UNIVERSITATEA DE VEST DIN TIMIȘOARA

FACULTATEA DE FIZICĂ

A. Neagu

PATH INTEGRALS
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Path Integrals 1

Adrian Neagu
University of Timişoara
Theoretical and Computational Physics Department

Abstract

Feynman's path-integral formulation of a quantum theory is considered. We start with the case of nonrelativistic quantum mechanics in one space dimension, generalize to more dimensions, and by analogy we formulate the path integral quantization of scalar field theory. A short account on analytical computation of Gaussian path integrals is also included. The quantum theory of Bose fields is presented in both the Euclidean- and Minkowski-space formulation, and it is pointed out, that the well-definiteness of the underlying path integral dictates Feynman's $i\epsilon$ prescription for the field propagator. Finally, the general procedure to derive Feynman rules in the theory of a self-interacting scalar field is discussed in detail, and the rules for the case of the ϕ^4 interaction are given.

1 Path integrals in quantum mechanics

The path-integral method due to Feynman is an alternative approach to a quantum theory. For the sake of simplicity we first illustrate it in the case of nonrelativistic quantum mechanics and then graduate to quantum field theory.

According to the generalized Huygens principle, the wavefunction $\Psi(q'', t'')$ which describes the state of a particle at the moment t'' stems from a superposition of earlier wavefunctions:

$$\Psi(q'',t'') = \int dq' K(q'',t'';q',t')\Psi(q',t') \qquad t'' > t' \qquad (1)$$

The Feynman kernel $K(q'',t'';q',t')$ is the probability amplitude for a particle that was observed in the location q' at the moment t' to be found in q'' at t'' .

In the framework of canonical quantization the states are represented by (generalized) Hilbert-space vectors and the observables are operators: $\hat{p},\hat{q}$. (From now on a hat crowns operators.) They satisfy:

$$[\hat{p},\hat{q}] = -i\hbar, \tag{2}$$

and we have the coordinate and momentum representations in which they are diagonal:

$$\hat{q} \mid q >= q \mid q > \tag{3}$$

$$\hat{p} \mid p >= p \mid p > \tag{4}$$

The normalization and completeness relations read :

$$< q'' \mid q' >= \delta(q'' - q') \tag{5}$$

$$1 = \int dq \mid q >< q \mid, \tag{6}$$

and

$$< p'' \mid p' >= \delta(p'' - p') \tag{7}$$

$$1 = \int dp \mid p >< p \mid . \tag{8}$$

We also have

$$< p \mid q >= \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}qp} \tag{9}$$

In this (Schrödinger) picture the state vector $\mid \Psi_t >$ is time-dependent and the wavefunction is

$$\Psi(q,t) =< q \mid \Psi_t > \tag{10}$$

The Heisenberg picture is obtained via a time-dependent unitary transformation, and is such that the state vector is time-independent, $\mid \Phi >$. We have:

$$\Psi(q,t) =< qt \mid \Phi > \tag{11}$$

where $| qt >$ are eigenstates of the Heisenberg coördinate operator

$$\hat{q}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{q}e^{-\frac{i}{\hbar}\hat{H}t} \tag{12}$$

with the eigenvalue q. In the above equation $\hat{H}$ is the time-independent Hamiltonian.

The constant state vector is

$$| \Phi >= e^{\frac{i}{\hbar}\hat{H}t} | \Psi_t > \tag{13}$$

and a comparison of Eqs.(10) and (11) yields the time evolution of the $| qt >$ states:

$$| qt >= e^{\frac{i}{\hbar}\hat{H}t} | q > \tag{14}$$

We may rewrite Eq.(1) as

$$< q''t'' | \Phi >= \int dq' < q''t'' | q't' >< q't' | \Phi > \tag{15}$$

because of the completeness property of the $\{| qt >\}$ basis for any given time, t.

Our aim is to express the transition amplitude in terms of the classical Hamiltonian, $H(q,p)$ without reference to operators and states in a Hilbert space.

To proceed, we divide the time interval $[t', t'']$ into N equal pieces.

$$\Delta t = \frac{t'' - t'}{N}; \qquad N \to \infty, \tag{16}$$

denote

$$\epsilon = \frac{\Delta t}{\hbar}, \tag{17}$$

and write the Feynman kernel as

$$\begin{aligned} K(q'', t''; q', t') \equiv < q''t'' | q't' > \; = & \int dq_1 \cdots dq_{N-1} < q''t'' | q_{N-1}t_{N-1} > \cdots \\ & \cdots < q_2t_2 | q_1t_1 >< q_1t_1 | q't' > \end{aligned} \tag{18}$$

A typical factor in the integrand is

$$
\begin{aligned}
< q_n t_n \mid q_{n-1} t_{n-1} > &= < q_n \mid e^{-\frac{i}{\hbar}\hat{H}(t_n - t_{n-1})} \mid q_{n-1} > \\
&= < q_n \mid (1 - i\epsilon\hat{H}) \mid q_{n-1} > \\
&= \int dp_n < q_n \mid p_n >< p_n \mid (1 - i\epsilon\hat{H}) \mid q_{n-1} > \quad (19)
\end{aligned}
$$

If $\hat{H}$ contains no cross products of $\hat{q}$ and $\hat{p}$, i. e. $\hat{H}$ is a sum of $\hat{q}$-dependent and $\hat{p}$-dependent terms, then

$$
< p_n \mid \hat{H} \mid q_{n-1} >=< p_n \mid q_{n-1} > H(p_n, q_{n-1}) \quad (20)
$$

and in the coordinate-dependent part of the classical Hamiltonian,H, we may replace q_{n-1} by the midpoint value:

$$
\bar{q}_n = \frac{q_n + q_{n-1}}{2} \quad (21)
$$

because of a Dirac δ-function stemming from the momentum integration in Eq.(19)

Hence,

$$
< q_n t_n \mid q_{n-1} t_{n-1} >= \int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar} p_n (q_n - q_{n-1})} (1 - i\epsilon H(p_n, q_n)) \quad (22)
$$

This relation holds also for $\hat{H}$ containig Weyl-ordered cross products of momentum and coordinate operators[1].

We next substitute (22) into (18):

$$
\begin{aligned}
< q'' t'' \mid q' t' > &= \int dq_1 \cdots dq_{N-1} \frac{dp_1}{2\pi\hbar} \cdots \frac{dp_N}{2\pi\hbar} \\
&\quad \prod_{n=1}^{N} e^{\frac{i}{\hbar} p_n (q_n - q_{n-1})} (1 - i\epsilon H(p_n, \bar{q}_n)) \quad (23)
\end{aligned}
$$

where $q_0 = q'$ and $q_N = q''$.

In the product from the integrand we encounter factors which, to first order in the infinitesimal parameter ϵ, coincide with an exponential. However,

[1] See [8], p.61

in the limit of $N \to \infty$ the product consists of an infinity of such factors, so this coincidence to first order doesn't justify the substitution $(1 - i\epsilon H) \to \exp(-i\epsilon H)$. For instance, another replacement, namely $(1 - i\epsilon H) \to (1 + i\epsilon H)^{-1}$, would give incorrect result.

There is a theorem which allows us to do the first substitution[2].

