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Reconstruction of Initial Cosmological Density Fluctuations by the Pseudo-Spectral Method

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ABSTRACT

The pseudo-spectral method is proposed for following the evolution of density and velocity fluctuations at the weakly non-linear stage in the expanding universe with a good accuracy. In this method, the evolution of density and velocity fluctuations is integrated in the Fourier space with using FFT and this method is very useful to investigate accurately the dynamics in the weakly non-linear evolution. Furthermore, when we integrate the evolution of density and velocity fluctuations back in time by the pseudo-spectral method with using the fitting formula for the relation between density and velocity fluctuations at the weakly non-linear stage, the density and velocity fluctuations at the early time could be recovered.

Subject headings: cosmology:theory - dark matter-gravitation-large-scale structure of universe

1. INTRODUCTION

The problem on the formation of the large-scale structures is now one of the most interesting and mysterious problems in the cosmology that has not yet completely understood. In the standard scenario of the large-scale structures formation, it is considered that very small density fluctuations at the very early time in the universe grow to the structure observed today by the gravitational instability(Peebles 1993). Hence it is very important task to analyze the growth of the density fluctuations

As is well known, the evolution of density fluctuations have some stages at which the dynamics in the growth of density fluctuations have different characters. The first stage is the linear growth of density fluctuations when the density fluctuation $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$ is much less than unity, here ρ is the density at some place $\mathbf{x}$ and time t and $\bar{\rho}$ is the mean density at the time t in the universe. In this stage, the dynamics is well understood because we can calculate analytically the dynamics(Peebles 1993).

The second stage is "weakly" non-linear stage. In this stage, the density fluctuation δ becomes around unity, but the caustics(divergence) of density do not appear. That is, orbits mixing does not occur above this scale and then the density fluctuations have memories of their initial states at the early universe. The final stage is "strongly" non-linear stage. In this stage after when the first caustics appear, the orbits mixing occur and then the initial memories have lost completely. Today some observations suggest that the density fluctuations on the scales above about $10h^{-1}$Mpc may stay today at the "weakly" non-linear regime. As we mentioned before, the initial memories might have been imprinted in the density fluctuations on these scales even now. So it might be possible to reconstruct the initial density fluctuations from the observed density fluctuations. Initial fluctuations contain many important theoretical informations on dark matters, inflation scenarios and so on. Then it is very useful to reconstruct the initial density fluctuations by using the present density fluctua-

tions on the weakly non-linear scales. Moreover the relation between the density fields and the peculiar velocity fields could be analyzed simply on these scales. Recently the velocity fields are obtained by the POTENT method(Bertchinger & Dekel 1989; Dekel, Bertchinger, & Faber 1990; Bertchinger et al 1990) and then using this relation the density parameter, Ω_0, is often investigated(Nusser & Dekel 1993). Furthermore we can analyze more accurately the dynamics of the "weakly" non-linear growth of density fluctuations than those at the "strongly" non-linear stage. As for an example, we might understand well the dependence of the growth rate of density fluctuations on the initial density spectrum and the scale of the fluctuations. From the above facts, it is concluded that it is very interesting and important task to analyze the "weakly" non-linear evolution of the density fluctuations. In analyzing the "weakly" non-linear evolution, the second order perturbations(Suto & Sasaki 1991) and Zel'dovich approximation (Zel'dovich 1970) are often used until now. And in recovering the initial density fluctuations by tracing the large-scale fluctuations back in time, the Zel'dovich-Bernoulli equation(Nusser & Dekel 1993) and its improved equations(Gramann 1993) are used. Zel'dovich approximation is certainly very useful. However it is the *approximate* formula and in some cases it does not trace necessary the evolution of the density field correctly. On the other hand, N-body simulations are very strong tool to analyze the non-linear evolution, but it is very difficult to trace the large-scale fluctuations back in time by performing the numerical simulations because the observational noise or numerical errors in the today's density fluctuations would act as the artificial decaying mode(Nusser & Dekel 1992). In this paper, we propose the new method for analyzing correctly the weakly non-linear evolution. That is the pseudo-spectral method by which the density and velocity fields are calculated by solving the continuous equation of density, Euler equation and the Poisson field equation in the Fourier space(This method is often used also in the fluid dynamics(Orszag 1971)). These equations are the partial differential equations in the real space, but in the Fourier space they are the ordinal differential equations only with respect to time t. Then

we can integrate these equations in time with more accuracy. Hence it is very useful to analyze the dynamics of the weakly non-linear evolution such as the growth rate of the density fluctuations and the validity of Zel'dovich approximation. Moreover, in the pseudo-spectral method, even if the observed present density fields would have large noise, we can integrate the equations back in time by picking up only the growing mode using the fitting formula for the relation of density and velocity fields at the weakly non-linear stage(Nusser et al 1991) as shown in §4. Furthermore, in this method, the numerical computation time is proportional to $\sim N\ln N$ by using the Fast Fourier Transformation(FFT) for calculations of the non-linear coupling terms in the equations for the evolution of the density and velocity fields(see §2), where N is $(2k_{max})^3$ and k_{max} is the maximum wavenumber which we consider in the numerical calculation. The purpose of this paper is that we explain the pseudo-spectral method and investigate how the initial density fluctuations are reconstructed by this method.

