Padé Approximants - Type I and Type II - and Their Application

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ABSTRACT

Padé approximants (P.A.) are considered as a method to accelerate convergence of an infinite sequence. First type I P.A. are introduced and studied and used in several examples. Then Type II P.A. are considered as a method of accelerating convergence. Later Type II P.A. are used to find the zeros of a function. Finally we discuss the question of how to choose the best coordinates for Padé II.

I. TYPE I PADÉ APPROXIMANTS

Definition: Let f(z) be an analytic function defined by its Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n Z^n$$

The Padé approximate to the function f(z) is defined as

$$f^{[n,m]}(z) = \frac{P_n(z)}{Q_m(z)} = f(z) + 0(z^{n+m+1}) \tag{1}$$

$P_n(z)$ and $Q_m(z)$ are polynomials of degree n and m. e.g. $P_n(z) = b_1 + b_{2z} + \cdots + b_N z^n$

The meaning of $\frac{P_n(z)}{Q_m(z)}$ is that we can write

$$f(z) = \frac{P_n(z)}{Q_m(z)} + \underbrace{d_{n+m+1}\ z^{n+m+1} + d_{n+m+z}\ z^{n+m+z} + \cdots}_{0(z^{n+m+1})} \tag{2}$$

where the d's and also the coefficients of z in the Padé approximate are functions of the coefficients in the Taylor series expansion.

For n = m, $f^{[n,n]}(z)$ is called a diagonal P.A.

Rationale for making this kind of approximation

1) A rational fraction can approximate a function near singularities where the approximation by polynomials breaks down.

2) Will find that we can often get good approximations to the function value from a Padé approximant constructed from the 1st several Taylor series coefficients.

3) Recursion formulas have been developed that make it relatively easy to construct the Padé approximant from the Taylor series coefficients.

We can organize the different Pade approximates into a table $F^{[n,m]}(x)$:

$$
\begin{array}{cccccc}
 & & M \rightarrow & & & \\
 & [0,0] & [0,1] & [0,2] & \cdots & \\
 & [1,0] & [1,1] & [1,2] & & \\
N & [2,0] & [2,1] & [2,2] & & \\
\downarrow & \vdots & & [3,2] & & \\
 & & & & \ddots &
\end{array}
$$

Example: Generate the Padé Table for

$$f(x) = \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad Q_0(x) \equiv 1 \tag{3}$$

$$f^{[0,0]}(x) = \frac{P_0(x)}{Q_0(x)} = a_0 = 0 \tag{4}$$

$$f^{[0,1]}(x) = \frac{P_0(x)}{Q_1(x)} = \frac{a_0}{1+b_1 x} = f(x) + 0(x^2) \tag{5}$$

$$a_0 = (1 + b_1 x)(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots) + 0(x^2)$$

$$= x + b_1 x^2 + 0(x^2)$$

$$= x + 0(x^2) \tag{6}$$

We can't find a value for a_0 that is independent of x so let's look at the lower half of the Padé table.

$$f^{[1,0]}(x) = \frac{P_1(x)}{Q_0(x)} = \frac{a_0 + a_1 x}{1} = (x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots) + 0(x^2) \tag{7}$$

$$\Rightarrow \quad a_0 = 0$$
$$a_1 = 1 \tag{8}$$

$$f^{[1,1]}(x) = \frac{P_1(x)}{Q_1(x)} = \frac{a_0 a_1 x}{1+b_1 x} = (x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots) + 0(x) \tag{9}$$

$$a_0 + a_1 x = x + (b_1 - \frac{1}{2})x^2 - \underbrace{b_1 \frac{x}{2}^3}_{0(x^{n+m+1}) = 0(x^3)} \tag{10}$$

$$\left.\begin{array}{l} a_0 = 0 \\ a_1 = 1 \\ b_1 = 1/2 \end{array}\right\} \longrightarrow \qquad f^{[1,1]}(x) = \frac{x}{1 + \frac{1}{2}x} \tag{11}$$