Theorem 1 *Given N numbers $z_1, \ldots, z_N$ such that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} z_n = X < \infty$, we have:*

$$\lim_{N\to\infty} \prod_{n=1}^{N} (1 + \frac{z_n}{N}) = \lim_{N\to\infty} \prod_{n=1}^{N} e^{\frac{z_n}{N}} = e^X \tag{24}$$

It can be proven by power series expansion in z_n. Using this theorem we rewrite Eq.(23)as

$$\begin{aligned} < q''t'' \mid q't' > &= \int dq_1 \cdots dq_{N-1} \frac{dp_1}{2\pi\hbar} \cdots \frac{p_N}{2\pi\hbar} \\ &\quad e^{\frac{i}{\hbar}\sum_{n=1}^{N}\left[p_n \frac{q_n - q_{n-1}}{\Delta t} - H(p_n, \bar{q}_n)\right]\Delta t} \end{aligned} \tag{25}$$

We adopt the view that the set of values $\{q_1, \ldots, q_{N-1}\}$ and $\{p_1, \ldots, p_N\}$ are successive values of certain continuous functions of time, $q(t)$ and $p(t)$, and denote:

$$t_n = t' + n\Delta t; \qquad n = 0, 1, \ldots, N \tag{26}$$

$$q_n = q(t_n) \tag{27}$$

$$p_n = p(t_n) \tag{28}$$

while taking the limit $N \to \infty$ (i. e. $\Delta t \to 0$) we have

$$\frac{q_n - q_{n-1}}{\Delta t} \xrightarrow{\Delta t \to 0} \dot{q}(t_n) \tag{29}$$

$$\sum_{n=1}^{N} f(t_n)\Delta t \xrightarrow{\Delta t \to 0} \int_{t'}^{t''} dt f(t), \tag{30}$$

[2]See [4], p.124

and for the integration measure we introduce the notations

$$dq_1 \cdots dq_{N-1} = \prod_{n=1}^{N-1} dq(t_n) \equiv \prod_t q(t) \equiv (Dq) \tag{31}$$

and

$$\frac{dp_1}{2\pi\hbar} \cdots \frac{dp_N}{2\pi\hbar} = \prod_{n=1}^{N} \frac{dp(t_n)}{2\pi\hbar} \equiv \prod_t \frac{dp(t)}{2\pi\hbar} \equiv (Dp) \tag{32}$$

In this way we obtained the phase-space path integral formula for the transition amplitude:

$$\begin{aligned} < q''t'' \mid q't' > &= \int_{q(t')=q'}^{q(t'')=q''} (Dp)(Dq) e^{\frac{i}{\hbar}\int_{t'}^{t''} dt[p\dot{q} - H(q,p)]} \\ &\equiv \int (Dp)(Dq) e^{\frac{i}{\hbar} S[q,p]} \end{aligned} \tag{33}$$

The integrations are performed over "paths",i. e. over the continuous set of variables $q(t)$ and $p(t)$ with $t \in [t', t'']$. The gaussian integrals of this kind can be performed analytically in a similar fashion as the usual ones.

In the case of classical Hamiltonians of the form

$$H(p,q) = \frac{p^2}{2m} + V(q) \tag{34}$$

the momentum integrations in (25) can be performed (formally), yielding Feynman's famous configuration-space path-integral formula. To this end, let us take the integral

$$\int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}\Delta t\left[p_n \frac{q_n - q_{n-1}}{\Delta t} - \frac{p_n^2}{2m} - V(\bar{q}_n)\right]} \tag{35}$$

We complete the square by making the change of variables

$$p_n' = p_n - m\frac{q_n - q_{n-1}}{\Delta t}, \tag{36}$$

and obtain for the integral (35)

$$\frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}\Delta t\left[\frac{m}{2}\left(\frac{q_n - q_{n-1}}{\Delta t}\right)^2 - V(\bar{q}_n)\right]} \int dp_n' e^{-\frac{i\Delta t}{2m\hbar} p'^2_n} \tag{37}$$

We now intend to integrate over p'_n, but the integrand is purely oscillatory. We have two choices: either we perform the Gaussian integral by taking formally $i\Delta t$ to be real (continuation to imaginary time), or we introduce by hand a convergence factor[3], $e^{-\delta p'^2_n}$.

Treating $i\Delta t$ as a real constant we obtain for the p'_n-integral the expression:

$$\sqrt{\frac{2m\pi\hbar}{i\Delta t}} \tag{38}$$

Hence (25) becomes

$$< q''t'' \mid q't' >= (\frac{m}{2\pi\hbar i\Delta t})^{\frac{N}{2}} \int_{q_0=q'}^{q_N=q''} dq_1 \cdots dq_{N-1} e^{\frac{i}{\hbar}\sum_{n=1}^{N}\left[\frac{m}{2}\left(\frac{q_n-q_{n-1}}{\Delta t}\right)^2 - V(\bar{q}_n)\right]\Delta t} \tag{39}$$

and in the limit $N \to \infty$, according to Eqs.(29) and (30), this gives the desired result

$$< q''t'' \mid q't' >= \mathcal{N} \int_{q(t')=q'}^{q(t'')=q''} (Dq) e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(\frac{m}{2}\dot{q}^2 - V(q))} \equiv \mathcal{N} \int (Dq) e^{\frac{i}{\hbar}S[q]} \tag{40}$$

which is Feynman's formula. The quantity $S[q]$ is the action, a functional of the particuar path, $q(t)$.

The normalization constant, $\mathcal{N}$, is infinite but it doesn't appear in expressions with physical meaning, because these have the form of a matrix element $< q''t'' \mid \mathcal{O} \mid q't' > / < q''t'' \mid q't' >$, with $\mathcal{O}$ some operator. It follows that it is enough to define the integration measure up to a multiplicative constant. From now on, we shall include any constant in the measure.

In words, the above formula means that the probability amplitude for the particle to be in q'' at time t'', given that it was in q' at time t', is a sum over all possible paths that start from q' at t' and end in q'' at t'', weighted by the exponential of $\frac{i}{\hbar}$ times the action evaluated for the particular path.

[3]See [5] p.7 , [6] p.77

2 Quantum Field Theory: Bose Fields

Fields are nothing but nondenumerably infinite variable systems. The first step in generalizing the path-integral formula (33) is to consider the D-dimensional case with coordinates $\{q_\alpha\}_{\alpha=1,\dots,D}$. The integration measure $(Dq)(Dq)$ contains an extra product over the index α. It is:

$$\prod_{\alpha,t} dq_\alpha(t)\frac{dp_\alpha(t)}{2\pi\hbar} \tag{41}$$

Next we replace the discrete index α by a continuous one: $\vec{x}$, and denote the value of the "coordinate" by ϕ. Thus a classical Bose field $\phi(\vec{x})$ can be viewed as the $\vec{x}$-component of a continuously infinite dimensional vector. The cooresponding path-integral measure becomes:

$$(D\Pi)(D\phi) \equiv \prod_{\vec{x},t} d\phi(\vec{x},t)\frac{d\Pi(\vec{x},t)}{2\pi\hbar}, \tag{42}$$

with

$$\Pi(\vec{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \tag{43}$$

the conjugate momentum.

In canonical quantization the coordinates are Schrödinger field operators, $\hat{\phi}(\vec{x})$ with eigenstates $\mid \phi >$. In the Heisenberg picture the field operator

$$\hat{\phi}(\vec{x},t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{\phi}(\vec{x})e^{-\frac{i}{\hbar}\hat{H}t} \tag{44}$$

which involves the (time independent) Hamiltonian of the system. If $\hat{H}$ were time dependent, $\exp(\frac{i}{\hbar}\hat{H}t)$ would be replaced by $\mathrm{T}\exp(\frac{i}{\hbar}\int_0^t d\tau\hat{H}(\tau))$ with T-the time ordering operator. We consider the time-independent case.