In §2, the pseudo-spectral method is explained. In §3, the numerical results are shown and also the numerical error of this method is tested. In §4, this model is applied to the reconstruction of the initial density fields and finally §5 is devoted to the discussions.

2. Pseudo-Spectral Method

The equations governing the evolution of a pressureless gravitating fluid in a standard cosmological background until the first appearance of divergence of density (or the orbits mixing) are the following three equations, those are, the continuity equation, the Euler equation of motion and the Poisson field equation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla(\frac{1}{2}\mathbf{v}^2) + \mathrm{rot}\mathbf{v} \times \mathbf{v} = -\nabla \Psi, \tag{2}$$

$$\Delta \Psi = \frac{4\pi G}{(1+z)}(\rho - \bar{\rho}), \tag{3}$$

where the spatial derivation is respect to the comoving coordinate $\mathbf{x}$. And ρ is the comoving density, $\bar{\rho}$ is the mean comoving density in the universe and z is

the redshift. And $\mathbf{x}, \mathbf{v}$ and t are defined as follows:

$$\mathbf{x} = (1+z)\mathbf{r}, \qquad \mathbf{v} = (1+z)^{-1}(\dot{\mathbf{r}} - H\mathbf{r}) \quad (H \equiv \frac{da}{dt'}/a)$$
$$t = -\int_t^\infty (1+z(t'))^2 dt'$$

where $\mathbf{r}$ is the proper distance from some chosen origin and a is the scale factor. In this paper, we consider only the case that the density parameter at the present Ω_0 is 1.0 and the cosmological constant term Λ equals 0. Then $t = -\frac{2}{H_0}\sqrt{1+z}$, where H_0 is the Hubble constant at the present. We assume that the velocity filed has no vorticity because the rotational mode of the velocity is the decaying mode on the weakly non-linear scales and then we can write $\mathbf{v} = \nabla\Phi$. Here Φ is called the velocity potential. So the Euler equation (2) reduces to the following equation:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 = -\Psi \qquad (4).$$

Equations (1),(3) and (4) are the ones we must solve for following evolution of density fluctuations. However they are the partial differential equations and so there exist some technical difficulties in the numerical integrations of these equations depending on the numerical scheme. Instead we calculate these equations in the Fourier space imposing the periodic boundary condition. The periodic boundary condition is normally used in N-body simulations in the expanding universe. Then $\rho(\mathbf{x},t), \mathbf{v}(\mathbf{x},t), \Phi(\mathbf{x},t)$ and Ψ have the discrete Fourier representations as follows:

$$\rho(\mathbf{x},t) = \sum_{\mathbf{k}} \rho_{\mathbf{k}}(t)e^{i\mathbf{kx}}, \qquad \mathbf{v}(\mathbf{x},t) = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}(t)e^{i\mathbf{kx}},$$
$$\Phi(\mathbf{x},t) = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}(t)e^{i\mathbf{kx}}, \qquad \Psi(\mathbf{x},t) = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}(t)e^{i\mathbf{kx}}. \qquad (5)$$

The wavenumber $\mathbf{k} = (k_x, k_y, k_z)$ is truncated at $|k_x|, |k_y|, |k_z| < k_{max}$. It is found that the Fourier component with $-\mathbf{k}$ is the same as the complex conjugate one

with $\mathbf{k}$, as for an example, $\rho_{-\mathbf{k}} = \rho_{\mathbf{k}}^*$ because these variables in real space are real. Then the basic equations for the evolution of the density and velocity fields in the Fourier space reduce to the following ones:

$$
\begin{aligned}
&\frac{d\rho_{\mathbf{k}}(t)}{dt} + i\mathbf{k} \cdot \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \rho_{\mathbf{k}'}(t)\mathbf{v}_{\mathbf{k}''}(t) = 0 \\
&\frac{d\Phi_{\mathbf{k}}(t)}{dt} + \frac{1}{2} \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \mathbf{v}_{\mathbf{k}'}(t) \cdot \mathbf{v}_{\mathbf{k}''}(t) = -\Psi_{\mathbf{k}} \\
&\Psi_{\mathbf{k}} = \begin{cases} -G\rho_{\mathbf{k}}/|\mathbf{k}|^2 & (\mathrm{k}=|\mathbf{k}| \neq 0) \\ 0 & (\mathrm{k}=0) \end{cases} \\
&\mathbf{v}_{\mathbf{k}}(t) = i\mathbf{k}\Phi_{\mathbf{k}}(t) \qquad (7)
\end{aligned}
$$