$$f^{[1,1]}(1) = 2/3 \tag{12}$$

Note: $\ln 2 = .69314$ (13)

$$f^{[2,1]}(x) = \frac{P_2(x)}{Q_1(x)} = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x} = (x - \frac{x^2}{2} + \frac{x^3}{3} + 0(x^4) \tag{14}$$

$$a_0 + a_1x + a_2x^2 = x + (b_1 - \tfrac{1}{2})x^2 + (-\tfrac{b_1}{2} + \tfrac{1}{3})x^3 + 0(x^4) \tag{15}$$

$$\left.\begin{array}{l} a_0 = 0 \\ a_1 = 1 \\ b_1 = \frac{2}{3} \\ a_2 = 1/6 \end{array}\right\} \longrightarrow \qquad f^{[2,1]}(x) = \frac{x + \frac{1}{6}x^2}{1 + \frac{2}{3}x} \tag{16}$$

$$f^{[2,1]}(1) = .7 \tag{17}$$

$$f^{[2,2]}(x) = \frac{P_2(x)}{Q_2(x)} = \frac{a_0 + a_1x + a_2x^2}{1 + b_1x + b_2x^2} = (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}) + 0(x^5) \tag{18}$$

$$a_0 + a_1x + a_2x^2 = x + (b_1 - \tfrac{1}{2})x^2 + (b_2 - \tfrac{b_1}{2} + \tfrac{1}{3})x$$

$$+ (\tfrac{b_1}{3} - \tfrac{b_2}{2} - \tfrac{1}{4})x^4 + 0(x^5) \tag{19}$$

$$\left.\begin{array}{rcl} a_0 & = & 0 \\ a_1 & = & 1 \\ b_1 - \frac{1}{2} - a_2 & = & 0 \\ b_2 - \frac{b_1}{2} + \frac{1}{3} & = & 0 \\ \frac{b_1}{3} - \frac{b_2}{2} - \frac{1}{4} & = & 0 \end{array}\right\} \longrightarrow \qquad f^{[2,2]}(x) = \frac{x + \frac{1}{2}x^2}{1 + x + \frac{1}{6}x^2} \tag{20}$$

$$f^{[2,2]}(1) = .6923 \tag{21}$$

$$\text{cf.} \qquad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} = .58 \tag{22}$$

Discussion of the convergence properties of the Padé Type 1 approximants and recursive schemes to calculate the coefficients are given by J. Zinn-Justin and Baker.

II. Padé Type II Approximants (Introduction)

Definition: Let $z_1, z_2, \cdots, z_p$ be p numbers for which the analytic function f(z) is defined.

The Padé Type II Approximent is defined to be:

$$f^{[n,m]}(z) = \frac{P_n(z)}{Q_m(z)} \tag{23}$$

with $f^{[n,m]}(z_i) = f(z_i)$ for

$i = 1,2,\cdots,p$ and $p = m + n + 1$

As before $P_n(z)$ and $Q_m(z)$ are polynomials of degree n and m.

Let us restrict our attention to the diagonal Padé approximants for which $n = m \equiv N$.

Then for each of the 2N + 1 coordinate z_k we can write

$$f^{[N,N]}(z_k) = \frac{\sum_{i=o}^{N} a_i \; z_k^i}{\sum_{j=o}^{N} b_j \; z_k^j} \equiv S_k = f(z_k) \tag{24}$$

For many problems of practical interest, we may know the values of the function ($S_1 \cdots S_{2N+1}$) corresponding to a range of coordinates

$z_1 > z_2 > \cdots > z_{2N+1} > 0$, for example.