The Heisenberg field operator has eigenstates $\mid \phi, t >$:

$$\hat{\phi}(\vec{x},t) \mid \phi,t >= \phi(\vec{x},t) \mid \phi,t > \tag{45}$$

and the transition amplitude is defined by

$$< \phi''t'' \mid \phi't' >\equiv< \phi'' \mid e^{-\frac{i}{\hbar}(t''-t')\hat{H}} \mid \phi' > \tag{46}$$

By analogy, its path-integral expression reads:

$$< \phi''t'' \mid \phi't' >= \int_{\phi'}^{\phi''} (D\phi)(D\Pi) e^{\frac{i}{\hbar}\int_{t'}^{t''} dt \int d^3x [\Pi(\partial_0\phi) - \mathcal{H}]} \tag{47}$$

In what follows we shall restrict ourselves to the case when the Hamiltonian density has the particular form:

$$\mathcal{H} = \frac{1}{2}\Pi^2(x) + \mathcal{V}(\phi(x), \nabla\phi(x)) \tag{48}$$

where x denotes the space-time point $x^\mu \equiv (t, \vec{x})$. We use units in which $c = \hbar = 1$, but we keep $\hbar$ in the exponent displayed because it will serve later as a parameter in the loop expansion.

In the case under consideration one integrates formally over the Πs and obtains the Feynman formula

$$< \phi''t'' \mid \phi't' >= \mathcal{N} \int_{\phi'}^{\phi''} (D\phi) e^{\frac{i}{\hbar}\int_{t'}^{t''} dt \int d^3x \mathcal{L}(x)} \tag{49}$$

The integration limits in (47) and (49) denote the endpoint constraints $\phi(\vec{x}, t') = \phi'(\vec{x})$ and $\phi(\vec{x}, t'') = \phi''(\vec{x})$.

The functional integration may be defined by first considering x to be a discrete variable, performing the multiple integration and then approaching the continuum limit[4] Alternatively, we may enclose the system in a large, but finite space-time volume, integrate over the discrete Fourier components of ϕ, and then approach the limit of infinite volume.

The transition amplitude can be continued analytically to complex times. To see this, consider the eigenstates $\mid n >$ of $\hat{H}$ and assume that there is a unique vacuum state with zero energy:

$$\begin{aligned} \hat{H} \mid n > &= E_n \mid n > \quad E_n \geq 0 \\ \hat{H} \mid 0 > &= 0 \qquad\qquad < 0 \mid 0 >= 1 \end{aligned} \tag{50}$$

Eq. (46) gives

$$< \phi''t'' \mid \phi't' >= \sum_{n=0}^{\infty} < \phi'' \mid n >< n \mid \phi' > e^{-\frac{i}{\hbar}(t''-t')E_n} \tag{51}$$

[4]See [7],p.218 and [10],p.72.

It can be continued for $(t''-t')$ a complex parameter with negative imaginary part. This amounts to rotating clockwise the time axis by an angle $\frac{\pi}{2}$. We shall return to this point later. Setting $t''=-i\tau$ and $t'=i\tau$ and taking the limit $\tau \to \infty$ we have:

$$\lim_{\substack{t'' \to -i\infty \\ t' \to i\infty}} < \phi'' t'' \mid \phi' t' > = \lim_{\tau \to \infty} \sum_{n=0}^{\infty} < \phi'' \mid n >< n \mid \phi' > e^{-\frac{2}{\hbar}\tau E_n} \qquad (52)$$

This is a useful property.

A very fruitful idea due to Schwinger[5] is to minimally couple the system to an external source $J(x)$, and study the response of the vacuum state to this (arbitrary) driving force. The whole dynamics of the system can be extracted from that.

To show this, let us consider the transition amplitude in the presence of a source:

$$< \phi'' t'' \mid \phi' t' >_J = \mathcal{N} \int_{\phi'}^{\phi''} (D\phi) e^{\frac{i}{\hbar}\int_{t'}^{t''} dt \int d^3x [\mathcal{L}(x) + J(x)\phi(x)]} \qquad (53)$$

We denote the coupling term by

$$\int_{t'}^{t''} dt \int d^3x J(x)\phi(x) \equiv (J, \phi). \qquad (54)$$

It is a functional of both the field and the source.

The differential calculus can be extended to the case of functionals. We call functional a real-valued function,

$$F : \mathrm{C}^\infty \to \mathrm{R}; \qquad F[f] \in \mathrm{R},$$

defined on the space, C^∞, of indefinitely differentiable functions with continuous derivatives. One defines the functional (or variational) differentiation by[6]:

$$\frac{\delta F[f]}{\delta f(t_0)} \equiv \lim_{\epsilon \to 0} \frac{F[f(t) + \epsilon\delta(t - t_0)] - F[f(t)]}{\epsilon} \qquad (55)$$

[5] J. Schwinger Proc. Nat. Acad. Sci. 37, 452 (1951)

[6] See [7], p.208

For example, this definition gives

$$\frac{\delta(J,\phi)}{\delta J(y)} = \phi(y), \tag{56}$$

and if we functionally differentiate the transition amplitude (53), we obtain the expression

$$\frac{\delta^n < \phi''t'' \mid \phi't' >_J}{\delta J(y_1)\cdots\delta J(y_n)} = (\frac{i}{\hbar})^n \mathcal{N}\int (D\phi)\phi(y_1)\cdots\phi(y_n)e^{\frac{i}{\hbar}[S+(J,\phi)]} \tag{57}$$

which resembles the matrix elements of time-ordered products of Heisenberg field operators

$$< \phi''t'' \mid \mathrm{T}(\hat{\phi}(y_1)\cdots\hat{\phi}(y_n)) \mid \phi't' >= \mathcal{N}\int (D\phi)\phi(y_1)\cdots\phi(y_n)e^{iS} \tag{58}$$

A comparison of the right hand sides of Eqs.(57) and (58) suggests that the transition amplitude in the presence of external sources is related to the Green's functions of the system. But who are the physical states of the system? The answer depends on the success of a perturbative approach, but nevertheless the state of least energy, called the vacuum, is thought to exist.

We intend to relate the vacuum-vacuum amplitude to the Green's functions, and to find a path-integral representation of it. Let us first define this amplitude.

In the presence of an external source the Hamiltonian of the system becomes time dependent:

$$\hat{H}_J(t) = \hat{H} - \int d^3x J(x)\hat{\phi}(x) \equiv \hat{H} + \hat{H}_1(t) \tag{59}$$

where $\hat{\phi}(x) = \hat{\phi}(\vec{x},t)$ is a Heisenberg field operator and $J(x)$ is a c-number function. It is supposed to vanish in the infinite past and infinite future:

$$J(x) \overset{|x^0|\to\infty}{\longrightarrow} 0, \tag{60}$$

a necessary condition for the system to be in the vacuum state, $\mid 0 >$, of $\hat{H}$ (sic) at these times.

We call vacuum-vacuum amplitude and denote by $< 0^+ \mid 0^- >_J$ the probability amplitude for the system to be in the state $\mid 0 >$ at time $x^0 = \infty$, given that it was in the state $\mid 0 >$ at $x^0 = -\infty$, in spite of the presence of the source J.

