Indeed, by definition the $\bar{I}$
numbers satisfying

$$Z_N(x_h; v_1, v_2, \ldots, v_M) = \frac{2\pi \bar{I}_h}{N} \quad ; \quad h = 1, .$$

where $Z_N(x; v_1, \ldots, v_M)$ is the "counting function" defined
the positive integers $\bar{I}_1, \ldots, \bar{I}_\nu$ are all distinct from the intege
the r ($r \le M$) real v–roots of the BAE. For N very large, J
$x < (\gamma/\pi)\ln N$, the counting function can be approximated a

$$Z_N(x; v_1, \ldots, v_M) = Z_\infty(x) + N^{-1}F(x; x_1, \ldots, x_M; \chi_1, \ldots,$$

where $Z_\infty(x)$ is the ground state counting function *at* the the

$$Z_\infty(x) = 2\arctan\left(\frac{\sinh \pi x/\gamma}{\cosh \pi\Theta/\gamma}\right)$$

and

$$F(x; x_1, \ldots, x_M; \chi_1, \ldots, \chi_M) = -i\log\prod_{h=1}^{\nu} S_0\left(\tfrac{\gamma}{\pi}(x - x_h)\right) S_0$$

$$- i\log\prod_{j=1}^{\hat{M}} \frac{\sinh\alpha(x - \chi_j + i\gamma/2)\,\sinh\alpha(x}{\sinh\alpha(x - \chi_j - i\gamma/2)\,\sinh\alpha(x}$$

In the last expression, the numbers χ_j are the roots of the hig
(5.19), while $S_0(\theta)$ coincides with the soliton-soliton scatterin
model

$$S_0(\theta) = \exp i\int_0^\infty \frac{dk}{k}\frac{\sinh(\pi/2\gamma - 1)k\,\sin k\theta}{\sinh(\pi k/2\hat{\gamma})\quad\cosh k}$$

under the standard identification $\gamma/\pi = 1 - \beta^2/8\pi$.

Combining eqs. (6.5) and (6.6), and taking the continuum
"higher level" expression

$$\exp(-imL\sinh\theta_h) = \prod_{\substack{n=1\\ n\neq h}}^{\nu} S_0(\theta_h - \theta_n)\; S_0(\theta_h + \theta_n)\prod_{j=1}^{\hat{M}} \frac{\xi_j w_h}{\xi_j w_h \hat{q}}$$

where, according to (5.19), $\xi_j = e^{-2\alpha\chi_j}$ and $w_h = e^{-2\hat{\gamma}\theta_h/\pi}$. To
level BAE (5.18), this last equation provides the exact Beth
tion of the commuting family formed by the ν renormalized

)

$$\theta_{h-1})\ldots S_{h,1}(\theta_h-\theta_1)\,g_h(-\theta_h)\,S_{h,1}(\theta_h+\theta_1)\ldots S_{h,\nu}(\theta_h+\theta_\nu)\times$$
$$_h-\theta_\nu)\ldots S_{h,h+1}(\theta_h-\theta_{h+1}) \tag{6.11}$$

σ_h^z and $S(\theta)$ is the complete 4×4 SG soliton $S-$matrix in the $<\pi/2$ (for brevity we set $\hat\theta=\gamma\theta/(\pi-\gamma)$):

$$)\begin{pmatrix}1&0&0&0\\0&\hat b&\hat c&0\\0&\hat c&\hat b&0\\0&0&0&1\end{pmatrix}\qquad \begin{aligned}\hat b&=b(\hat\theta,\hat\gamma)=\frac{\sinh\hat\theta}{\sinh(i\hat\gamma-\hat\theta)}\\ \hat c&=c(\hat\theta,\hat\gamma)=\frac{\sinh i\hat\gamma}{\sinh(i\hat\gamma-\hat\theta)}\end{aligned} \tag{6.12}$$

re the values at $\theta=\theta_h$ of the fully inhomogeneous Sklyanin $T(\theta;\theta_1,\ldots,\theta_\nu)$ constructed with $S(\theta-\theta_h)$ as local vertices. he higher-level Bethe Ansatz structure just described follows cation of the BAE solution corresponding to the ground state er assumptions, from the "bare" BA structure of the light-cone cular, this provides a derivation "from first principles" of the want to remark that $S(\theta)$ is also the exact $S-$matrix for the ns of the 6V model *on the infinite lattice*: it is a bona fide *lattice* e, in this case $\theta=\theta_1-\theta_2$ is the difference of *lattice* rapidities, the energy and momentum of the scattering particles through ation

$$\begin{aligned}&=a^{-1}Z_\infty(\gamma\theta_j/\pi)=2a^{-1}\arctan\left(\frac{\sinh\theta_j}{\cosh\pi\Theta/\gamma}\right)\\ &=a^{-1}\epsilon(\gamma\theta_j/\pi)=2a^{-1}\arctan\left(\frac{\cosh\theta_j}{\sinh\pi\Theta/\gamma}\right)\end{aligned} \tag{6.13}$$

dispersion relation

$$\cos aE_j/2=\tanh(\pi\Theta/\gamma)\cos aP_j/2 \tag{6.14}$$

e continuum limit only changes these energy-momentum re- ard relativistic form, $(E_j,\,P_j)=m(\cosh\theta_j,\,\sinh\theta_j)$, without rix as a function of θ.

7. SOS and RSOS reductions and kink interpretation

Under the vertex-face correspondence previously described, the light-cone 6V model is mapped into the SOS model. At the level of the Hilbert space, this corresponds to the restriction to the highest weight states of $SU(2)_q$: each $(2J+1)-$dimensional multiplet of spin J (for generic, non rational values of γ/π) is regarded as a single state of the SOS model. In particular, in such a state the local height variables ℓ_n have well defined boundary values $\ell_0=1$ and $\ell_N=2J+1$. The ground state of the 6V model, which is a $SU(2)_q$ singlet, is also the ground state of the SOS model: it has $\ell_0=\ell_N=1$ and is "dominated", in the thermodinamic limit $N\to\infty$ by the see-saw configuration depicted in fig. 1. Our boundary conditions allow for only one such "ground state dominating" configuration, with $\ell_n=(3-(-)^n)/2$, while periodic b.c. *on the SOS variables* ℓ_n would allow any possibility: $\ell_n=(2\ell+1\pm(-)^n)/2$, with ℓ any positive integer.

