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LIGHT FRONT TAMM-DANCOFF METHOD HYDROGEN ATOM AND 2D YUKAWA MODEL

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Abstract

The light front frame and Tamm-Dancoff method is reviewed briefly and the light front Tamm-Dancoff method is discussed. The method is applied to the hydrogen atom and the 2D Yukawa model.

The light front Tamm-Dancoff (LFTD) method is introduced as an alternative approach to lattice gauge theory to understand relativistic bound states.[1,2]

To alleviate the case the coupling parameter is not small, Tamm and Dancoff[3] (TD) independently considered the possibility of expanding in terms of amplitude that represent a finite number of particles, and solve the coupled set of integral equations of a small number of these amplitudes. This method was applied to a variety of problems in strong interactions, but it was unsuccessful.

In the light front (LF) or null plane formalism, a physical state is given by the evolution variable that plays the role of time $x^+ = (x^0 + x^3)/\sqrt{2} =$ const, and $x^- = (x^0 - x^3)/\sqrt{2}$, $x_\perp = (x^1, x^2)$ are spatial variables. There are seven generators P^1, P^2, P^+, M_{12}, M_{1+}, M_{2+}, and M_{-+} that leave the LF plane invariant.[4]

The momentum of a particle of mass m is

$$p^\mu = \left(p^+,\ p^- = \left(m^2 + p_\perp^2\right)/2p^+,\ p_\perp = \left(p^1, p^2\right)\right) \qquad \mu = +, -, 1, 2$$

where $p^\pm = (p^0 \pm p^3)/\sqrt{2}$ and $p^2 = m^2 + p_\perp^2 - p_\perp^2 = m^2$, where

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

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The p^+ and $p_\perp$ are conserved at each vertex and particles are on mass shell. We require spectrum of p^μ be contained in the forward cone

$$p^0 > 0, \text{ so } p^+ = \left(\sqrt{m^2 + p^{3^2} + p_\perp^2} + p^3\right)/\sqrt{2} > 0\,.$$

The advantages of the LF formalism is that[5] (1) If the LF wave function of a system is known at rest, it can be obtained at other momentum by boost operators. (2) The property $p^+ > 0$, and its conservation reduces the number of Fock space states needed to generate a covariant result. (3) The bare vacuum is equal to the physical vacuum.

But these advantages are accompanied by some disadvantages, (1) It is necessary to require for renormalization, mass and counter terms that depend on the sector of Fock space within which they act. (2) Note if the momentum p^3 is cut off at Λ, $p^+ = (p^0 + p^3)/\sqrt{2} = (2\Lambda + \frac{m^2}{2\Lambda})/\sqrt{2}$. There is an infrared divergence as $p^+ \to$ small. So ultraviolet and infrared divergences have to be treated in a consistent manner. (3) Renormalization 'constants' Z_1 and Z_2 are momentum dependent.[6] (4) How to handle zero modes and gluons in QCD is not worked out.
Consider $H_I = g\phi^3$ as an example.[7]

If one restricts the TD amplitudes to less than two particles, then the second Z diagram is dropped. Both are needed for covariance in the equal time frame. In the LF frame the Z diagram does not exist, as the leg going backward has negative energy and $p^+ = (-p^0 + p^3)/\sqrt{2}$ cannot be positive. But in the LF the first diagram alone is covariant

$$\begin{aligned} p_1 &= px + p_{1\perp} \\ p_2 &= p(1-x) + p_{2\perp} \\ p &= p_1 + p_2 \qquad \therefore p_{1\perp} + p_{2\perp} = p_\perp = 0 \end{aligned}$$

$$M = g^2 \frac{1}{p^+} \frac{1}{2\left(p_{in}^- - p_{int}^-\right)} \qquad p_1^- = \frac{m^2 + p_{1\perp}^2}{2p_1^+}$$

$$
\begin{aligned}
M &= \lim_{p\to\infty} g^2 \left[\frac{1}{p^+\left(\frac{m^2+p_{1\perp}^2}{p_1^+}\right) + p^+\left(\frac{m^2+p_{2\perp}^2}{p_2^+}\right) - p^+\left(\frac{m^2}{p^+}\right)} \right] \\
&= g^2 \left[\frac{1}{\left(m^2+p_{1\perp}^2\right)/x + \left(m^2+p_{2\perp}^2\right)/\left(1-x\right) - m^2} \right] = \frac{g^2}{s^2-m^2} \qquad (1)
\end{aligned}
$$