By unitarity, this amplitude can only be a phase factor. It is also denoted by

$$< 0^+ \mid 0^- >_J \equiv Z[J] \equiv e^{\frac{i}{\hbar}W[J]} \tag{61}$$

Using the evolution operator in the interaction picture with respect to $\hat{H}_1$, the state of the system at $x^0 = 0$ is

$$\mid 0^- >_J = \mathrm{T} e^{-\frac{i}{\hbar}\int_{-\infty}^{0} dt \hat{H}_1(t)} \mid 0 > \tag{62}$$

and the state at $x^0 = 0$ which will evolve into $\mid 0 >$ at $x^0 = \infty$ is given by

$$\mid 0^+ >_J = \mathrm{T} e^{-\frac{i}{\hbar}\int_{\infty}^{0} dt \hat{H}_1(t)} \mid 0 > . \tag{63}$$

Thus

$$\begin{aligned} Z[J] &= < 0 \mid \mathrm{T} e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \hat{H}_1(t)} \mid 0 > \equiv \\ &\equiv < 0 \mid \mathrm{T} e^{\frac{i}{\hbar}\int d^4x J(x)\hat{\phi}(x)} \mid 0 > \end{aligned} \tag{64}$$

which is Dyson's definition for the generating functional of the Green's functions. Indeed, expanding the right hand side of (64) we obtain

$$Z[J] = 1 + \sum_{n=1}^{\infty} \left(\frac{i}{\hbar}\right)^n \int d^4x_1 \cdots d^4x_n J(x_1) \cdots J(x_n) < 0 \mid \mathrm{T}(\ \hat{\phi}(x_1) \cdots \hat{\phi}(x_n)\ \mid 0 \tag{65}$$

where we identify the Green's functions which can be "extracted" from $Z[J]$ by functional differentiation:

$$G(x_1, x_2, \ldots, x_n) \equiv < 0 \mid \mathrm{T}(\ \hat{\phi}(x_1) \cdots \hat{\phi}(x_n)\ \mid 0 > = \left(\frac{\hbar}{i}\right)^n \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{66}$$

In this sense is the vacuum-vacuum amplitude the generating functional for the Green's functions.

It can be shown[7] that the phase functional $W[J]$ in turn is the generating functional for the connected Green's functions

[7] See [4] p.131

$$G_c(x_1,\ldots,x_n) = \left(\frac{\hbar}{i}\right)^{n-1} \left.\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} \tag{67}$$

These are the sum of all connected Feynman graphs with n external legs terminating at $x_1,\ldots,x_n$.

We next seek for a path-integral expression of the generating functional. To satisfy (60) we consider a source which is turned on during a large, finite time interval $[-\tau,\tau]$:

$$J(x) = 0 \quad \text{for} \quad |\, x_0 \,| > \tau, \tag{68}$$

and take the limit $\tau \to \infty$ later.

The transition amplitude between the moments $t' < -\tau$ and $t'' > \tau$ is

$$<\phi''t'' \mid \phi't'>_J = \int (D\phi_1)(D\phi_2) < \phi''t'' \mid \phi_2\tau >< \phi_2\tau \mid \phi_1, -\tau >_J< \phi_1, -\tau \mid \phi't' > \tag{69}$$

Here we used the completeness of the $\mid \phi t >$ states.

By continuation to imaginary times[8], the source-free amplitudes can be computed as follows:

$$\begin{aligned} < \phi''t'' \mid \phi_2\tau > &= < \phi'' \mid e^{-\frac{i}{\hbar}(t''-\tau)\hat{H}} \mid \phi_2 >= \\ \sum_{n=0}^{\infty} < \phi'' \mid n >< n \mid \phi_2 > e^{-\frac{i}{\hbar}(t''-\tau)E_n} &\overset{t''\to -i\infty}{\longrightarrow} < \phi'' \mid 0 >< 0 \mid \phi_2 > \end{aligned} \tag{70}$$

and

$$< \phi_1, -\tau \mid \phi', t' > \overset{t'\to i\infty}{\longrightarrow} < \phi_1 \mid 0 >< 0 \mid \phi' > . \tag{71}$$

We substitute them in (69) and express the source-dependent amplitude in terms of the time-dependent Hamiltonian.

$$\begin{aligned} \lim_{\substack{t''\to -i\infty \\ t'\to i\infty}} < \phi''t'' \mid \phi't' >_J &= \int (D\phi_1)(D\phi_2) < \phi'' \mid 0 >< 0 \mid \phi_2 > \\ &= < \phi_2 \mid \mathrm{T}e^{-\frac{i}{\hbar}\int_{-\tau}^{\tau} dt \hat{H}_J(t)} \mid \phi_1 >< \phi_1 \mid 0 >< 0 \mid \phi' > \end{aligned} \tag{72}$$

[8] See Eq.(52)

Using the completeness of the $\mid \phi >$ states we obtain

$$\lim_{\substack{t'' \to -i\infty \\ t' \to i\infty}} < \phi'' t'' \mid \phi' t' >_J = < \phi'' \mid 0 >< 0 \mid \mathrm{T} e^{-\frac{i}{\hbar}\int_{-\tau}^{\tau} dt \hat{H}_1(t)} \mid 0 >< 0 \mid \phi' > \tag{73}$$

where we have also used the fact that $\exp(-\frac{i}{\hbar} 2\tau \hat{H}) \mid 0 >= 1$. Taking the limit $\tau \to \infty$ we recognize the expression (64) of the vacuum-vacuum amplitude. Using also Eq.(52) we conclude that

$$Z[J] = \lim_{\substack{t'' \to -i\infty \\ t' \to i\infty}} \frac{< \phi'' t'' \mid \phi' t' >_J}{< \phi'' t'' \mid \phi' t' >} \tag{74}$$

This expression can be written as a path integral over fields defined in Euclidean 4-space, a space obtained from the Minkowski space by the Wick rotation: a clockwise rotation of the real axis of the complex x^0- plane into the negative imaginary axis. This procedure is needed to make the oscillating path integrals from Eqs.(53) and (49) well defined.

We denote a point in the Euclidean 4-space by x_E. It is related to the point $x \equiv (x^0, \vec{x})$ of the Minkowski space by:

$$\begin{aligned} x_E &\equiv (\vec{x}, x_4) \text{ with } x_4 = ix^0 \text{ real} \\ d^4x &= -id^4x_E \\ x_E^2 &= x_1^2 + x_2^2 + x_3^2 + x_4^2 = -x^2 \end{aligned} \tag{75}$$

The corresponding Euclidian momentum space is defined so that $k_4 x_4 = k_0 x^0$. This convention assures that the propagation of a plane wave in the positive sense of x_4 corresponds to the positive sense of x^0. This convention implies that we have to rotate the k_0 axis counter-clockwise into the positive imaginary axis, as indicated in Fig.1.

We have

$$\begin{aligned} k_E &\equiv (\vec{k}, k_4) \text{ with } k_4 = -ik_0 \text{ real} \\ d^4k &= id^4k_E \\ k_E^2 &= k_1^2 + k_2^2 + k_3^2 + k_4^2 = -k^2 \end{aligned} \tag{76}$$

Note that $k \cdot x = k^0 x^0 - \vec{k}\vec{x}$ goes into $k_4 x_4 - \vec{k}\vec{x}$ but in taking the Fourier transform of a function of k^2 we may replace $k \cdot x$ by $k_E \cdot x_E = k_4 x_4 + \vec{k}\vec{x}$.

Continuation to Euclidean space means that we consider the dynamical evolution of the system in imaginary time, i. e. we must solve the equation of motion in which the time, x^0, is replaced by $-ix_4$ with x_4 a real parameter.

A Lorentz-invariant real scalar field, $\phi(x)$, turns into a real scalar field, $\phi(x_E)$, invariant under the rotation group in four dimensions, $O(4)$.

A massive vector field, $A^\mu(x)$, with real components is replaced by a Euclidean vector field, $A^\mu(x_E)$, with real components according to the rule:

$$
\begin{aligned}
A^j(x) &\rightarrow A^j(x_E) \quad j = 1, 2, 3 \\
A^0(x) &\rightarrow iA_4(x_E)
\end{aligned}
$$

Note that A^0 continues to A_4 with sign opposite to that of x^0 because it should transform like $\frac{\partial}{\partial x_0}$. The Lorentz condition $\partial_\mu A^\mu = 0$ is replaced by

$$\nabla\vec{A} + \frac{\partial A_4}{\partial x_4} = 0 \tag{77}$$

For Euclidean vectors there is no distinction between upper an lower indices.