We may not know the value of the function at $z = 0$ however. We can obtain an approximation to the value f(0) by constructing the Padé Type II approximant using the known values of the function $f(z_k)$

$$f^{[N,N]}(0) = a_0 \tag{25}$$

Solve for coefficient a_0:

$$S_k \sum_{j=0}^{N} b_j z_k^j = S_k(1 + \sum_{j=1}^{N} b_j z_k^j) = \sum_{i=0}^{N} a_i z_k^i \tag{26}$$

We have 2N + 1 equations for the unknown a_i and b_i which we can write as a matrix equation

$$\begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^N & -S_1 z_1 & \cdots & -S_1 z_1^N \\ 1 & z_2 & & & \vdots & \vdots & & -S_2 z_2^N \\ \vdots & & & & & & & \\ & & & & & & & \\ 1 & z_{2N+1} & & & z_{2N+1}^N & -S_{2N+1} Z_{2N+1} & \cdots & -S_{2N+1} z_{2N+1}^N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \\ b_1 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ \\ S_{2N+1} \end{bmatrix} \tag{27}$$

Using Cramer's Rule solve for a_0

$$a_o = \frac{\begin{vmatrix} S_1 & z_1 & z_1^2 \cdots & z_1^N & -S_1 z_1 \cdots & -S_1 z_1^N \\ S_2 & & & & & \\ \vdots & & & & & \\ S_{2N+1} & z_{2N+1} & & & & -S_{2N+1} z_{2N+1}^N \end{vmatrix}}{\begin{vmatrix} 1 & z_1 \cdots & z_1^N & -S_1 z_1 \cdots & -S_1 z_1^N \\ S_2 & & & & \\ \vdots & & & & \\ 1 & z_{2N+1} & & & -S_{2N+1} z_{2N+1}^N \end{vmatrix}} \tag{28}$$

III. Example for Pade Type II

Suppose that we want to find the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{29}$$

Regarding S as the limit of a sequence of functions (Partial Sums).

$$S_k = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is the } k^{th} \text{ partial sum.} \tag{30}$$

So that for example the first 3 partial sums are

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2^2} = 5/4$$
$$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2} = 49/36 \tag{31}$$

Having 3 partial sums we just need 3 coordinates z_1, z_2, z_3 to construct the $f^{[1,1]}$ Padé approximant.

A valid choice for the coordinate associated with the kth partial sum of this particular series is

$$z_k = \frac{1}{k} \tag{32}$$

Then for the [1,1] approximate

$$f^{[1,1]}(z) = \frac{a_0 + a_1 z}{1 + b_1 z} \tag{33}$$

We demand that the P.A. be equal to the partial sums at $z_1 = 1, z_2 = \frac{1}{2}, z_3 = \frac{1}{3}$

$$f^{[1,1]}(1) = \frac{a_0 + a_1}{1 + b_1} = 1 \tag{34}$$

$$f^{[1,1]}(\frac{1}{2}) = \frac{a_0 + a_{1/2}}{1 + b_{1/2}} = \frac{5}{4} \tag{35}$$

$$f^{[1,1]}\left(\frac{1}{3}\right) = \frac{a_0 + a_{1/3}}{1 + b_{1/3}} = \frac{49}{36} \tag{36}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -5/8 \\ 1 & 1/3 & -49/108 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/4 \\ 49/36 \end{bmatrix} \tag{37}$$

$$a_0 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 5/4 & 1/2 & -5/8 \\ 49/36 & 1/3 & -49/108 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1/2 & -5/8 \\ 1 & 1/3 & -49/108 \end{vmatrix}}$$

$$\frac{\begin{vmatrix} 1 & 1 & -1 \\ 0 & -3/4 & 5/8 \\ 0 & 0 & \frac{11}{216} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1/2 & 3/8 \\ 0 & 0 & 5/108 \end{vmatrix}} = \frac{33}{20} = 1.65 \tag{38}$$

This is to be compared with the "exact" answer.

$$\xi(2) = \pi^2/6 \simeq 1.6449 \tag{39}$$

Also note that the 3rd partial sum $S_3 = 1.3$

IV. Using Pade II approximates to find the zero of a function

Let f(z) be the function for which we want to find the value of z that satisfies the equation

$$f(z) = 0 \tag{40}$$

We start with 3 points z_1. z_2, z_3 for which we evaluate

$$\begin{aligned} f_1 &= f(z_1) \\ f_2 &= f(z_2) \\ f_3 &= f(z_3) \end{aligned} \tag{41}$$