These equations are the ordinal differential equations. So we can integrate the evolution of the density and velocity fields with a good accuracy via, for an example, the 4-th order Runge-Kutta scheme. In calculating the non-linear coupling terms in eq.(7), $\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \rho_{\mathbf{k}'}(t)\mathbf{v}(\mathbf{k}'', t)$ and $\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \mathbf{v}_{\mathbf{k}'}(t) \cdot \mathbf{v}_{\mathbf{k}''}(t)$, we use the Fast Fourier Transformation(FFT). At first, we transform $\rho_{\mathbf{k}}(t)$ and $\mathbf{v}_{\mathbf{k}}(t)$ to the real ones, $\rho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$, via FFT. And then we get the production of these terms, those are, $\rho(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t)$. And finally we transform again these products into the ones in the Fourier space and then we can get the coupling terms in eq.(7). This method is called "pseudo-spectral method". In adopting this method, the computation time for numerical calculations is proportional to $\sim N\mathrm{lnN}$ owing to FFT. However if we calculate directly the non-linear coupling terms, the computation time is proportional to $\sim N^2$. Here, however, it must be noticed that there exists a serious problem in calculating numerically the non-linear coupling terms by using FFT according to the procedure mentioned above. This problem is called "alias" problem. There appear unphysical Fourier spectra in this procedure because we must cut the Fourier spectrum at the finite wavenumber, k_{max}. In explaining the alias problem, we consider the one-dimensional system for simplicity. As mentioned before, when we calculate the coupling term, $F(k) \equiv \sum_{k=k'+k''} \rho_{k'} v_{k''}$, at first we calculate

$\rho(x) = \sum_{k'=-k_{max}}^{k_{max}} \rho_{k'} e^{ik'x}$ and $v(x) = \sum_{k"=-k_{max}}^{k_{max}} v_{k"} e^{ik"x}$ and product these two terms and then get $F(x) = \rho(x) \cdot v(x) = \sum_{k',k"} \rho_{k'} v_{k"} e^{i(k'+k")x}$. Finally we get the Fourier spectrum of $F(x)$ at k via FFT. At this time, the product of $\rho_{k'}$ and $v_{k"}$ with $k' + k" < -k_{max}$ (or $k' + k" > k_{max}$)contributes unphysically to $F(k)$ at $k = 2k_{kmax} - k' - k" > 0$ (or $k = k' + k" - 2k_{max}" < 0$) because of the periodicity of the Fourier transformation with the finite resolution for wavenumber($k \leq k_{max}$). So we must remove this spurious contributions, "alias", to $F(k)$. There is a good procedure for removing the alias as follows: when we need to get $F(k)$ with $k \geq 0$, we consider only the following variables, $\tilde{\rho}(x) \equiv \sum_{k'=0}^{k_{max}} \rho_{k'} e^{ik'x}, \tilde{v}(x) \equiv \sum_{k"=0}^{k_{max}} v_{k"} e^{ik"x}$ and $G(x) = \tilde{\rho}(x)\tilde{v}(x) + \tilde{\rho}^*(x)\tilde{v}(x) + \tilde{\rho}(x)\tilde{v}^*(x)$ instead of the product, $\rho(x)v(x)$. Here * represents the complex conjugate. In transforming $G(x)$ to the Fourier space, we can get $G(k)$ and it is easily found that $G(k) = F(k)$ with $k \geq 0$. And in this procedure, we remove the contributions from $\rho_{k'} v_{k"}$ with $k' < 0$ and $k" < 0$, and so the spurious Fourier components do not appear for $F(k)$ with $k \geq 0$. When we need to get $F(k) = G(k)$ with $k < 0$, we consider only the $\bar{\rho}(x) \equiv \sum_{k<0} \rho_{k'} e^{ik'x}$, $\bar{v}(x) \equiv \sum_{k"<0} v_{k"} e^{ik"x}$ and transform th product, $\bar{\rho}(x)\bar{v}(x) + \bar{\rho}^*(x)\bar{v}(x) + \bar{\rho}(x)\bar{v}^*(x)$, into the Fourier space. In the same procedure, we can remove the alias also from the coupling term, $\sum_{k=k'+k"} v_{k'} v_{k"}$. Also in the real 3-dimensional systems, we can remove the alias by following the same procedure which are extended to the 3-dimensional case.

3. Numerical Results

In this section, we show the numerical results for following the evolution of density and velocity fields by the pseudo-spectral method. As for initial conditions as the redshift z is larger, the power spectrum of density fields($P(k) = \langle |\rho_{\mathbf{k}}|^2 \rangle$) are given by any shape and the phases of the Fourier component are randomly distributed. The initial Fourier spectrum of the velocity potential (or velocities) are given by using the linear approximation for the relation between the growing mode of the velocity fields and the density fields. As for examples, we consider the following initial power spectrum of density fields at $z = 99$.