Consider now a BA state with one hole. Then N is necessarily odd while $J=1/2$ and $\ell_N=2J+1=2$. From the SOS point of view this state is "dominated" by height configurations of the type depicted in fig.4. Upon "renormalization", we can replace the ground state and one hole configurations with the smoothed ones of fig. 5. The hole corresponds, in configuration space, to a kink of the SOS model which interpolates between two neighboring vacuum states: the vacuum on the left of the kink has $\ell_n=(3-(-)^n)/2$, corresponding to a constant "renormalized" height $\hat\ell_n=1$, while the vacuum on the right has $\ell_n=(5+(-)^n)/2$, corresponding to $\hat\ell_n=2$. Of course, many kink configurations like that depicted in fig. 4 are to be combined into a standing wave (a plane wave in the infinite volume limit which turns the one hole BA state into an eigenstate of momentum). Let us observe that, by expressions like "dominating configuration", we do not mean that, e.g. the ground state, becomes an eigenstate of ℓ_n as $N\to\infty$. We expect local height fluctuations to be present even in the thermodynamic limit or, in other words, that the ground state remains a superpositions of different height configurations. The identification of a dominating configuration is made possible by the integrability of the model, which guarantees the existence of the higher-level BA. In turns, the higher-level BA allows us to consistently interpret the ground state or the one hole state as in figs. 1, 4 and 5, since the presence of fluctuations only renormalizes in a trivial way the scattering of physical excitations relative to the bare, or microscopic, $R-$matrix (2.17), as evident from eq.(6.12). Therefore, we can reinterpret at the renormalized level the vertex-face correspondence: the holes, that is the solitons of the SG model, are $SU(2)_q$ doublets acting as SOS kinks that increase or decrease by 1 the renormalized heights $\hat\ell_n$. The higher-level BA makes sure that the total number of internal states of ν kinks interpolating between $\hat\ell=1$ and $\hat\ell=2J$ is just the number of highest weight states of spin J in the tensor product of ν doublets

$$d_\nu(J)=\binom{\nu}{\hat M}-\binom{\nu}{\hat M-1}\quad;\quad \hat M=[\nu/2]-J \tag{7.1}$$

This parallels exactly the original BA, which provides the

$$d_N(J) = \binom{N}{M} - \binom{N}{M-1} \quad ; \quad \hat{M} = [N/2] - J \tag{7.2}$$

highest wight states of N doublets with total spin J.

Comparing eqs. (2.17), (4.28) and (6.12), we can directly write down the $S-$matrix of the SOS kinks:

$$S(\theta)^{is,sj}_{ir,rj} = S_0(\theta) \left\{ \delta_{rs} + \delta_{ij}\, b(\hat{\theta}, \hat{\gamma}) \left[\frac{\sin \hat{\gamma} r \, \sin \hat{\gamma} s}{\sin \hat{\gamma} i \, \sin \hat{\gamma} j} \right]^{1/2} \right\} \tag{7.3}$$

It defines the two-body scattering as follows. The first ingoing kink interpolates between the local vacuum with $\hat{\ell} = i$ and that with $\hat{\ell} = r \quad (r = i \pm 1)$, while the second interpolates between $\hat{\ell} = r$ and $\hat{\ell} = j \quad (j = r \pm 1)$. The outgoing kinks are interpolating between $\hat{\ell} = i$ and $\hat{\ell} = s \quad (s = i \pm 1, i.e, s = r, r \pm 2)$ and between $\hat{\ell} = s$ and $\hat{\ell} = j \quad (j = s \pm 1)$.

When $\gamma = \pi/p$ for $p = 3, 4, \ldots$, the SOS models can be restricted to the RSOS(p) models, by imposing $\ell_n < p$. In the vertex language of the 6V model this corresponds to the Hilbert space reduction to the subspace formed by the so-called type II representations of $SU(2)_q$ with $q^p = -1$ [5,6]. In the next section we shall describe in detail how the restriction takes place in our BA framework. Here we simply observe that the local height restriction, when combined with the finite renormalization $\gamma \to \hat{\gamma}$ of eq.(5.20), provides a strong support for the kink interpretation presented above. Indeed, if $\gamma = \pi/p$, then $\hat{\gamma} = \pi/(p-1)$ and the renormalized heights $\hat{\ell}_n$ can take the values $1, 2, \ldots, p-2$. This appears now obvious, since each constant configuration of $\hat{\ell}_n$ corresponds to an ℓ_n- configuration oscillating two neighboring values. Moreover under the standard identification of the critical RSOS(p) models with the minimal CFT series M_p, we see that each light-cone RSOS(p) model has kink excitations whose $S-$matrix (7.3) is proportional to the (complex) microscopic Boltzmann wights of the RSOS($p-1$) model, under the replacement of $2\Theta = \Theta - (-\Theta)$= rapidity difference of light-cone right and left movers, with $(p-1)$ times $\theta = \theta_1 - \theta_2$ =rapidity difference of physical particles. This is exactly the pattern found by bootstrap techniques [21] for the minimal model M_p perturbed by the primary operator $\phi_{1,3}$ (with negative coupling).

8. BA roots when q is a root of unity and Quantun

As previously explained to each M roots solution of the l
a highest weight state of the quantum group $SU(2)_q$ with s
is well known that when q is a root of unity, say $q^p = \pm 1$, t
and the representations of $SU(2)_q$ divide into two very di
representations are reducible and generally indecomposable.
as pairwise mixings of standard irreps (that is the irreducibl
not a root of unity) with spin J and J' such that

$$|J - J'| < p\,, \qquad J + J' = p - 1 \ (\mathrm{mod}$$

Notice that the sum of $q-$dimensions for this pair of reps va

Type II representations are all the others. They are sti
structured just like the usual $SU(2)$ irreps. Since $(J_\pm)^p = 0$,
have necessarily dimension smaller than p, that is

$$J < \frac{p-1}{2}$$

We shall now show how the BAE reflect these peculiar pr
representations for q a root of unity. First of all let us stres
root of unity ($i.e. \gamma/\pi$ is irrational), then the BAE cannot po
infinity. Indeed, since $\hat{\mathcal{B}}(\infty)$ is proportional to J_- (eq.(5.1
means that the corresponding BA state is obtained by the
operator J_- on some other state with higher spin projection
of unity, the BA states have $J = J_z$ and therefore cannot be
J_- on any other state. On the other hand, if we assume that
the BAE (4.12) lay at the origin, $i.e. Re\, v_1 = +\infty$, then eqs.

$$q^{4(J+1)} = 1$$

That is, q must be a root of unity. It is now crucial to obser
equations for the non-zero roots (those labelled by $j = 2, 3$
the BAE for $M-1$ unknowns. This invariance property holds
group covariant, fixed boundary conditions BAE. It does n
BAE where $v-$roots at infinity twist the remaining equation
twisting reflect the fact that the corresponding periodic row-
is not quantum group invariant but gets twisted under $SU(2$
Thus, when q is a root of unity, BA states (4.11) with one
the form

$$\Psi(v_1 = \infty, v_2, \ldots, v_M) = (1 - q^2) J_- \Psi(v_2, v_3, .$$

Let us recall that the BA state on the l.h.s. is annihilated by
q a root of unity) [3]. Hence eq.(8.4) represents the mixing

a type I representation. Indeed, applying the mixing rule condition (8.3) for one root at infinity, yields an identity as

$$1 = q^{4(J+1)} = q^{2(J+J'+1)} = (q^p)^{2n} = 1 \tag{8.5}$$

J'– dependent) positive integer.