We then build the [1,1] Padé Type II approximate

$$f_i = \frac{a_0 + a_1 z_i}{1 + b_1 z_i} \qquad i = 1,2,3 \tag{42}$$

writing these 3 equations as a matrix equation:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \underset{\sim}{M} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix} \tag{43}$$

$$\underset{\sim}{M} = \begin{bmatrix} 1 & z_1 & -f_1 z_1 \\ 1 & z_2 & -f_2 z_2 \\ 1 & z_3 & -f_3 z_3 \end{bmatrix} \tag{44}$$

Solving for the coefficients a_0 and a_1

$$a_0 = \frac{\begin{vmatrix} f_1 & z_1 & -f_1 z_1 \\ f_2 & z_2 & -f_2 z_2 \\ f_3 & z_3 & -f_3 z_3 \end{vmatrix}}{\left| \underset{\sim}{M} \right|} \tag{45}$$

$$a_1 = \frac{\begin{vmatrix} 1 & z_1 & -f_1 z_1 \\ 1 & z_2 & -f_2 z_2 \\ 1 & z_3 & -f_3 z_3 \end{vmatrix}}{\left| \underset{\sim}{M} \right|} \tag{46}$$

The 1st approximation to the zero of f(z) is given by the solution to the simpler equation

$$f^{[1,1]}(z) = 0 \tag{47}$$

or

$$a_0 + a_1 z_{NEW} = 0 \tag{48}$$

$$z_{NEW} = -a_0/a_1 \tag{49}$$

We can then set up an iterative scheme replacing the coordinates z_1, z_2, and z_3 by the z_{NEW}

V. Type II Padé Approximants (continued)

We now discuss methods of extrapolating certain types of convergent sequences $\{S_1, S_2, \cdots, S_K, \cdots\}$ to the limit

$$S = \lim_{k \to \infty} S_k$$

using Padé type II approximants. An excellent review is given by J. Zinn-Justin[1].

Definition: Let $z_1, \cdots, z_p$ be p complex numbers f(z) a given analytic function. The [n,m] Padé tyope II approximant is the rational fraction:

$$f^{[n,m]}(z) = \frac{P_n(z)}{Q_m(z)} \tag{50}$$

with $f^{[n,m,]}(z_i) = f(z_i)$ for $i = 1, 2, \cdots, p$ and $n + m + 1 = p$. P_n and Q_m are polynomials of degree n and m respectively.

J. Zinn Justin[1] gives the following convergence theorem analogous to a convergence theorem for the usual Padé approximants proved by Nutall[2].

Theorem: If f(z) is a function of the Stieltjes[3] type, and if the points z_i are chosen on a compact set of the real axis on the right of all singularities of the function, then the sequence of Padé approximations converges toward the function, because the zero of the denominators are on the cut of f(z).

In particular we consider the application of Padé type II approximants to the problem of summing an infinite series $S = \sum_{i=1}^{\infty} U_i$ whose kth partial sum is $S_k = \sum_{i=1}^{k} U_i$. The diagonal approximant is defined to be

$$S^{[n,n]}(z) = \frac{a_0 + a_1 Z + \cdots + a_n Z^n}{1 + b_1 Z + \cdots + b_n Z^n} \tag{51}$$

with $S^{[n,n]}(Z_k) = S_k$; $k = 1, 2n + 1$. We define the Padé approximant to the sum S to be

$$S^{[n,n]}(o) = a_0$$

Consideration of a simple example shows that the rate of convergence of the approximants to the sum S is sensitive to the choice of the $\{Z_K\}$. For example, consider $\xi(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Take as co-ordinates $Z_k = \frac{1}{k^P}$, $P \epsilon \{\frac{1}{2}, \frac{3}{4}. 1, \frac{3}{2}, 2\}$. The results shown in Table I for $S^{[2,2]}(O) \Big|_{Z_k = \frac{1}{k^P}}$ show that the simple choice[4,5] $Z_k = \frac{1}{k}$ gives the best agreement with the exact result.

$$\xi(2) = 1.64493406684822643637 \cdots . \tag{52}$$

In fact, if we evaluate $f^{[5,5]}(O) \Big|_{Z_k = \frac{1}{k}}$, we obtain 11-place agreement with the exact result!