(1)

$$P(k) \propto k \qquad (k \leq 8)$$
$$P(k) = 0 \qquad (8 < k \leq k_{kmax} = 15)$$

(2)

$$P(k) \propto k \qquad (k \leq 8)$$
$$P(k) \propto k^{-3} \qquad (8 < k \leq k_{max} = 15)$$

We follow the evolution of the density and velocity fields until $z = 0$ by the pseudo-spectral method. The normalization of the initial density is determined by the criterion that the first caustics of density would appear around a little after $z = 0$. The time evolution of the power spectrum for case (1) and (2) are shown in Figures 1 and 2, respectively. The rms of the density fluctuations at $z = 99$ for case (1) and (2) are 0.58 and 0.48, respectively. As a numerical test, we confirmed that the power spectrum at smaller wavenumbers follow the linear growth, that is, $P(k) \propto a^2$, where a is the scale factor. Furthermore we tested the time reversal. We followed the evolution backward in time from $z = 0$ until $z = 99$. To compare the statistical properties of the recovered density fields at $z = 99$ with the true initial density fields, we use the following static(Gramann 1993):

$$\epsilon(k) = \sqrt{\frac{\sum |\rho^r_{\mathbf{k}} - \rho^t_{\mathbf{k}}|^2}{\sum |\rho^r_{\mathbf{k}}|^2 + \sum |\rho^t_{\mathbf{k}}|^2}}, \qquad (8)$$

where $\rho^r_{\mathbf{k}}$ and $\rho^t_{\mathbf{k}}$ are the values of the Fourier component at the wavenumber k in the reconstructed density fields and in the true initial density fields, respectively. The sum in eq.(8) is taken over the spherical layer with the radii $k - 0.5 < |\mathbf{k}| < k + 0.5$. When the value of the phases and amplitude of the Fourier components in the recovered fields exactly coincide with their true initial values, $\epsilon(k)$ vanishes. But when the recovered phases are totally scrambled with respect to their true initial ones, $\epsilon(k)$ is unity. We found that in both case (1) and (2), $\epsilon(k) \sim 10^{-3} - 10^{-2}$ even with the relatively large time step Δt. The reversal

evolutions are shown in figures 3 and 4 for case (1) and (2), respectively. Of course, when we take the smaller time step, $\epsilon(k)$ becomes more smaller (in this case the number of time step is 160 and then the time step $\Delta t = 3.9 \times 10^{-2}$).

We tried the numerical tests for the different shapes of the initial power spectrum of density and got almost the same results as for case (1) and (2) though we do not show the results in this paper. So we can conclude the pseudo-spectral method is a good numerical method for following the evolution of density and velocity fields in the weakly non-linear regime.

4. Reconstruction of Initial Density Fluctuations

As mentioned in §1, it is very useful to reconstruct the initial density fields because it would have many important informations about the dark matter, the inflation models and so on. In principle, as shown in the numerical test for the time reversal, we could get correctly the initial density fields on the weakly non-linear scales by following numerically the evolution back in time if we could get the exact values of the present density and velocity fields on the weakly non-linear scales which are given by smoothing fields on the strongly non-linear scales with using some smoothing windows(Melott 1993). In fact, however, there exist noise or the observational errors in the observed present density and velocity fields. The decaying mode picks up these noise and it amplifies them into artificial fluctuations of high amplitude which eventually dominate the density and velocity fluctuations at the early times when we attempt to integrate equations backward in time(Nusser & Dekel 1992). We tried to test this effect of noise in the reconstruction of the initial states as follows: as for an example, we make the artificial "noise" in the density field at $\mathbf{x}$, $\delta\rho(\mathbf{x})$, and one in velocity potential, $\delta\Phi(\mathbf{x})$, given by the Gaussian distribution with the dispersions $\sigma_\rho(x) = f \cdot |\rho(\mathbf{x})|$ and $\sigma_v(\mathbf{x}) = f \cdot |\Phi(\mathbf{x})|$, respectively. Here f is a constant parameter which represents the ratio of the noise to the true one. We found that f must be less than 10^{-4} to get the recovered initial density fields with $\epsilon(k) \leq 0.1$ even when we integrate the evolution back in time until the time at which the scale factor is 0.1 times than the present time. This result is also deduced with using the linear approxima-

tion. So it is practically very difficult to recover the initial density fields from the present density and velocity fields with the observational noise(in which maybe $f > 10^{-4}$). In order to recover the initial density fields, it is necessary to pick up only the growing mode. We tried to use the fitting formula for the relation between the density and velocity fields for their growing mode in the weakly non-linear regime:

$$\nabla \frac{\mathbf{v}}{a^2} = -\frac{\Gamma^{-1}\delta}{1+0.18\delta},$$
$$\Gamma = a^2 G(t) / \frac{dG(t)}{dt}. \qquad (9)$$

in our variables(refer to eq.(42) in Nusser et al (1991)). It is a good fitting formula checked by N-body simulations and/or perturbation method(Nusser et al 1991). We integrate numerically the evolution of only the density fields.(or velocity potential) back in time and get the velocity potential(or density) using the above formula at each time step. When we integrate from the present density fields without noise($f = 0$) using this formula, we get the recovered initial density fields which coincide the true ones within $\epsilon(k) \sim 10^{-2}$. This result is shown in figure 5 for case (1). This result also means that the fitting formula(eq.(9)) is correct within a few per cent error.