We begin with Einstein's equation to consider the mass renormalization of the electron.

$$
\left(M^2 + P_\perp^2 - 2P^+P_0^-\right)\left|\Psi\right> = 2P^+P_I^-\left|\Psi\right> \qquad (2)
$$

We outline the calculation[8] for the hydrogen atom of state $|H>$

$$
\left|H\right> = C_0\left|Pe\right> + C_1\left|Pe\gamma\right> \qquad (3)
$$

Project out C_0 amplitude from Eqs. (2) and (3).

$$
FC_0 = e\int L_{01}C_1 + e^2\int L_{00}C_0\,, \qquad (4)
$$

where L_{00} represents the instantaneous photon exchange and

$$
F = \left[M^2 + (p_\perp + k_\perp)^2 - \left(p^+ + k^+\right)\left\{\frac{k_\perp^2 + M_{e0}^2}{k^+} + \frac{p_\perp^2 + M_{P0}^2}{p^+}\right\}\right].
$$

Project out C_1 from Eqs. (2) and (3), and obtain

$$
HC_1 = eL_{10}C_0 + e^2\int L_{11}C_1 \qquad (5)
$$

Eliminate C_1 from Eq. (4) with the aid of Eq. (5)

$$
FC_0 = e^2\int L_{01}H^{-1}L_{10}C_0 + e^2\int L_{00}C_0\,. \qquad (6)
$$

In the nonrelativistic limit, $|p|,|q| < \Lambda \ll M_e$, after mass renormalization we obtain, with $M^2 = (M_e + M_P - B)^2 = (M_P + M_e)^2 - 2(M_P + M_e)\overline{B}$

$$
\left(\overline{B} + \frac{p^2}{2\mu}\right)\Phi(P) = \frac{\alpha}{2\pi^2}\int_{-\infty}^{\infty} d^3k\,\frac{\Phi(k)}{(p-k)^2} \qquad (7)
$$

μ is the reduced mass $\alpha = e^2/4\pi$. This is the Lippman-Schwinger equation and its solution[9] is $\overline{B}_n = \dfrac{\alpha^2\mu}{2n^2}$.

A poor person's solution when $p = 0$ can be found by using the trial function $\Phi(k) = C/(k^2+a^2)^2$, where a is to be determined.

We find $\overline{B} = \frac{\alpha}{2}\, a$ where a has dimensions of mass. Put $a = \sqrt{2\overline{B}\mu}$, then

$$\begin{aligned} \overline{B} = \frac{\alpha^2\mu}{2} &= \frac{\alpha^2 m_e}{2} \quad \text{hydrogen} \quad n=1 \\ &= \frac{\alpha^2 m_e}{4} \quad \text{positronium} \quad n=1 \end{aligned} \tag{8}$$

We consider next fermion-fermion interaction via meson exchange in two dimensions[10]. The fermions have equal mass, different flavor and are neutral. Expand the two fermion bound state $|\psi>$

$$\left|\psi\right> = C_0\left|ff\right> + C_1\left|ffb\right> \tag{9}$$

It is shown in Ref. 10 that the self-energy corrections can be ignored in the following approximation in the non-relativistic limit. We let the fractions of the momentum of the two fermions be x and $(1-x)$, so that $p^+/p_1^+ = 1/x$, $p^+/p_2^+ = 1/(1-x)$ and $p = p_1 + p_2$. We put $P_\perp = 0$, project out C_0 and C_1 from Eqs. (2) and (9), and obtain[10]