TABLE I

PADÉ TYPE II APPROXIMANTS TO $\xi(2)$
FOR DIFFERENT CHOICES
OF CO-ORDINATES

P	$S^{[2,2]}(O) \Big\|_{Z_k = \frac{1}{k^P}}$
1/2	1.64668
3/4	1.65594
1	1.64489
3/2	1.61145
2	1.58065

In order to arrive at a rule for systematically choosing the $\{Z_k\}$, let us consider the expression for a_0

$$a_0 = \frac{\begin{vmatrix} S_1 & z_1 & \cdots z_1^n & -S_1 z_1 & \cdots & -S_1 Z_1^n \\ S_2 & z_2 & & & & \\ \vdots & \vdots & & & & \\ S_{2n+1} & z_{2n+1} & \cdots z_{2n+1}^n & & & -S_{2n+1} Z_{2n+1}^n \end{vmatrix}}{\begin{vmatrix} 1 & z_1 & \cdots & -S_1 z_1^n \\ 1 & & & \\ \vdots & & & \\ 1 & z_{2n+1} & \cdots & -S_{2n+1} z_{2n+1}^n \end{vmatrix}} \tag{53}$$

We notice that $a_0 = S$ if the co-ordinates z_k are chosen as

$$z_k = \frac{S - S_k}{S} \tag{54}$$

We then have $S^{[n,n]}(o) = S$ and the P.A. is exact! We see that the set $\{z_k\}$ is on a compact set of the real axis with the origin as an accumulation point. For example, if the given series is convergent and monotonically increasing such that $O < U_{k+1} < U_k$, then we have $O < z_{k+1}$ and we can choose the z_k's on [0,1].

We can now understand why $z_k = \frac{1}{k}$ is an appropriate choice of co-ordinates for summing $\xi(2)$. Consider the kth partial sum S_k. The remainder is

$$R_{k+1} = \sum_{m=k+1}^{\infty} \frac{1}{m^2}$$

We can easily show that $\frac{1}{k+1} < R_{k+1} < \frac{1}{k}$. The point is that for large k, $R_{k+1} \sim \frac{1}{k}$. Hence for large k the co-ordinates are approximately

$$z_k \sim \frac{\frac{1}{k}}{S} \qquad k >> 1 \tag{55}$$

Now $S^{[n,n]}(O)$ is invariant under the transformation $\{z_k\} \rightarrow \{C\, z_k\}$, where C is an arbitrary normalizing constant; consequently, we may simply choose

$$z_k \sim \frac{1}{k} \qquad k >>1 \tag{56}$$

An example for which we may determine the co-ordinates exactly is the geometric series. We have $S_k = \frac{1-r^k}{1-r}$, $S = \frac{1}{1-r}$, and $z_k = r^k$. In general we can not find the "correct" co-ordinates $\{z_k\}$ since we would need to know the sum of the series; however, as a method of approximating the sum of a series of positive terms, we can try to determine the asymptotic form of the remainder, and then use the asymptotic approximation for the $\{z_k\}$ for all k.

We now consider an example that is particularly relevant to the evaluation of an integral whose integrand has a singularity in its zeroth or first derivative at one of the endpoints of the integration interval.