In the case that $f = 0.1$, the time reversal evolution of the power spectrum for case (1) is shown in figure 6. The recovered initial density fields at $z = 99$ agree with the true one within $\epsilon(k) \sim 10^{-1}$. This error might be practically enough to get some informations from the recovered initial states. So we conclude that even if the observed present density(or velocity) fields have large observational error, we can recover the initial density fields by the pseudo-spectral method with using the fitting formula for the relation between the density and velocity fields.

5. Discussions

We propose the pseudo-spectral method for following the evolution of density and velocity fields in the expanding universe until the first appearance of

divergence of the density(or orbits mixing). This method with the fitting formula(eq.(9)) is good one for recovering the initial density and velocity fields on the weakly non-linear scales which have very important informations. Nusser & Dekel (1993) and Gramann (1993) proposed the Zel'dovich-Bernoulli equation and its improved one , respectively, for reconstruction of initial density fields. They showed that the initial fluctuations can be recovered with good accuracy although Zel'dovich approximation is not so correct in some cases. Our method do not use the Zel'dovich approximation and this method might be complementary to others.

Furthermore, we found that by the pseudo-spectral method we can integrate the evolution forward in time with a good accuracy in the weakly non-linear regime and then this method is very useful to investigate the dynamics in this regime such as the growth rate of the power spectrum of the density. And by the pseudo-spectral method, we shall investigate the validity of Zel'dovich approximation systematically which is often used in many works.

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Figure Captions

FIG. 1.-Time evolution of the power spectrum for case (1). The initial power spectrum at $z = 99$ and final one at $z = 0$ are shown.

FIG. 2.-The same as FIG.1, but for case (2).

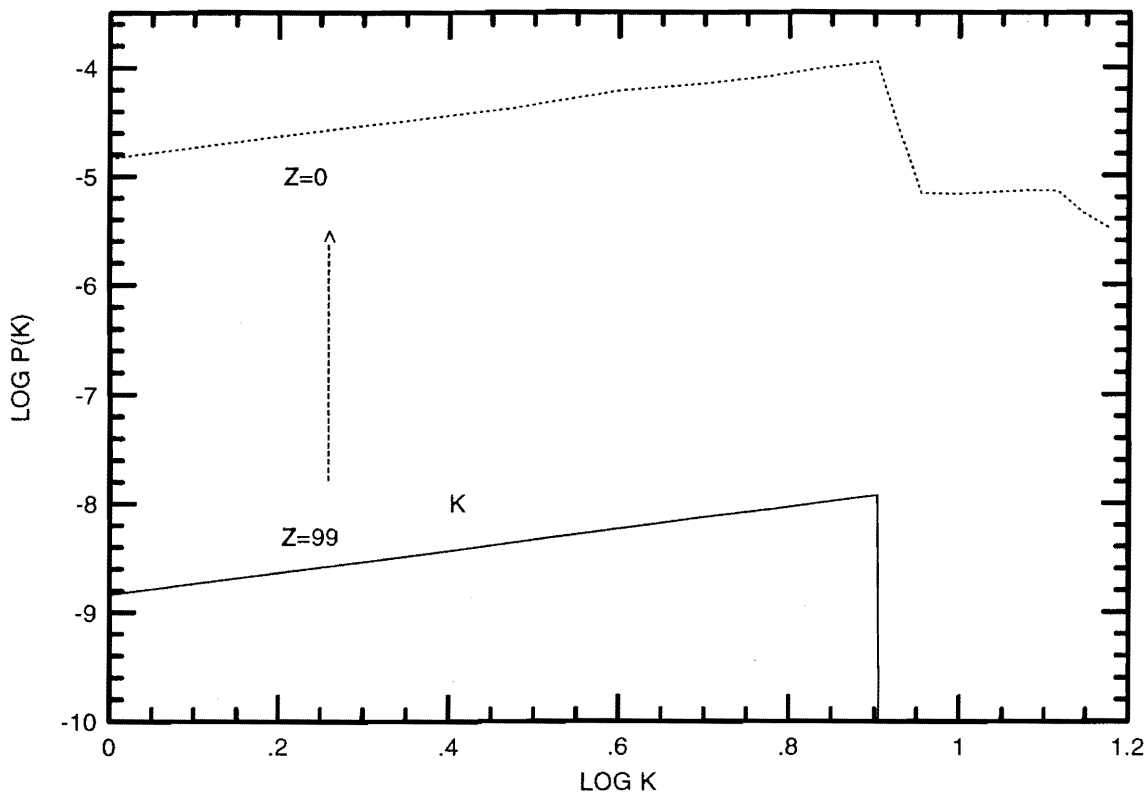
FIG.3-Time reversal evolution of the power spectrum for case (1). The recovered initial power spectrum at $z = 99$ integrated the power spectrum at $z = 0$ backward in time and one at $z = 0$ are shown.

FIG. 4-The same as the FIG.3, but for case (2).

FIG. 5.-The recovered power spectrum at $z = 99$ from the power spectrum with $f = 0$ at $z = 0$ by using the fitting formula (eq.(9)) and one at $z = 0$ are shown for case (1).

FIG. 6-The same as FIG.5, but the recovered power spectrum from the power spectrum with $f = 0.1$ for case (1) is shown.