Rewriting Eqs.(49) and (53) in Euclidean space and introducing the results in (74) yields the functional-integral expression for the vacuum-vacuum amplitude

$$e^{\frac{i}{\hbar}W[J]} = \frac{\int(D\phi)e^{-\frac{1}{\hbar}[S_E[\phi]+(J,\phi)_E]}}{\int(D\phi)e^{-\frac{1}{\hbar}S_E[\phi]}}, \tag{78}$$

where the denominator is a constant which eventually can be absorbed in the integration measure from the numerator. Equation (78) involves the Euclidean action functional

$$S_E[\phi] \equiv \int d^4x_E \mathcal{L}(x_E) = -iS[\phi] \tag{79}$$

and the coupling

$$(J,\phi)_E \equiv \int d^4x_E J(x_E)\phi(x_E) = -i(J,\phi). \tag{80}$$

In the Feynman-graph expansion of the vacuum-vacuum amplitude the Euclidean prescription merely supplies the correct $i\epsilon$ in the propagator. This point will be dealt with later in the case of the real scalar field.

An equivalent Minkowski-space path integral expression of the generating functional is[9]:

$$e^{\frac{i}{\hbar}W[J]} = \frac{\int(D\phi)e^{\frac{i}{\hbar}[S^\epsilon[\phi]+(J,\phi)]}}{\int(D\phi)e^{\frac{i}{\hbar}S^\epsilon[\phi]}} \tag{81}$$

where

$$S^\epsilon[\phi] \equiv \int d^4x[\mathcal{L}(\phi,\partial^\mu\phi) + \frac{i}{2}\epsilon\phi^2] \tag{82}$$

is the action supplemented with an extra convergence factor which renders the oscillating path integral and also provides the correct $i\epsilon$ in the Feynman propagator.

3 Functional Integration

The functional- (or path-) integral[10] representation for the generating functional of a quantum field theory has many virtues. First of all, it makes especially easy to see how the theory changes if we make non-linear transformations on its dynamical variables. It also permits to easily introduce auxiliary fields, passing in this way to a new theory with the same dynamical properties.

The first part of this section will be devoted to calculating Gaussian path integrals[11]. They can be performed exactly and are important because they can be used also in approximation schemes when the exact path integral is intractable. In our approach we shall use the analogy with ordinary integration without any attempt at mathematical rigour.

Let us start with the ordinary Gaussian integral

$$\int dx e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} \tag{83}$$

[9] See [7],p.210

[10] For the definition of path integrals and their use in quantum mechanics see the classical book of Feynman and Hibbs [1]

[11] We closely follow S.Coleman's lectures as represented in [2], cap.5

where a is a positive real number. By analytic continuation the formula is also true for complex a, provided it has a positive real part. The above equation can readily be generalized to n-dimensional space. The inner product of two vectors x and y of this space will be denoted by (x, y).

For a real symmetric positive-definite (and thus nonsingular) matrix, A, there holds:

$$\int d^n x e^{-\frac{1}{2}(x,Ax)} = (2\pi)^{\frac{n}{2}}(detA)^{-\frac{1}{2}} \tag{84}$$

as can be seen by diagonalizing A. As before, Eq. (84) holds also for a complex symmetric matrix with positive-definite real part, by analytic continuation. Denoting

$$(2\pi)^{-\frac{n}{2}} d^n x \equiv (dx) \tag{85}$$

we have

$$\int (dx) e^{-\frac{1}{2}(x,Ax)} = (\det A)^{-\frac{1}{2}} \tag{86}$$

We can also integrate exponentials of general quadratic forms

$$Q(x) = \frac{1}{2}(x, Ax) + (b, x) + c \tag{87}$$

where b is some vector and c is a number. Let $\tilde{x}$ be the location of its minimum

$$\tilde{x} = -A^{-1}b \tag{88}$$

Then $Q(x)$ can be written in canonical form

$$Q(x) = Q(\tilde{x}) + \frac{1}{2}(x - \tilde{x}, A(x - \tilde{x})\,) \tag{89}$$

with

$$Q(\tilde{x}) = -\frac{1}{2}(b, A^{-1}b) + c \tag{90}$$

Whence,

$$\int (dx) e^{-Q(x)} = e^{-Q(\bar{x})} (\det A)^{-\frac{1}{2}} \tag{91}$$

Using Eq. (91) we can do the integral of any analytic function, f, of the n coordinates x_i times the exponential of a quadratic form, just by differentiating with respect to b_i

$$\int (dx) f(x_i) e^{-Q(x)} = f(-\frac{\partial}{\partial b_i}) \int (dx) e^{-Q(x)} \tag{92}$$

We will need later also formulae for integrating over an n-dimensional complex space, not in any contour-integral sense, but merely in the sense of integrating separately over imaginary and real parts. A vector of this space is written as $z \equiv (x_1 + iy_1, \ldots, x_n + iy_n)$, and the usual Hermitian inner product of two such complex vectors, z and w, is denoted by (z^*, w). If A is a positive-definite Hermitian matrix

$$\int (dz^*)(dz)\, e^{-(z,Az)} = (\det A)^{-1}, \tag{93}$$

as can easily be seen by diagonalizing A. The integration measure is a symbol for the $2n$-dimensional real integration with respect to the measure

$$(dz^*)(dz) \equiv \pi^{-n} d^n x d^n y \qquad x_i, y_i \quad \text{real} \tag{94}$$

The change in the power of the determinant stems from the fact, that each eigenvalue contributes twice to the integral, once from the integration over the real part, and once from the integration over the imaginary part. The missing factors of 2 in Eq. (94) are related to the missing 1/2 in the exponent.

A general quadratic form may be written as

$$Q(z, z^*) = (z^*, Az) + (b^*, z) + (z^*, b) + c \tag{95}$$

with b -some complex vector and c -a number. It is minimal for

$$\tilde{z} = -A^{-1} b \tag{96}$$

with the value

$$Q(\tilde{z}, \tilde{z}^*) = -(b^*, A^{-1} b) + c \tag{97}$$

Thus

$$Q(z,z^*) = Q(\tilde{z},\tilde{z}^*) + (\,(z-\tilde{z})^*, A(z-\tilde{z})\,) \tag{98}$$

which yields the complex analogue of Eq. (91)

$$\int (dz^*)(dz)e^{-Q(z,z^*)} = e^{-Q(\tilde{z},\tilde{z}^*)}(\det A)^{-1} \tag{99}$$

We remark that the above integration formulae hold for any (finite) dimension of the vector space, therefore we extend them to infinite-dimensional vector spaces. To point out the mathematical linchpin of this step we describe it using Eq. (86) as an example. Given a quadratic form (x, Ax), defined by a linear operator, A, on a real Hilbert space, we first restrict the form to some finite-dimensional subspace. On this subspace both the integral and the determinant are well-defined. Then we consider an increasing sequence of such finite-dimensional subspaces such that their union is dense in the Hilbert space. The limit of this sequence will be the whole space.

This limit defines both the functional determinant and the functional integral. From the mathematical point of view the key problem is to find out if, for a given operator A, the limit exists and is independent of the chosen sequence of subspaces.

We will simply generalize our finite dimensional formulae assuming that the involved expressions are well defined[12]. The infinite-dimensional spaces we will be most concerned with will be spaces of functions, for example, the space of functions, $\phi(x)$, of space-time points.