his analysis to any number of vanishing BA z–roots. Let ing to the origin with $z_1, z_2, \ldots, z_r$, $1 \le r \le M$. The BAE zero roots $z_{r+1}, \ldots, z_M$ take the standard form (4.12) valid by $M - r$ $\hat{\mathcal{B}}$–operators, which has therefore spin $J + r$. The roots take the form

$$z_j^{2(r+1)} = F_j \equiv \prod_{\substack{k=1 \\ k\neq j}}^{r} \frac{z_j q^2 - z_k}{z_j - z_k q^2}\,, \quad 1 \le j \le r \tag{8.6}$$

F_j is understood in the limit of vanishing $z_1, \ldots, z_r$. We can ent with the quantum group mixing rules (8.1) by setting natural choice

$$F_j = 1\,, \quad 1 \le j \le r \tag{8.7}$$

vanishing z–roots imply the mixing of two representations $+ r$, so that

$$q^{4J+2(r+1)} = q^{2(J+J'+1)} = (q)^{2n} = 1 \tag{8.8}$$

nding on J and J'. Eq. (8.7) ha a very simple solution for of the vanishing roots. We find that if

$$z_j = \omega^{j-1} z_1\,, \quad 1 \le j \le r \tag{8.9}$$

ically

$$\prod_{\substack{k=1 \\ k\neq j}}^{r} \frac{\sin[-\gamma + \pi(k-j)/r]}{\sin[\gamma + \pi(k-j)/r]} = 1\,, \quad 1 \le j \le r \tag{8.10}$$

hat eq.(8.10) should hold for generic values of $q = e^{i\gamma}$ is studying the approach of the roots to the origin when γ/π ber. We would like to remark that the limiting behaviour of ing BA z–roots depend only on their number r and neither value of γ/π nor on the other non vanishing roots.

In the v–plane, the roots $v_1, v_2, \ldots, v_r$ go to infinity as a r–string with spacing $i\pi/r$. For example, when $r = 2$ we have a pair of complex roots with imaginary parts tending to $\pm i\pi/4$ as their real parts simultaneously diverge. For $r = 3$ there will be a limiting 3-string with a real member and a complex pair at $\pm i\pi/3$. For these two examples, we have performed an explicit numerical test when $\gamma \to (\pi/3)^-$ and $\gamma \to (\pi/4)^-$, respectively, finding perfect agreement with the picture proposed here. We shall present more detail on the special cases in the next section. Let us anticipate here that the numerical results suggest the following conjecture on the behaviour of the singular roots: $|z_j| \approx O(\epsilon^{1/r})$ and $\mathrm{Im}[v_j - v_j(\epsilon = 0)] \approx O(\epsilon^{1/r})$ where $\epsilon = \gamma - \pi/(r+1)$.

When $q^p = \pm 1$ and the $SU(2)_q$ reps divide into type I and type II, it is possible to perform the consistent RSOS(p) reduction of the Hilbert space. This consists in keeping, among all states annihilated by J_+ which form the SOS subspace, only the type II states. It is known that this corresponds to SOS configurations with local heights ℓ_n restricted to the set $1, 2, \ldots, p-1$ [(8.3)6]. In our BA framework, therefore, the RSOS reduction is obtained by retaining only those states with spin $J < (p-1)/2$, that is with $M > (N - p + 1)/2$, such that all BA roots $z_1, z_2, \ldots, z_r$ are non-zero. This constitues the general and simple prescription to select the RSOS(p) subspace of eigenstates of the evolution operator when $q^p = \pm 1$.

We have considered up to here the RSOS reduction at the microscopic level, that is for the BA states (4.11) described by the bare BAE (4.12). It should be clear, however, that the analysis of the singular BAE solutions when q is a root of unity holds equally well for the higher-level BAE (5.18), due to their structural identity with the bare BAE. The crucial problem is whether a quantum group reduction carried out the higher level would be equivalent to that described above in the "bare" framework. This equivalence can actually be established as follows. Suppose that γ/π is irrational, but as close as we like to a specific rational value. Then all solutions of the bare BAE are regular and can be correctly analyzed, in the limit $N \to \infty$, using the density description for the real roots. As described in paper I, this yields the higher-level BAE (5.18), those roots are in a precise correspondance with the *complex* roots of the bare BAE. When γ tends to π/p, then $\hat{\gamma}$ tends to $\pi/(p-1)$ and singular solutions of the bare BAE with two or more singular roots are in one-to-one correspondance with singular solutions of the higher-level BAE. Thus in this case quantum group reduction and renormalization of the BAE commute. The only potentially troublesome cases are those of singular solutions with only one real singular root. Indeed the density description of the v–roots cannot account, by construction, for *real* roots at infinity, and no sign of the type I nature of the corresponding BA state would show up at the higher level. Notice that spin J BA states with a single singular root are mixing with spin $J+1$ states, so that, by the quantum group mixing rule (8.1), necessarily $J = p/2 - 1$. On the other hand, a direct application of the constraint (8.2) at the higher level, that is with the replacement $p \to \hat{p} = p - 1$, would overdo the job, by incorrectly ruling out all BA states with $J = p/2 - 1 = (\hat{p} - 1)/2$. To be definite, consider N

even. Then the type II BA states with $J = p/2 - 1$, which ar e superpositions of SOS configurations with $\ell_n \leq p - 1 = 2J + 1 = \ell_N$, have an effective, higher-level spin $\hat{J} \equiv (\hat{\ell}_N - 1)/2 = J - 1/2$. This higher-level spin does satisfy (8.2). The type I states are those in which ℓ_n somewhere exceeds $p - 1$. In particular, the simplest change on a type II configuration, turning it into type I, is to flip the oscillating ℓ_n close to the right wall (as shown in fig. 6). ℓ_N and hence the spin J are left unchanged, but clearly $\hat{\ell}_N$ increases by one, causing $\hat{J}$ to violate the bound (8.2). In other words, RSOS(p)-acceptable BA states with $J = p/2 - 1$ have one kink less than the corresponding SOS states. ¿From the detailed analysis of the $p = 4$ case, to be discussed below, it appears that the removal of one kink corresponds to giving infinite rapidity to the hole representing that kink in the higher-level BAE.

Thank to the possibility of performing the restriction directly at the renormalized level, the kink S-matrix of the RSOS(p) model follows by direct restriction on the SOS S-matrix given by eq.(7.3). Namely, one must set $\gamma = \pi/p$, that is $\hat{\gamma} = \pi/(p-1)$, and consider all indices as running from 1 to $p - 2$.