FIG. 1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-8
-9
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K
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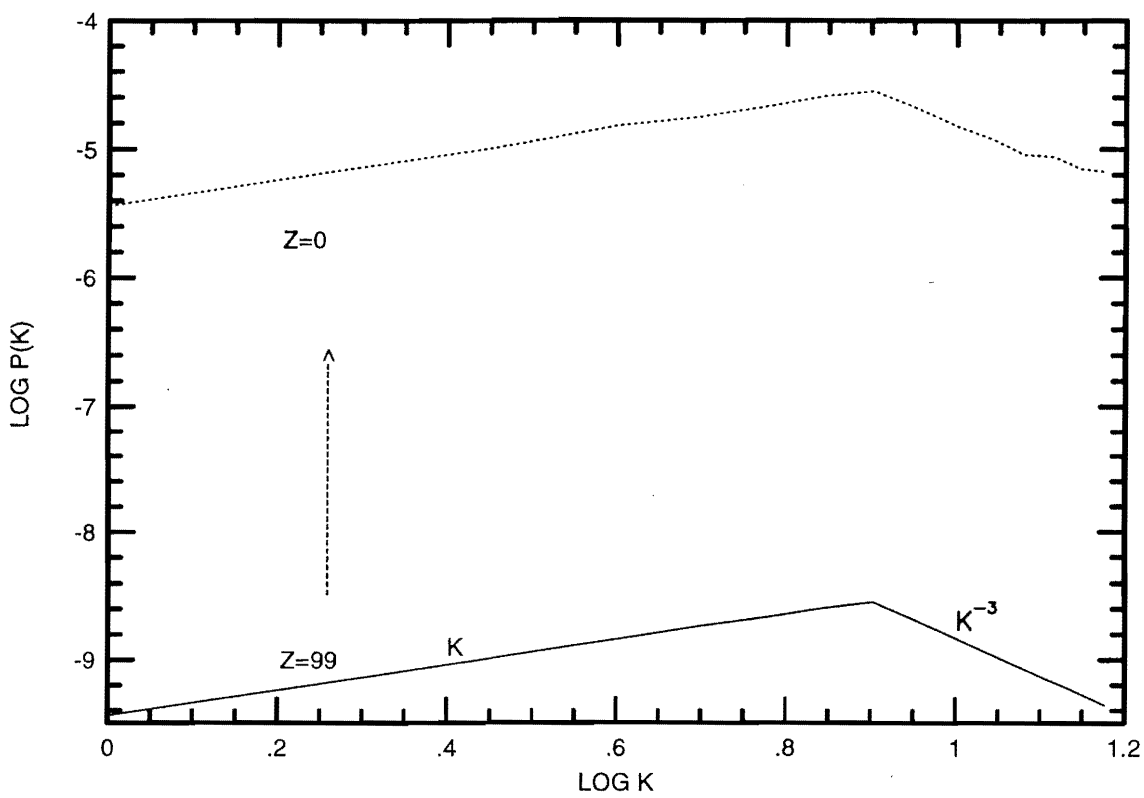
1
1.2
LOG K

FIG. 2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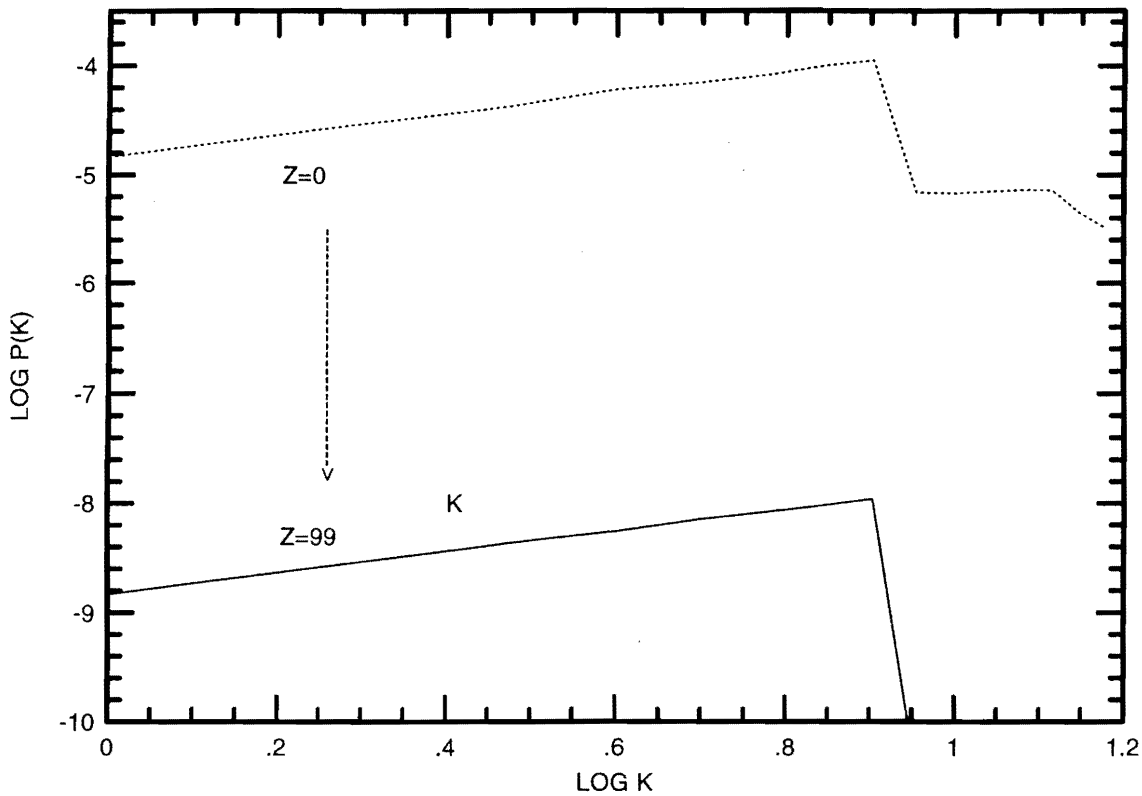
1.2
LOG K