$$\begin{aligned} &\left(M^2 - m^2/(1-x) - m^2/x\right) C_0(1-x,x) \\ &= \lambda m \left\{ \int_0^{1-x} \frac{dy}{\sqrt{4\pi y}} \left[1/(1-x) + 1/(1-x-y)\right] C_1(1-x-y,x,y) \right. \\ &\quad \left. + \int_0^{x} \frac{dy}{\sqrt{4\pi y}} \left[1/x + 1/(x-y)\right] C_1(1-x,x-y,y) \right\} \end{aligned} \tag{10}$$

$$\begin{aligned} &\left(M^2 - m^2/(1-x-y) - m^2/x - m_B^2/y\right) C_1(1-x-y,x,y) \\ &= \frac{\lambda m}{\sqrt{4\pi y}} \left\{ \left[1/(1-x-y) + 1/(1-x)\right] C_0(1-x,x) \right. \\ &\quad \left. + \left[1/x + 1/(x+y)\right] C_0(1-x-y,x+y) \right\} \end{aligned} \tag{11}$$

Eliminate C_1 from Eqs. (10) and (11),

$$\begin{aligned} &\left[M^2 - m^2/(1-x) - m^2/x\right] C_0(1-x,x) \\ &= \lambda^2 m^2 \left\{ \int_0^{1-x} \frac{dy}{4\pi y} \frac{\left[1/(1-x) + 1/(1-x-y)\right]\left[1/x + 1/(x+y)\right]}{M^2 - m^2/(1-x-y) - m^2/x - m_B^2/y} C_0(1-x-y,x+y) \right. \\ &\quad \left. + \int_0^{x} \frac{dy}{4\pi y} \frac{\left[1/(1-x) + 1/(x-y)\right]\left[1/(1-x) + 1/(1-x+y)\right]}{M^2 - m^2/(1-x) - m^2(x-y) - m_B^2/y} C_0(1-x+y,x-y) \right\} \end{aligned} \tag{12}$$

Change variables in Eq. (12), $1-x=x_1$, $x=x_2$, $1-x-y=y_1$, $x+y=y_2$

$$x_{1,2}=\frac{1}{2}\left(1\mp\frac{q}{\sqrt{m^2+q^2}}\right)\qquad y_{1,2}=\frac{1}{2}\left(1\mp\frac{\ell}{\sqrt{m^2+\ell^2}}\right)$$

$$\phi(q)=4\sqrt{\frac{m^2+q^2}{m}}\,C_0(x_1x_2)\quad \phi(\ell)=4\sqrt{\frac{m^2+\ell^2}{m}}C_0(y_1y_2)$$

and the resulting equation with $M=2m-B$, $\overline{B}=B-(B^2/4m)$ is[10]

$$\left(\frac{q^2}{2\mu}+\overline{B}\right)\phi(q)=\frac{\lambda^2}{2\pi}\int d\ell\,\frac{1}{(q-\ell)^2+m_B^2}\Phi(\ell) \tag{13}$$

This is remarkably similar to the hydrogen atom equation (7). The correspondence is $\frac{d\ell}{2\pi}\to\frac{d^3k}{(2\pi)^3}$ and $m_B=0$ as expected. To solve the 2D Eq. (13) at $q^2=0$, choose a trial wave function $\Phi(\ell)=\frac{C}{\sqrt{\ell^2+a^2}}$ and minimize $\overline{B}$ with respect to a. The solution is $\overline{B}=\frac{\lambda^2}{\pi m_B}$ where λ has the dimension of mass in 2D.

One would like to regard the 2D Eq. (13) extended to 4D as a beginning equation to calculate the deuteron bound state energy. But the problem is much more complicated because of a repulsive core and D wave contributions. The relativistic limit of one boson exchange in the Yukawa model is discussed in Ref. 11.

For fun if one substitutes in Eq. (13)

$$\int\frac{d\ell}{2\pi}\to\int\frac{d^3\ell}{(2\pi)^3}$$

$$\phi=\frac{C}{\sqrt{k^2+a^2}}\to\phi=\frac{C}{(k^2+a^2)^{3/2}}$$

one obtains after minimizing with respect to a

$$\overline{B}=\frac{\lambda^2}{3\pi^2}\,m_B\,.$$

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