9. The models RSOS(3) and RSOS(4)

In order to clarify matters about the BA RSOS reduction discussed in the previous section, we present here the more details about the two simplest examples, when $\gamma = \pi/3$ and $\gamma = \pi/4$. For these two cases the RSOS reductions correspond, respectively, to a trivial one-state model and to the Ising model (at zero external field and non-critical temperature). Let us recall that the light-cone approach yields in the continuum limit massive field theories with the same internal symmetry of the corresponding critical regimes. Therefore the RSOS(4) reduction of the light cone 6V model coincides with the $\mathbb{Z}_2$- preserving perturbation of the $c = 1/2$ minimal model (which is obtained upon quantum group restriction of the critical 6V model).

9.1 The case $p = 3$

From the RSOS viewpoint, the case $\gamma = \pi/3$ is particulary simple. Since $p = 3$ the local hight variables ℓ_n can assume only the values 1 and 2. Then the SOS adjacency rule $|\ell - \ell'| = 1$ implies that the global configuration is completely determined once the height of any given site is chosen. In our light-cone formulation the first height on the left is frozen to the value 1 (see eq.(4.18) and fig. 2), so that the restricted Hilbert space contains only one state: an $SU(2)_q$ singlet when N is even and the spin up component of a $J = 1/2$ doublet when N is odd.

For $\gamma = \pi/3$ the BAE still have many solutions which reproduce the full SOS Hilbert space. Indeed, from the SOS point of view, nothing particular happens when $\gamma \to \pi/3$. However, at this precise value of the anisptropy, only one BAE solution corresponds to the unique type II representation: for N even (odd) it contains $N/2\,(N/2 - 1/2)$ non-zero roots in the z-plane. All other BAE solutions

with $M = [N/2]$ contain at least one vanishing z-root and
I SOS states. Moreover, there unique RSOS state is the grou
model and is formed by real positive roots labelled by consecut
(no holes). We reach therefore the following rather non-tri
complicated system of algebraic equations (4.12) admit, for
one and only one solution with $M = [N/2]$ non-zero roots w
In addition, these roots are real and positive. For few specific
verified this picture numerically.

9.2 The case $p = 4$

The local height variables can assume now the three values
each configuration can be decomposed into two sub-configu
two sublattices formed by even and odd faces, respectively.
the SOS adjacency rule $|\ell - \ell'| = 1$ freezes the local heights
value $\ell = 2$. Then on the other sub-lattice we are left with tw
and 3. Moreover, the interaction round-a-site of the original
way to a nearest-neighbor interaction in the vertical and horiz
framework is that of the Ising model.

To obtain the standard Ising formulation, we can set

$$\sigma_n(t) = \ell_{2n}(t) - 2$$

where the numbering of the lattice faces can be read from fig
the ℓ_n now correspond to

$$\sigma_0(t) = -1\,, \quad \sigma_R(t) = (-1)^{J-1} \qquad (N = 2$$
$$\sigma_0(t) = -1\,, \quad \sigma_R(t) = \pm 1 \qquad (N = 2R$$

where $J = 0, 1$ for N even, due to the bound (8.2). For N odd
the possibility $J = 1/2$, which implies $\ell_N = 2$, since the line
the extreme right belong to the frozen sublattice. Then ℓ_{N-1}
between 1 and 3, leading to free b.c. on σ. The matrix elem
evolution operator $\tilde{U}$ can be calculated from the explicit for
(4.27)-(4.28). They can be written in the "lagrangian" form

$${}_{t+1}\langle \sigma'_0, \sigma'_1, \ldots \sigma'_R | \, \tilde{U} \, | \sigma_0, \sigma_1, \ldots \sigma_R \rangle_t = e^{iL}$$

where

$$L(t) = \beta_v \sum_{n=1}^{R-1} \sigma_n(t)\sigma_n(t+1) - \beta_h \sum_{n=0}^{R-1} \sigma_n(t)\sigma_{n+1}($$

$$\tfrac{1}{4}(\pi + 2i \ln \tanh 2\Theta)\ , \quad \beta_h = \arctan \tanh 2\Theta \tag{9.5}$$

d simple manipulations allow to rewrite $\tilde{U}$ explicitly in terms

$$\begin{aligned} \tilde{U} &= e^{-i\beta_h H_2} e^{-i\beta_h H_1} \\ H_1 &= \sum_{n=1}^{R-1} (\sigma_n^x + 1)\ , \quad H_2 = \sum_{n=0}^{R-1} \sigma_n^z \sigma_{n+1}^z \end{aligned} \tag{9.6}$$

es that the *complex* Boltzmann weights *formally* belong to the

$$\sin 2\beta_v \sin 2\beta_h = 1 \tag{9.7}$$

s from the original definition of the light-cone 6V model in onometric Boltzmann weights, which, under the replacement pond to the critical standard 6V model. Nevertheless, just as also in this "light-cone" Ising model, a massive field theory the $\mathrm{Re}\,\Theta \to +\infty$ limit.

lization of $\tilde{U}$, we must restrict ourselves to the BAE solutions $= R - 1$ for even N and $M = R$ for odd N. For sufficently ınd state has R real roots with consecutive quantum integers. ated above, this is a general fact valid for all RSOS(p) models. olume ground state of th SG model, of the SOS model and RSOS(p) is the same f.b.c. BA state. It is the unique $SU(2)_q$ sitive roots and no holes. It is described in the thermodynamic l) by the density of roots given in eq.(5.7) of paper I.

h $J = 0$ have an even number of holes and a certain number that no z-root lays at the origin. For instance a two-particle es (holes are naturally identified with the particles) and a two- parts $\pm[\pi/8+$ corrections exponentially small in $N]$ laying . This state (that is this precise choice of quantum integers) is tate of the SG or SOS model, with γ fixed to the precise value ly checked, by numerically solving the BAE for various values z-roots stay away from the origin as γ crosses $\pi/4$ while the kept fixed to the two-hole configuration. Now consider a state t from the multiplicity of the rapidity phase space, there are ıch states: in v-space one contains two two-strings, while the le wide pair, that is a complex pair with imaginary part larger n holds for generic values of γ/π and simply reflects the fact $2)_q$ doublets. The crucial point is that, as γ reaches $\pi/4$ (from noves towards infinity, while its real part gets closer and closer

to the (diverging) value of the largest real root and the imaginary part approaches $\pi/3$. This picture is confirmed by a careful numerical study of the BAE and is in perfect agreement with the general picture presented in sec. 5. In addition, there exists numerical evidence that the real parts diverge like $\log(\pi/4 - \gamma)^{-1/6}$ while the imaginary part of the wide pair goes to $\pi/3$ like $(\pi/4 - \gamma)^{1/3}$. When $\gamma > \pi/4$, the v-root corresponding to the largest quantum integer has the largest real part, but is no longer real. having an imaginary part equal to $\pi/2$. As $\gamma \to (\pi/4)^+$, this real part again diverges together with real part of the wide pair. Hence the four-hole state with a wide pair is type I when $\gamma = \pi/4$. On the other hand, the four-hole state with two two-string is type II, since all its v-roots stay finite.