The inner product is defined by

$$(\phi, \phi') \equiv \int d^4x \phi(x)\phi'(x) \tag{100}$$

and the analogue of Eq. (86) is:

$$\int (D\phi)e^{-\frac{1}{2}(\phi,A\phi)} = (\mathrm{Det}A)^{-\frac{1}{2}}. \tag{101}$$

The measure in the above formula is that of a path-integral, and Det$A$ denotes the functional determinant of the linear operator A. The evaluation of such determinants is a problem we will refer to later.

[12]For a good mathematical reference see [3]

For complex fields we have

$$\int (D\phi^*)(D\phi)e^{-(\phi^*,A\phi)} = (\mathrm{Det}A)^{-1} \tag{102}$$

From these formulae equations analogous to those derived before follow directly. We write out explicitely just one of them

$$\int (D\phi)f(\phi(x))e^{-Q[\phi]} = f\left(-\frac{\delta}{\delta b(x)}\right)\int (D\phi)e^{-Q[\phi]} \tag{103}$$

which tells us how to obtain the others. Note that ordinary derivatives are also replaced by functional ones, because the quadratic form itself is a functional of ϕ and b.

The operators we will be mostly interested in are differential operators, some polynomials, $P(\partial_\mu)$, in space-time derivatives. The continuous matrices associated to them are diagonal [5].

$$A(x,y) = \delta^4(x-y)P(\frac{\partial}{\partial x^\mu}) \tag{104}$$

Their action is defined by

$$(A\phi)(x) \equiv \int d^4y A(x,y)\phi(y) = P(\frac{\partial}{\partial x^\mu})\phi(x) \tag{105}$$

We assume that they are invertible, i.e. that there exists $A^{-1}(x,y)$ so that

$$\int d^4z A(x,z)A^{-1}(z,y) = \delta^4(x-y) \tag{106}$$

For the special case of operators like (104) the inverse is a function satisfying the equation

$$P(\frac{\partial}{\partial x^\mu})A^{-1}(x,y) = \delta^4(x-y) \tag{107}$$

A momentum-space representation of the operator can be obtained by Fourier- transforming (twice) the function, $\phi(y)$, it acts on. This gives:

$$A(x,y) = \int \frac{d^4k}{(2\pi)^4}P(-ik_\mu)e^{-ik(x-y)} \tag{108}$$

and

$$A^{-1}(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{P(-ik_\mu)} \tag{109}$$

The functional determinant is defined by the product of the eigenvalues. We remind that A is already diagonal with elements given by Eq.(108). Hence,

$$\mathrm{Det} A \equiv \prod_x A(x,x) = \prod_x \int \frac{d^4k}{(2\pi)^4} P(-ik_\mu) \tag{110}$$

It seems to be a divergent quantity because $A(x,x)$ are strictly positive numbers for the operators we are interested in. (From the positive-definiteness follows, that all the principal minors are strictly positive). Actually, we don't know how to compute the product with respect to continuous index like x. This is the reason of the ambiguous formulation:"it seems to be".

A more practical formula for a functional determinant arises from the generalization of the identity

$$\ln \det A = \operatorname{tr} \ln A \tag{111}$$

valid for real, symmetric, positive-definite, finite-dimensional matrices. This can easily be verified by diagonalizing A.

Whence,

$$\mathrm{Det} A = e^{\mathrm{Tr} \ln A} \tag{112}$$

where Tr stands for a functional trace defined by

$$\mathrm{Tr} \ln A \equiv \int dx (\ln A)(x,x) = \int dx \ln(A(x,x)) = \int dx \ln \int \frac{d^4k}{(2\pi)^4} P(-ik_\mu) \tag{113}$$

This last form is a manifestly divergent quantity. In usual field-theoretical applications the determinant may be included in the overall normalization factor which drops out from expressions of physical interest.

However, functional determinants are not entirely useless. When we integrate over some fields which appear quadratically in the Lagrangian of a system, we are left with an effective action. [See e. g. [2, ?]] To one loop order, it is the logarithm of a functional determinant. One tries to find the effective lagrangian by bringing the effective action to the form of an integral over the space-time variables. (In our case this would amount in an evaluation of the momentum-space integral.) The problem in those computations is that P is not just a polynomial in derivatives, but it contains also x-dependent fields which does not commute with the derivatives. How to disentangle such functional traces is a relative recently solved problem [11].

4 The Feynman Propagator

We shall use the techniques discussed in the last section to evaluate the generating functional for a real scalar field whose dynamics is governed by the action

$$S[\phi] = \int d^4x \, [\mathcal{L}_0(\phi, \partial_\mu \phi) + \mathcal{L}_i(\phi)] \tag{114}$$

Here $\mathcal{L}_0$ is the Lagrangian density of the free field,

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \tag{115}$$

and the interaction Lagrangian, $\mathcal{L}_i$, is some polynomial of the c-number field ϕ, but independent of its derivatives.

The generating functional has a path-integral expression

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N} \int (D\phi) e^{\frac{i}{\hbar}\{S[\phi]+(J,\phi)\}} \tag{116}$$

in which the normalization constant, $\mathcal{N}$, is chosen so that $W[J]$ vanishes when the external source $J(x)$ vanishes.

In this form the integrand in Eq.(116) is oscillating, nothing like the nicely damped gaussians of the previous section. There are two ways out.

First, in the Minkowski-space formulation, we may introduce by hand a convergence factor:

$$\mathcal{L}_0 \longrightarrow \mathcal{L}_0 + \frac{1}{2} i\epsilon\phi^2 \tag{117}$$

which creates the positive-definite real part of the "matrix" that appears in the quadratic form from the exponent [13]. [See the comment after Eq.(84)].

The second approach to make the integral in Eq. (116) well-defined is to rotate the time axis with $\theta_{x_0} = -\frac{\pi}{2}$ and the energy-variable axis with $\theta_{k_0} = \frac{\pi}{2}$, obtaining thus the Euclidean-space formulation of the theory [14]. Then, we perform the (Gaussian) integral and rotate back to Minkowski space.

Remarkably enough, both of the above approaches specify the correct prescription for the Minkowski-space Feynman propagator. The well-definiteness of the path-integral dictates the correct Green's functions!

To see this, let us evaluate the generating functional for the noninteracting field. In the Minkowski-space formulation we have

$$e^{\frac{i}{\hbar}W_0[J]} = \mathcal{N}\int (D\phi) e^{\frac{i}{\hbar}\{S_0^{(\epsilon)}[\phi]+(J,\phi)\}} \tag{118}$$

where $S_0^{(\epsilon)}$ is the free action supplemented with the convergence factor

$$S_0^{(\epsilon)} \equiv \int d^4x \left[\mathcal{L}_0 + \frac{1}{2}i\epsilon\phi^2\right] = -\int d^4x \phi(\Box + m^2 - i\epsilon)\phi. \tag{119}$$

The last expression arises because

$$(\partial_\mu\phi)(\partial^\mu\phi) = \partial_\mu(\phi\partial^\mu\phi) - \phi(\Box\phi), \tag{120}$$

and the total-derivative term doesn't contribute to the action.

The integral obtained is just like Eq. (91):

$$e^{\frac{i}{\hbar}W_0[J]} = \mathcal{N}\int (D\phi) e^{-Q[\phi]} = \mathcal{N}(\mathrm{Det}A)^{-\frac{1}{2}} e^{-Q[\tilde{\phi}]}, \tag{121}$$

where the quadratic form is

$$Q[\phi] = \frac{1}{2}(\phi, A\phi) + (b,\phi); \qquad b(x) = -\frac{i}{\hbar}J(x). \tag{122}$$

and $\tilde{\phi}$ denotes the location of its minimum, i. e. that particular field for which the functional $Q[\phi]$ is minimal.