FIG. 3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1
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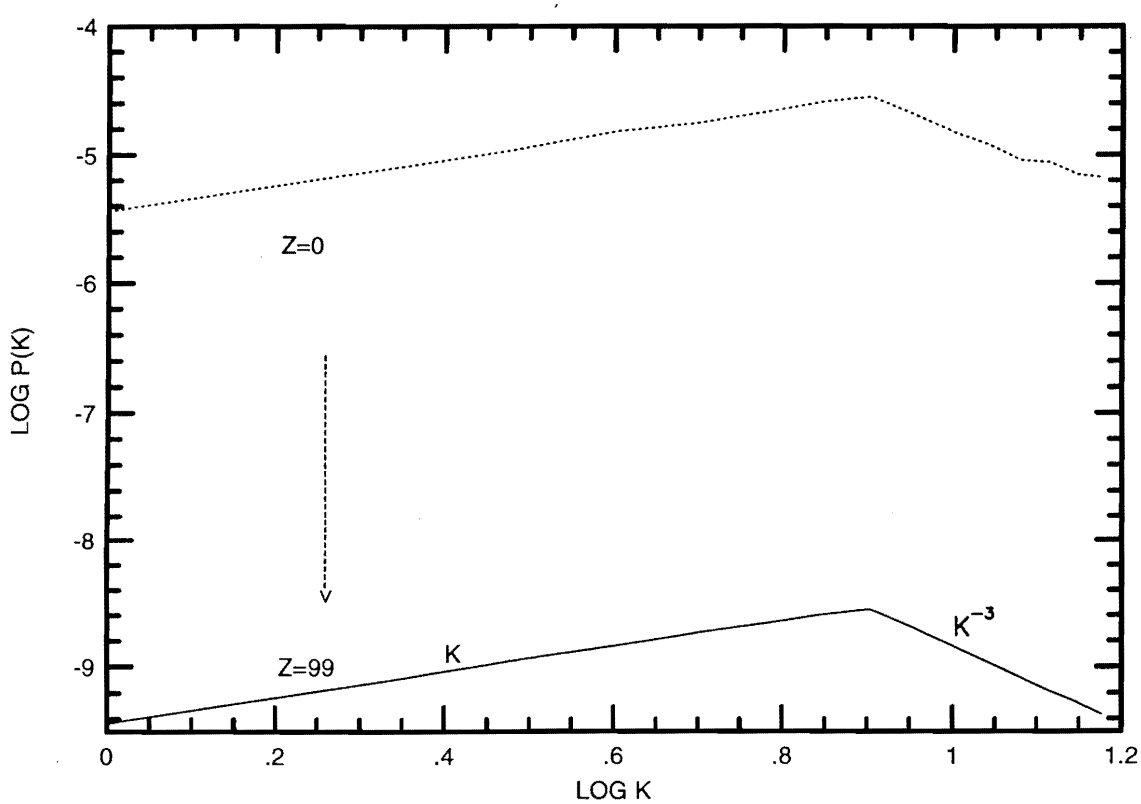
LOG K