¿From the study of the two- and four-hole states, we are led to the following general conjecture: in the BA framework, the RSOS(4) $J = 0$ states are all and only the states with $\nu = 2k$ holes and k two-strings. Then, in the higher-level BAE (5.18), we recognize this state as that corresponding to the unique solution with k real χ-roots. In other words, this BA state is completely determined once the location of the holes (that is the rapidities of the physical particles) is given.

Consider now the states with $J = 1$. In this sector, the lowest energy type II state contains exactly one hole. This is an unusual situation for BA systems on lattices with even N, where holes are always treated in pairs. As long as $\gamma \neq \pi/4$, the same is true in our f.b.c. BA: the lowest energy state with $J = 1$ contains two holes, in agreement with the interpretation of holes as SG solitons with quantum spin 1/2. For $\gamma < \pi/4$ all roots are real. For $\gamma > \pi/4$ the root $v_{N/2-1}$ corresponding to the largest quantum integer $I_{N/2-1} = N/2 + 1$ aquires an imaginary part $\pi/2$ (see the appendix for details). But when $\gamma = \pi/4$ then $\mathrm{Re}\, v_{N/2-1} = +\infty$, and the $J = 1$ two-hole state is mixed with some $J = 2$ state into a type I representation. It does not belong to the RSOS(4) Hilbert space. This picture can be easily verified numerically.

To prevent the largest root $v_{N/2-1}$ from diverging, it is sufficent to consider a $J = 1$ state with only one hole and $I_{N/2-1} = N/2$. From the point of view of the SG or SOS models this one-hole state has a cutoff-dependent energy which diverges as a^{-1} in the continuum limit. If x_1 is the position of the single hole, the energy relative to the ground state reads

$$E - E_0 = a^{-1} \mathrm{e}(x_1) + \pi a^{-1} \tag{9.8}$$

where the renormalized energy function $\mathrm{e}(x)$ is given in eq.(6.1). Notice that this result coincides with the limit $x_2 \to \infty$ of a two-hole state (cft. eq.(6.1)). Strictly speaking, however, the density method leading to (6.1) has no justification when "one hole is at infinity". We trust eq.(9.8) nonetheless because it passes all our numerical checks. In the continuum limit $a \to 0$, $\Theta \to \infty$, $4a^{-1}e^{-\pi\Theta/\gamma} = m$ (fixed), this one-hole state is removed from the physical spectrum, as required from the SG point of view: holes are spin 1/2 solitons while here $J = 1$.

The preceding discussion easily extends to generic states in the $J = 1$ sector containing an odd number of holes and a given set of complex pairs. The infinite-volume energy of these multiparticle states is given by

$$E = E_0 + a^{-1} \sum_{h=1}^{\nu} e(x_h) + \pi a^{-1} \tag{9.9}$$

with the same divergent constant πa^{-1} appearing irrespective of the physical content of the state. Our conjecture for the $J = 0$ states naturally extends to these $J = 1$ states: if there are $\nu = 2k+1$ holes, then the v-roots are all finite, implying that the state is type II, provided there are also k two-strings. Then the higher-level BAE imply that these states are completely defined by the hole rapidities. To retain these type II $J = 1$ BA states in the RSOS(4) model, an extra J-dependent subtraction is necessary to get rid of the divergent constant. Namely, for N even and $J = 0$ or 1, we set

$$H_{RSOS(4)} = H_{SOS}(\gamma = \pi/4) - \pi a^{-1} J \tag{9.10}$$

where H_{SOS} is given by eqs. (4.10), (4.14), (4.22)-(4.28).

Alltogether, we see that the BA picture for the excitations of the RSOS(4) model is fully consistent with the kink interpretation for the holes. Indeed $J = 0$ corresponds to even Dirichlet b.c. for the Ising field, while $J = 1$ corresponds to odd Dirichlet b.c. (cft. eqs. (9.2)). In the Ising model the kink description is valid below the critical temperature, with the disorder field as natural interpolator for the kinks. In this case the S-matrix must be -1 (it is $+1$ when the asymptotic particles are the free massive Majorana fermions) and this is exactly what follows from eq.(7.3) upon setting $\hat{\gamma} = \pi/3$ and restricting all indices to run from 1 to $p - 2 = 2$.

The analisys of the BA spectrum for odd N does not contain real new features. Now J is fixed to 1/2 and the b.c. on the Ising field are of mixed fixed-free type, as shown in the second of eqs. (9.2). The lowest energy state corresponds to the BAE solution formed by real roots and one single hole as close as possible to the the v-origin. In other words, the quantum integers are given by $I_j = j + 1$, for $j = 1, 2, \ldots, (N-1)/2$. By letting this hole to move away along the positive real axis, we reconstruct the energy spectrum of a one particle state. Of course, to compare this state to the global ground state, which contains no holes, a judicious choice of the odd value of N is required. If in the ground state $N = 2R$, then the new state is indeed an excited state if we choose $N = 2R - 1$, rather than $N = 2R + 1$, because of the antiferromagnetic nature of the interaction.

APPENDIX

In this appendix, we present for completeness a (rather non
of the BAE (4.12). As explained in paper I, it is convenient t
the p.b.c. form

$$\left[\frac{\sinh(\lambda_j - \Theta + i\gamma/2)\,\sinh(\lambda_j + \Theta + i\gamma/2)}{\sinh(\lambda_j - \Theta - i\gamma/2)\,\sinh(\lambda_j + \Theta - i\gamma/2)}\right]^N \frac{\sinh}{\sinh}$$
$$= - \prod_{k=-M+1} \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)}$$

where the numbers $\lambda_{-M+1}, \lambda_{-M+2}, \ldots, \lambda_M$ are related to the

$$\lambda_j = v_j = -\lambda_{1-j}\,, \quad j = 1, 2, \ldots, M$$

and $\mathrm{Re}\, v_j > 0$, thanks to the symmetries of the BAE (4.12). I
attention on those BAE solutions which are mostly real, that
an arbitrary but fixed number of complex pairs interspersed
of real roots. The reason for this restriction will become cle
counting function associated to a given solution $v_1, \ldots, v_M$ is

$$Z_N(x\,; v_1, \ldots, v_M) = \phi_{\gamma/2}(x + \Theta) + \phi_{\gamma/2}(x - \Theta) +$$
$$-N^{-1} \sum_{j=-M+1} \phi_\gamma(x - \lambda_j)$$

where

$$\phi_\alpha(x) \equiv i \log \frac{\sinh(i\alpha + x)}{\sinh(i\alpha - x)}$$

The logarithmic branch in eq.(A4) is chosen such that $\phi_\alpha(x)$
quence $Z_N(x\,; v_1, \ldots, v_M)$, are odd.