---

[13] See e. g. [6], p.74, and [7], p.218

[14] It is nicely described in [4],p.133 and [6], p.91

The involved operator

$$A(x,y) = \delta^4(x-y)\frac{i}{\hbar}(\Box + m^2 - i\epsilon) \tag{123}$$

has a positive-definite real part, $\epsilon\delta^4(x-y)/\hbar$, and its inverse is the solution of the equation

$$\frac{i}{\hbar}(\Box + m^2 - i\epsilon)A^{-1}(x,y) = \delta^4(x-y). \tag{124}$$

We denote

$$A^{-1}(x,y) \equiv i\hbar\Delta_F(x-y), \tag{125}$$

where Δ_F is the Feynman propagator. From the momentum space representation of A^{-1} [Eq. (109)]

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}. \tag{126}$$

We have also

$$Q[\tilde{\phi}] = -\frac{1}{2}(b, A^{-1}b) = \frac{1}{2\hbar^2}\int d^4x d^4y J(x)A^{-1}(x,y)J(y), \tag{127}$$

which gives the generating functional

$$e^{\frac{i}{\hbar}W_0[J]} = e^{-\frac{i}{\hbar}\frac{1}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}. \tag{128}$$

This is the standard expression involving the correct propagator. The normalization constant is just $(\mathrm{Det}A)^{-\frac{1}{2}}$. [See the comment after Eq. (116)]

In what follows we carry out the Wick rotation of the time axis, and perform the path integral over the Euclidian fields, after which we rotate back to Minkowski space.

$$S_0[\phi] = -\frac{1}{2}\int d^4x\phi(x)(\Box + m^2)\phi(x) = \frac{i}{2}\int d^4x_E\phi(x_E)(-\Box_E + m^2)\varphi(x_E) \tag{129}$$

where $\Box_E = \partial_4^2 + \nabla^2$ is the Euclidian-space d'Alembert operator. The inner product in terms of an Euclidean-space integral is written as

$$(J,\phi) \equiv \int d^4x J(x)\phi(x) = -i \int d^4x_E J(x_E)\phi(x_E). \tag{130}$$

Whence,

$$e^{\frac{i}{\hbar}W_0[J]} = \mathcal{N} \int (D\phi) e^{-Q[\phi]} \tag{131}$$

where

$$Q[\phi] = \frac{1}{2}(\phi, A_E\phi) + (b_E, \phi) \tag{132}$$

with

$$b_E(x) = -\frac{1}{\hbar} J(x_E) \tag{133}$$

and

$$A_E(x_E, y_E) = \delta^4(x_E - y_E)\frac{1}{\hbar}(-\Box_E + m^2). \tag{134}$$

Its inverse is

$$A_E^{-1}(x_E, y_E) = \hbar \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{-ik_E(x_E-y_E)}}{k_E^2 + m^2} \tag{135}$$

and, if we introduce the Euclidian propagator by

$$A_E(x_E, y_E) \equiv i\hbar\Delta_E(x_E - y_E), \tag{136}$$

we obtain

$$\Delta_E(x_E - y_E) = -i \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{-ik_E(x_E-y_E)}}{k_E^2 + m^2}. \tag{137}$$

The generating functional is found to be:

$$\begin{aligned} e^{iW_0[J]} &= \mathcal{N}(\mathrm{Det}A)^{-\frac{1}{2}} e^{\frac{1}{2}(b_E, A_E^{-1} b_E)} \\ &= e^{\frac{i}{2\hbar}\int d^4x_E d^4y_E J(x_E)\Delta_E(x_E-y_E)J(y_E)}. \end{aligned} \tag{138}$$

We now explain the way in which the Euclidian propagator (137) transforms into the Feynman propagator (126) while we perform the rotation back to the Minkowski space.

The direct rotation amounts to replacing the (initially real) energy variable by complex numbers like a real number times $e^{i\theta}$ with $\theta \in [0, \pi/2]$, and at the end of the rotation, $\theta_f = \pi/2$, the energy variable is pure imaginary:ik_4. The time variable suffers also a similar rotation, but of opposite sign: $x_0 \rightarrow -ix_4$. The variables k_4 and x_4 are real numbers.

At the beginning of the direct rotation (and at the end of the inverse one) the energy variable is $k_0 e^{i\theta} \approx k_0(1 + i\theta)$ with θ–a small *positive* angle. This means that we find the propagator by the end of the rotation backwards to the Minkowski space if we replace in Δ_E the Euclidean energy variable ik_4 by $k_0(1 + i\theta)$ i. e.

$$k_4 \longrightarrow -ik_0 + k_0\theta, \qquad \theta = 0_+. \tag{139}$$

This amounts to

$$k_E^2 + m^2 \longrightarrow -k_0^2 + \vec{k}^2 + m^2 - i\epsilon \tag{140}$$

with $\epsilon = 2k_0^2\theta$ a small positive number. This implies that

$$\Delta_E(x_E - y_E) \longrightarrow \Delta_F(x - y). \tag{141}$$

We conclude that the Euclidean-space formulation yields the contour of the k_0-integration in the usual manner. Feynman's $i\epsilon$-prescription arises when we rotate back to Minkowski space. (But without completing the rotation $\theta_{back} = \frac{\pi}{2} - \frac{\epsilon}{2k_0^2}$).

Under the inverse rotation the right hand side of Eq.(138) takes the usual form, and we find again the expression (128) for the generating functional of the free field Green's functions.

Thus, we have pointed out a beautiful feature of the path-integral formalism of a quantum field theory: once we have made the path-integral formula of the generating functional well-defined, we obtain the correct causal Green's functions from it.

5 Feynman rules

The generating functional (116) for a real scalar field in the presence of interaction

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N}\int(D\phi)e^{\frac{i}{\hbar}[\int\mathcal{L}_i(\phi(x))d^4x+S_0[\phi]+(J,\phi)]} \tag{142}$$

may be rewritten using the property

$$\frac{\hbar}{i}\frac{\delta}{\delta J(x_1)}e^{\frac{i}{\hbar}(J,\phi)} = \phi(x_1)e^{\frac{i}{\hbar}(J,\phi)} \tag{143}$$

which tells us that

$$e^{\frac{i}{\hbar}\int d^4x\mathcal{L}_i(\phi(x))}e^{\frac{i}{\hbar}(J,\phi)} = e^{\frac{i}{\hbar}\int d^4x\mathcal{L}_i(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})}e^{\frac{i}{\hbar}(J,\phi)} \tag{144}$$

The first factor on the right hand side is independent of the field, whence we pull it out from the path integral. This gives

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N}e^{\frac{i}{\hbar}\int d^4x\mathcal{L}_i(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})}e^{\frac{i}{\hbar}W_0[J]} \tag{145}$$

with the free-field generating functional, $\exp(W_0[J])$, given by Eq.(128).