FIG. 4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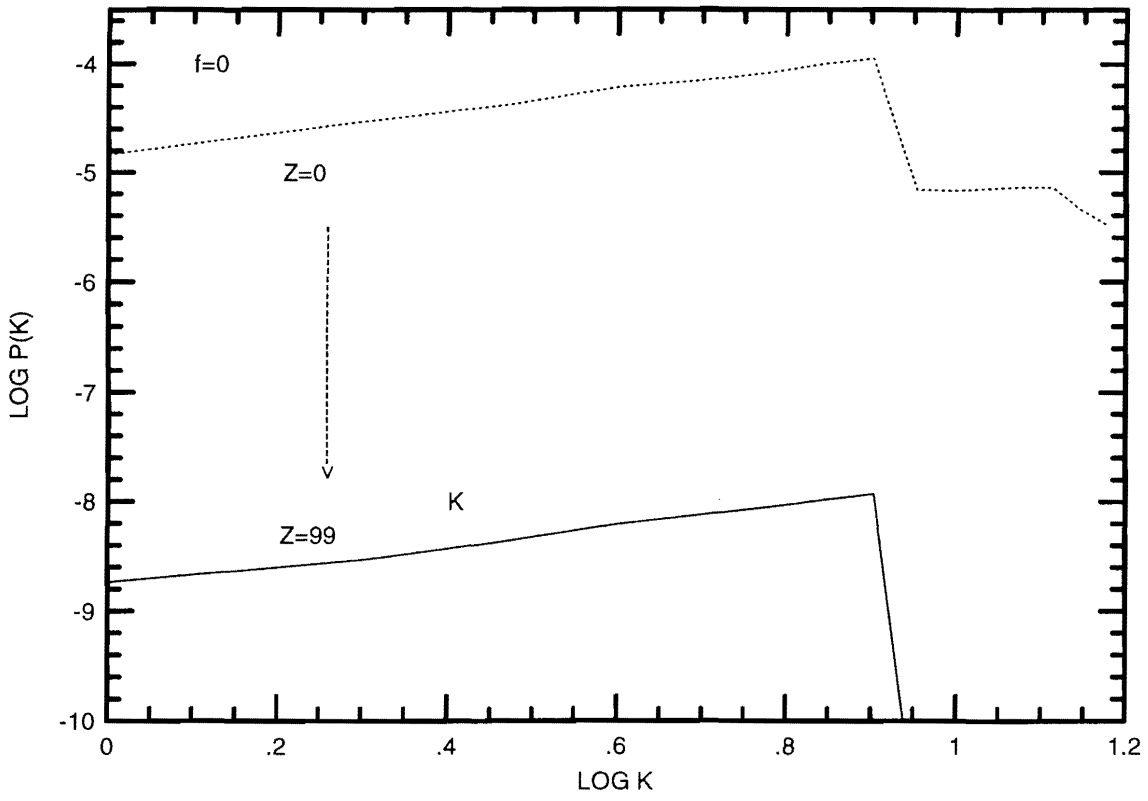
LOG K

FIG. 5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LOG K
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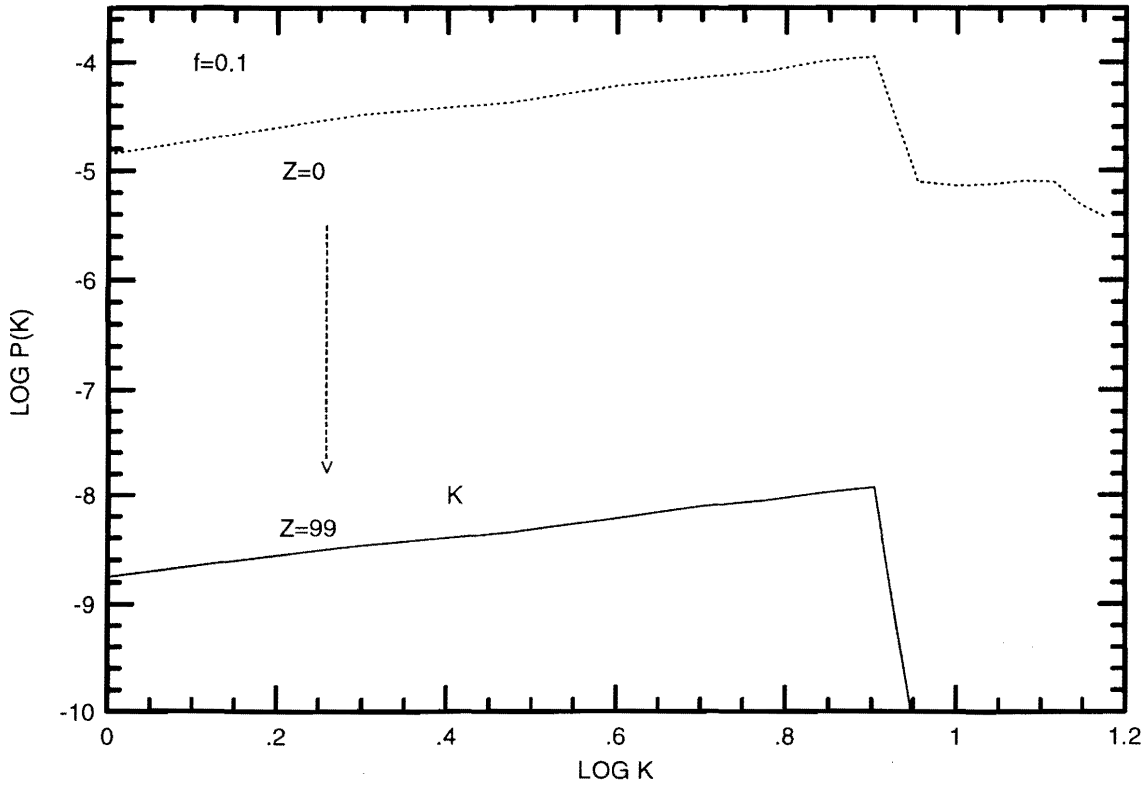
1.2

FIG. 6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1.2