The BAE (A1) can now be written in compact form

$$Z_N(\lambda_j\,; v_1, \ldots, v_M) = 2\pi N^{-1} I_j\,, \quad j = -M+1, -M$$

where the *quantum integers* I_j entirely fix the specific BAE s
the BA eigenstate, and by construction satisfy $I_{1-j} = -I_j$
Notice also that by definition $Z_N(0\,; v_1, \ldots, v_M)$=0, but $\lambda =$
(A2). Rather, $\lambda = 0$ is always a *hole*, from the p.b.c. point of

: the case when N is even, $M = N/2$ (*i.e.* $J = 0$) and all roots he groud state for even N) is unambiguously identified by the

$$I_j = j \qquad j = 1, 2, \ldots, N/2 \tag{A6}$$

; counting function is indeed monotonically increasing on the s name. To our knowledge, the existence itself of this BAE proven in a rigorous analytical way. But it is very easy to for values of N in the thousands and precisions of order 10^{-15} station.

ioving J roots from the ground state. J is the quantum spin ıew BA state. For γ sufficently small, one now finds that

$$+ J < Z_N(+\infty; v_1, \ldots, v_M) < N/2 + J + 1 \tag{A7}$$

the actual roots satisfying eq.(A5), there must exist positive ,, with $\nu \geq 2J$, satisfying

$$_h; v_1, \ldots, v_M) = 2\pi N^{-1} \bar{I}_h\,, \quad h = 1, 2, \ldots, \nu \tag{A8}$$

tive integers. If $Z_N(x)$ is monotonically increasing, then nec- he integers $\{I_1, \ldots, I_{N/2-J}, \bar{I}_1, \ldots, \bar{I}_{2J}\}$ are all distinct. The , are naturally called *holes*. For J held fixed as N becomes natural to expect that the counting function is indeed mono- nd numerical calculations confirm this expectation. For larger on becomes more involved. Numerical studies show that, first a local maximum beyond the largest root, while still satisfying exceeds a certain (J−dependent) value. For even larger val- ic value of $Z_N(x)$ becomes smaller than $Z^* \equiv 2\pi(1/2 + J/N)$, s larger than Z^*, provided there is indeed a root correspond- $I_M = N/2 + J$). Up to now, v_M is obviously located where J from below, and one could say that there exists an extra yond, where $Z_N(x)$ reaches Z^* from above. When γ reaches c, the local maximum lowers till Z^*. At this point the root change their places, and for sligtly larger values of γ the hole x) reaches Z^* from below, while the root lays further away, Z^* from above. As the numerical calculations show, however, he same quantum integer $I_M = N/2 + J$ of the largest root no energy increase is really associated to its presence. When ıas $N/2 + J$ as quantum integer, then $Z_N(x)$ does never reach e γ, and, strictly speaking there are only $2J - 1$ holes. This l that the energy increases with respect to the ground state in ere was "a hole at infinity", that is a hole beyond the largest ıys be regarded to be even, when N is even.

As an important example consider the definite choice $J = 1$. Then for $\gamma < \pi/6$ we find $Z_N(+\infty) > Z^*$, and there are two holes, with $1 \leq \bar{I}_1 < \bar{I}_2 \leq N/2 + 1$. Assume $\bar{I}_2 \leq N/2 + 1$ and consider the interval $\pi/6 < \gamma < \pi/4$. The counting function has a maximum Z_{max} (larger than Z^*) situated to the right of the largest root $v_{N/1-1}$, as long as γ is smaller than the critical value γ^* at which the maximum lowers to Z^*. For $\gamma > \gamma^*$, the maximum is still larger than Z^* but is located to the left of $v_{N/1-1}$. In any case, for $\pi/6 < \gamma < \pi/4$ the asymptotic value $Z_N(+\infty)$ is smaller than Z^*. When $\gamma \to (\pi/4)^-$, then the largest root as well as the maximum Z_{max} are pushed to infinity, and $Z_N(x)$ is once again monotonic with $Z_N(+\infty) = Z^*$. As γ exceeds $\pi/4$, the last root $v_{N/1-1}$ passes, through the point at infinity, from the real line to the line with $\mathrm{Im}\, v = \pi/2$. This pictures generalizes to arbitrary J with the two special values $\gamma = \pi/6$ and $\gamma = \pi/4$ replaced, respectively, by $\gamma = \pi/(4J + 2)$ and $\gamma = \pi/(2J + 2)$. In fig. 7 the salient portion of the numerically calculated counting function is plotted for $J = 1$, $N = 64$ and a specific choice of $\bar{I}_1$, $\bar{I}_2$. In this case we approximatively find $\gamma^* \simeq 0.21\pi$.

Finally, consider a BAE solution containing, in addition to a number of order N of real roots, also a certain configuration of complex roots. In the v−space, these complex roots appear either in complex conjugate pairs or with fixed imaginary part equal to $i\pi/2$, so that the counting function is real analytic: $\overline{Z_N(x)} = Z_N(x)$. Moreover, it is fairly easy to show, by looking at the value of the counting function a real infinity, that the presence of complex roots implies the existence of holes in the sea of real roots. For our next porposes, we shall now consider γ/π irrational, so that all v−roots are finite. Denoting with u_q, $q = 1, 2, \ldots, M_c$ the values of the complex roots and with $M_r = M - M_c$ the number of real roots, we now write the derivative of the counting function as

$$2\pi\rho_N(x) \equiv Z'_N(x) = F'_\Theta(x) + N^{-1}\left[F'_0(x) + F'_c(x) - F'_h(x)\right] + (K * \rho_\delta)(x) \tag{A9}$$

where

$$\begin{aligned}
F_\Theta(x) &= \phi_{\gamma/2}(x + \Theta) + \phi_{\gamma/2}(x - \Theta) \\
F_0(x) &= \phi_\gamma(2x) \\
F_c(x) &= \sum_{q=1}^{M_r} \left[\phi_\gamma(x - u_q) + \phi_\gamma(x + u_q)\right] \\
F_h(x) &= \phi_\gamma(x) + \sum_{n=1}^{\nu} \left[\phi_\gamma(x - x_n) + \phi_\gamma(x + x_n)\right] \\
N\rho_\delta(x) &= \sum_{j=1}^{M_r} \delta(x - \lambda_j) + \phi_\gamma(x) + \sum_{n=1}^{\nu} \left[\delta(x - x_h) + \delta(x + x_h)\right]
\end{aligned} \tag{A10}$$

and $K*$ is the convolution defined by

$$(K * f)(x) = \int_{-\infty}^{+\infty} dy\, \phi'_\gamma(x-y) f(y) \tag{A11}$$

The so–called "density approach" consists in replacing, as $N \to \infty$, both ρ_N and ρ_δ with the same smooth function ρ, representing the density of roots and holes on the real line. This function is therefore the unique solution of the *linear* integral equation (cft. eq.(A9))

$$2\pi\rho = F_\Theta + N^{-1}(F_0 + F_c - F_h) + K * \rho \tag{A12}$$

which can be easily solved by Fourier transformation. Combining this equation with eq.(A9), we now obtain, after some simple manipulations

$$\rho_N = \rho + G * (\rho_N - \rho_\delta) \tag{A13}$$

where $G* = (2\pi + K)^{-1} * K*$ stands for the convolution with kernel

$$G(x) = \int \frac{dk}{2\pi} e^{ikx} \frac{\sinh(\pi/2 - \gamma)k}{\sinh(\pi - \gamma)k/2 \cosh \gamma k/2} \tag{A14}$$