Suppose that a positive series S is asymptotically geometric. By this we mean $\underset{k\to\infty}{\ell t} \frac{U_{k+m}}{U_k} = r^m$ where $O < r < 1$ and $m \epsilon \{1, 2, \cdots\}$. Then the asymptotic form of the co-ordinates $z_k \equiv \frac{S-S_k}{S}$ for the Padé type II approximant is

$$z_k \xrightarrow{k\to\infty} U_k \tag{57}$$

For consider

$$z_k = \frac{U_{k+1} + U_{k+2} + \cdots}{S} \tag{58}$$

but since $\frac{U_{k+m}}{U_k} \to r^m$ as $k \to \infty$ we have

$$z_k \to U_k \frac{r}{(1-r)S} \tag{59}$$

Noting the invariance of $S^{[n,n]}(0)$ under $\{z_k\} \to \{C\, z_k\}$ we have the result.

As an example of the application of this result to the evaluation of an integral, we consider the integral representation of $\xi(2)$

$$S = \xi(2) = -\int_0^1 dt\, \frac{\ln(1-t)}{t} \tag{60}$$

We consider $\xi(2)$ as a limit of a sequence of partial sums

$$S_k = -\int_0^{1-\varepsilon_k} dt\, \frac{\ln(1-t)}{t} \tag{61}$$

where $0 < \varepsilon_k < 1$.

Defining $U_k = S_k - S_{k-1}$, we form $\frac{U_{k+m}}{U_k}$. Upon letting $\varepsilon_{k+m} = r^m \varepsilon_k$ with $0 < r < 1$, we find that

$$\lim_{k \to \infty} \frac{U_{k+m}}{U_k} = \lim_{\varepsilon_k \to 0} \frac{-\int_{1-r^{m-1}\varepsilon_k}^{1-r^m\varepsilon_k} \frac{\ln(1-t)}{t}\, dt}{-\int_{1-r^{m-1}\varepsilon_k}^{1-r^m\varepsilon_k} \frac{\ln(1-t)}{t}\, dt} = r^m \tag{62}$$

Consequently, the asymptotic form of the Padé type II co-ordinates is

$$Z_k \xrightarrow{k \to \infty} U_k$$

On the other hand, we can study the analytic structure of the sequence of functions $S_k (k = 1,2,\ldots)$ numerically. Let the interval [0,1] be divided into $m = \{1,2,4,8,16,32\}$ subintervals. This choice corresponds to setting $r = 1/2$. An eight point Gauss quadrature

is then applied to each subinterval, and the results of all the subintervals are summed. Thus a sequence of approximant partial sums S'_k shown in column 1 of Table II is obtained. The results shown in columns 2 or 3 of Table II clearly show the geometric character of the convergence. Using the co-ordinates $Z_k = U'_k/U'_1$, we obtain for the Padé extrapolation

$$S^{[2,2]}(0)\Big|_{Z_k = \frac{U_k}{U_1}} = 1.644934066848720$$

This remarkable result agrees with $\xi(2)$ to 13 decimal places (a 9 place improvement over the last quadrature approximation S_6)! It should be remembered; however, that 504 function evaluations were required to obtain this number. Also the convergence is not nearly so rapid for some other functions e.g., $\xi(3)$.

TABLE II

SEQUENCE OF QUADRATURE APPROXIMATIONS S'_K IS SHOWN ALONG WITH DIFFERENCES U'_k AND NORMALIZED CO-ORDINATES Z_k

k	S'_k	$U'_k = S'_{k+1} - S'_k$	$Z_k = U'_k/U'_1$
1	1.636221116771679	$4.344583280914582 \times 10^{-3}$	1.
2	1.640565700052593	$2.181170094296302 \times 10^{-3}$	.5020435685693985
3	1.642796870146890	$1.092839813718882 \times 10^{-3}$	.2515407696772230
4	1.643839709960609	$5.469881472146554 \times 10^{-4}$	.1259011766715422
5	1.644386698110782	$2.736367125502070 \times 10^{-4}$	.0629834197798145
6	1.644660334820374		

The geometric interval method is most useful when the integrand or its first derivative is singular at one of the endpoints of the integration interval.

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