Finally, a simple application of the residue theorem to the analytic function $\rho_N(1 - e^{-iNZ_N})^{-1}$ plus an integration by parts lead to the following formal *nonlinear* integral equation for the counting function

$$Z_N = Z + G * L_N \tag{A15}$$

where

$$L_N(x) = -iN^{-1} \log \frac{1 - e^{iNZ_N(x+i0)}}{1 - e^{-iNZ_N(x-i0)}} \tag{A16}$$

and Z is the odd primitive of $2\pi\rho$, namely

$$Z = Z_\infty + N^{-1}(2\pi + K) * (F_0 + F_c - F_h) \tag{A17}$$

with

$$Z_\infty(x) = ((2\pi + K) * F_\Theta)(x) = 2 \arctan\left(\frac{\sinh \pi x/\gamma}{\cosh \pi\Theta/\gamma}\right) \tag{A18}$$

Eq.(A15) is a formal integral equation since the knowledge of the exact position of holes and complex roots is required in Z. It becomes a true integral equation for the

ground state counting function. In any case it is an exact exp
where the number of site N enters only in an explicit, paramet
positions of the holes and of the complex roots. After having fi
quantum integers, these parameters retain an implicit, mild
large N.

We shall now show that eq.(A15) is very effective for estab
which is of crucial importance for the calculation of the $S-$
to check that the second nonlinear term in the r.h.s. of eq.(A
order in N^{-1} relative to the first. To this purpose observ
contour of the convolution in eq.(A15)can be deformed awa
lower edges of the real axis, since for sufficently large N no co
in the whole strip $|\mathrm{Im}\, x| < \gamma/2$ and $G(x)$ is analytic there. I
as $N \to \infty$, and one can eplicitly check that the imaginary
definite for $0 < \mathrm{Im}\, x < \gamma/2$ and negative definite for $0 > \mathrm{Im}$
implies that the contribution to the convolution integral is
N for all values of the integration variable (let's call it y) w
For $|\mathrm{Re}\, y|$ of order $\log N$, we find $\mathrm{Im}\, Z_\infty(y)$ of order N^{-1}, s
L_N, rather than exponentially small, is also of order N^{-1}. B
damping in y of the kernel $G(x-y)$ guarantees that the
globally of order N^{-2} or smaller, provided $|x|$ is kept sma
Hence we can write

$$Z_N = Z + O(N^{-2})$$

Finally, the coefficent of the N^{-1} term of Z, in eq.(A17),
the $N \to \infty$ limit, with the techniques described at length
straightforward albeit cumbersome algebra, this yields

$$\lim_{N\to\infty} ((2\pi + K) * (F_0 + F_c - F_h))(x) = F(x\,; x_1, \ldots, x_M;$$

where the higher–level quantity F is defined in eq.(6.8). To
and (A17), this proves eq.(6.6) of section 3, as claimed.

Let us now consider the problem of calculating the ener
through eq.(4.15). We rewrite first the "bare" energy functi

$$e_0(x) = -2\pi + \phi_{\gamma/2}(x + \Theta) - \phi_{\gamma/2}(x -$$

$$\rho_\delta(x)\phi_{\gamma/2}(x) + \sum_{q=1}^{M_c} \phi_{\gamma/2}(u_q) - \sum_{h=1}^{\nu} \phi_{\gamma/2}(x_h) - \tfrac{1}{2}\phi_{\gamma/2}(0)$$

$$\rho(x)\phi_{\gamma/2}(x+\Theta) + \sum_{q=1}^{M_c} \phi_{\gamma/2}(u_q) - \sum_{h=1}^{\nu} \phi_{\gamma/2}(x_h) - \tfrac{1}{2}\phi_{\gamma/2}(0)$$

$$[\rho_\delta(x) - \rho_N(x)]\left((1-G) * \phi_{\gamma/2}\right)(x) \tag{A22}$$

heorem and an integration by parts, the last term can be efore for the counting function, into an integral of the non- y the integral

$$\tfrac{1}{2}N \int_{-\infty}^{+\infty} \frac{L_N(x)}{\gamma \cosh \pi x/\gamma}$$

used above, this last expression is globally of order N^{-1}. xplicit form of the continuum density $\rho(x)$ into eq.(A22) and d (A21), for the energy we obtain eq.(6.1) of the main text

$$E = E_0 + a^{-1} \sum_{h=1}^{\nu} e(x_h) + O(a^{-1}N^{-1}) \tag{A23}$$

$$\frac{-\gamma)k/2 \sin k\Theta}{k/2 \cosh \gamma k/2} \left[2N \cos k\Theta + 1 + \frac{\sinh(\pi - 3\gamma)k/4}{\sinh(\pi-\gamma)k/4}\right] \tag{A24}$$

und state.

ose this appendix with a comment on the limitations of the e one deals only with the solution ρ of the linear equation as the actual density of real roots and holes in the $N \to \infty$ e it to replace summations with integrals. What one learns

from the exact treatment presented above as well as from computer calculations, is that the error made in such a replacement depends crucially on the large x behaviour of the quantity which is to be summed. This error is down by N^{-1} only when there is exponential damping in x. This means, for instance, that the integral of $\rho(x)$ does not reproduce in general the exact number of real roots and holes, but rather some γ–dependent quantity close to it. Misundertanding this for the actual number of real roots and holes would lead to the absurd result that the holes have a γ–dependent value of the $SU(2)_q$ spin, which is instead necessarily integer or half-integer and, in the particular case of the holes, just 1/2 for any value of γ.

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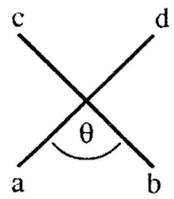
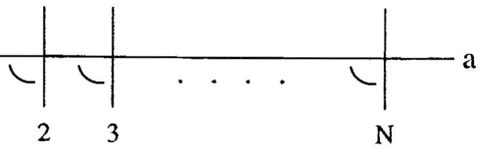
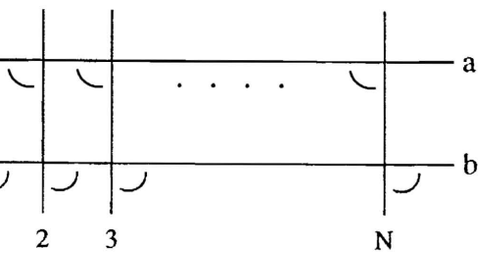


. Graphical representation of the R-matrix.



l row-to-row monodromy matrix. The numbers from 1 to N label
paces and each vertical line represents a couple of free indices in
nding vertical space. Indices on the internal horizontal lines are
.



monodromy matrix used to describe systems with fixed boundary
The correct contraction over the internal indices is dictated by the
he arcs.

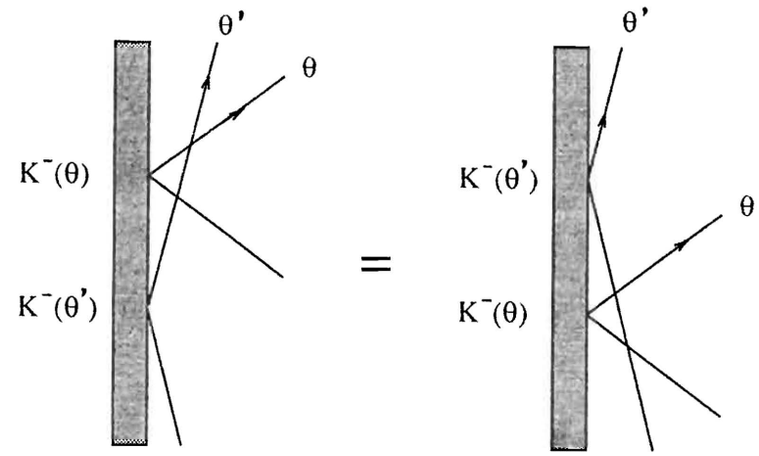


Fig.4. Graphical representation of the compatibility relations (2.12) between the two-body scattering and reflections on the left wall.

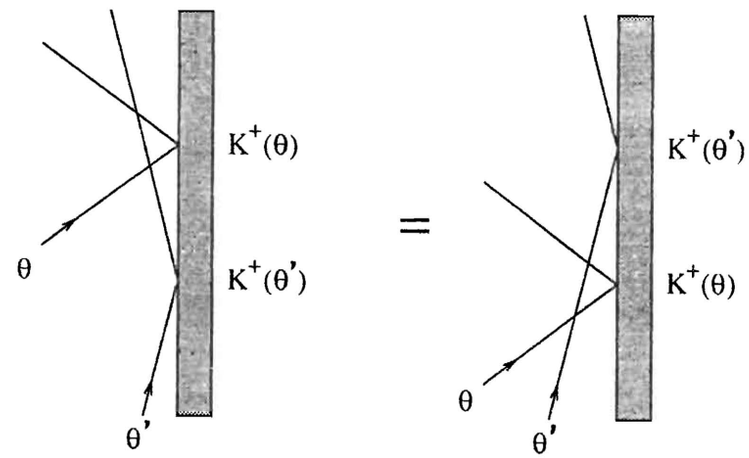


Fig.5. Graphical representation of the compatibility relations (2.15) between the two-body scattering and reflections on the right wall.

*Fig.*13. One of the local height configurations that dominate the one–hole BA state. of spin $J = 1$.

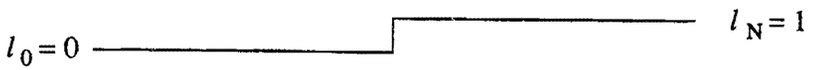


*Fig.*14. The "renormalized" version of Fig.13.

*Fig.*15. Flipping the last portion of the configuration from the solid to the dotted line does not change the quantum spin $J = p/2 - 1$, but tranform the type II state into type I.

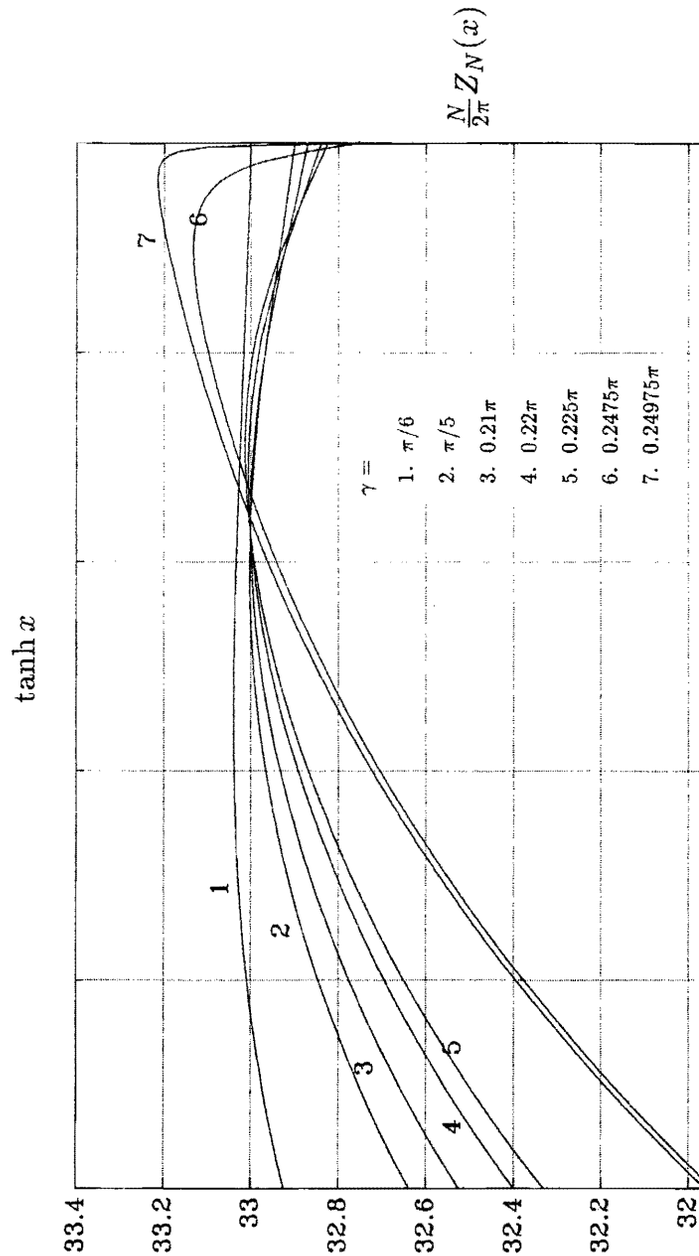


*Fig.*16. Plot of the $N/2\pi$ times the counting function $Z_N(x)$ versus tanh
$J = 1$, $\Theta = .15$ and various values of γ from $\pi/6$ to $0.999\pi/4$.
integers associates to the two holes are $\bar{I}_1 = 8$ and $\bar{I}_1 = 21$. The
of $(N/2\pi)Z_N$ is $N/2 + J = 33$, while that of γ is, roughly, $\gamma^* =$