Nevertheless, it is impossible to obtain an explicit closed formula from the above expression. It remains to look for a perturbative expansion in powers of the coupling constant involved by the interaction term, $\mathcal{L}_i$. Such an evaluation gives the ordinary Feynman rules. To see this we first "prove" a functional identity by verifying it in the finite-dimensional case. Let $F(\vec{u})$ and $G(\vec{u})$ be any two c-number functions on a vector space; then

$$F(\frac{1}{\alpha}\frac{\partial}{\partial\vec{u}})G(\vec{u}) = G(\frac{1}{\alpha}\frac{\partial}{\partial\vec{v}})F(\vec{v})\ e^{\alpha(\vec{u},\vec{v})}\Big|_{\vec{v}=0} \tag{146}$$

with $\vec{u},\vec{v}$ some vectors and α an arbitrary complex number. This is most easily proved by Fourier analysis, that is to say, by taking

$$F(\vec{u}) = e^{\alpha(\vec{a},\vec{u})} \qquad \text{and} \qquad G(\vec{u}) = e^{\alpha(\vec{b},\vec{u})}$$

with $\vec{a}$ and $\vec{b}$ fixed vectors. Then the left hand side of Eq.(146) becomes

$$e^{\alpha(\vec{a},\frac{1}{\alpha}\frac{\partial}{\partial\vec{u}})}e^{\alpha(\vec{b},\vec{u})} = e^{\alpha(\vec{a}+\vec{u},\vec{b})},$$

while on the right hand side we have

$$e^{\alpha(\vec{b},\frac{1}{\alpha}\frac{\partial}{\partial\vec{v}})}e^{\alpha(\vec{a},\vec{u})}e^{\alpha(\vec{u},\vec{v})}\Big|_{\vec{v}=0} = e^{\alpha(\vec{b}+\vec{v},\vec{u}+\vec{a})}\Big|_{\vec{v}=0},$$

which is the same.

We conclude, that in a function space holds the identity

$$F\left[\frac{1}{\alpha}\frac{\delta}{\delta J}\right]G[J] = G\left[\frac{1}{\alpha}\frac{\delta}{\delta\phi}\right]F[\phi]\,e^{\alpha(J,\phi)}|_{\phi(x)=0} \tag{147}$$

for any two functionals F and G.

Let us apply this to our case: [see Eqs.(145) and (128), and put $\alpha = \frac{i}{\hbar}$]

$$Z[J] \equiv e^{\frac{i}{\hbar}W[J]} = \mathcal{N}\exp\left[\frac{i}{\hbar}\int d^4x\mathcal{L}_i(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})\right]e^{-\frac{i}{\hbar}\frac{1}{2}\int d^4zd^4z'J(z)\Delta_F(z-z')J(z')} =$$

$$= \mathcal{N}\exp\left[\frac{1}{2}\int d^4zd^4z'i\hbar\Delta_F(z-z')\frac{\delta}{\delta\phi(z)}\frac{\delta}{\delta\phi(z')}\right]e^{\frac{i}{\hbar}\int d^4x[\mathcal{L}_i(\phi(x))+J(x)\phi(x)]}\Big|_{\phi=0} \tag{148}$$

This equation yields the Feynman rules for the vacuum-to-vacuum matrix element, $< 0^+ \mid 0^- >_J \equiv Z[J]$. To show this we expand the last exponential as

$$e^{\frac{i}{\hbar}\int d^4x[\mathcal{L}_i(\phi(x))+J(x)\phi(x)]} = \sum_{p=0}^{\infty}\frac{1}{p!}\left[\frac{i}{\hbar}\int d^4y\mathcal{L}_i(\phi(y))\right]^p\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{i}{\hbar}\int d^4xJ(x)\phi(x)\right]^n \tag{149}$$

and insert this into (148). We are left with the expansion of the generating functional in powers of the interaction

$$\begin{aligned}
Z[J] &= \mathcal{N}\sum_{n,p=0}^{\infty}\frac{1}{n!p!}\left(\frac{i}{\hbar}\right)^{n+p}\exp\left[\frac{1}{2}\int dzdz'[i\hbar\Delta_F(z-z')]\frac{\delta}{\delta\phi(z)}\frac{\delta}{\delta\phi(z')}\right] \\
&\quad \int dx_1\cdots dx_ndy_1\cdots dy_p\mathcal{L}_i(\phi(y_1))\ldots\mathcal{L}_i(\phi(y_p)) \\
&\quad \phi(x_1)\cdots\phi(x_n)J(x_1)\cdots J(x_n)|_{\phi=0} \equiv \sum_{n,p=0}^{\infty}Z^{(n,p)}[J]
\end{aligned} \tag{150}$$

where we have introduced a notation for the various terms appearing in the power series.

Let us recall that according to Eq.(66) the n-point Green's function is obtained from $Z[J]$ by n times differentiating with respect to the source,J, after which J is set to zero. Only terms which contain n J's survive both the operations. Hence, the n-point Green's function is generated by the terms $Z^{(n,p)}, p = 1, 2, \ldots$. A term with a given p yields the p-th order contributions in powers of the coupling constant which enters $\mathcal{L}_i$. While evaluating $Z^{(n,p)}$ one has to keep in mind the fact, that , in the expansion of the exponential in Eq.(150), only terms which contain an equal number of ϕ's and $\frac{\delta}{\delta\phi}$'s survive.

Let us next analyze a few simple examples in the case of the ϕ^4 interaction whith coupling constant g :

$$\mathcal{L}_i = -\frac{g}{4!}\phi^4 \tag{151}$$

The contribution to the two-point Green's function, to zeroth order in the coupling constant stems from

$$\begin{aligned} Z^{(2,0)} &= \frac{\mathcal{N}}{2}\left(\frac{i}{\hbar}\right)^2 \left[\frac{1}{2}\int dz dz' [i\hbar\Delta_F(z-z')] \frac{\delta}{\delta\phi(z)} \frac{\delta}{\delta\phi(z')}\right] \\ &\quad \int dx_1 dx_2 \phi(x_1)\phi(x_2) J(x_1) J(x_2) \\ &= \frac{\mathcal{N}}{2}\left(\frac{i}{\hbar}\right)^2 \int dx_1 dx_2 J(x_1)[i\hbar\Delta_F(x_1-x_2)] J(x_2) \end{aligned} \tag{152}$$

For the sake of bookkeeping we may associate to this term the diagram:

which consists of:

- external points

 $\frac{i}{\hbar}J(x_1)$

- a line connecting them

$i\hbar\Delta_F(x_1 - x_2)$

The two-point Green's function to this order ($p = 0$) is:

$$G(x_1, x_2) = \left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z^{(2,0)}[J]}{\delta J(x_1)\delta J(x_2)} = i\hbar\Delta_F(x_1 - x_2) \tag{153}$$

We next discuss the case $n = 0, p = 1$, i.e., the number of ϕ's in Eq.(150) is four. Thus

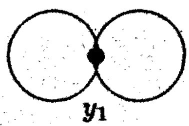
$$Z^{(0,1)}[J] = \mathcal{N}\frac{i}{\hbar}\left(-\frac{g}{4!}\right)\frac{1}{2!}\left[\frac{1}{2}\int dzdz'[i\hbar\Delta_F(z - z')]\frac{\delta}{\delta\phi(z)}\frac{\delta}{\delta\phi(z')}\right]^2 \int dy_1\phi^4(y_1) =$$

$$= \mathcal{N}\frac{-ig}{\hbar}\frac{1}{8\cdot 4!}\int dz_1dz_1'dz_2dz_2' i\hbar\Delta_F(z_1 - z_1')i\hbar\Delta_F(z_2 - z_2')$$

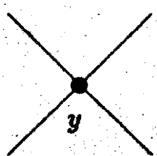
$$\int dy_1 4!\delta(z_1 - y_1)\delta(z_1' - y_1)\delta(z_2 - y_1)\delta(z_2' - y_1) =$$

$$= \mathcal{N}\frac{-ig}{8\hbar}\int dy_1[i\hbar\Delta_F(y_1 - y_1)]^2 = \mathcal{N}\frac{1}{8}\left(-\frac{i}{\hbar}g\right)\int dy_1[i\hbar\Delta_F(0)]^2 \tag{154}$$

The corresponding diagram is



which suggests that a vertex, (an interaction point), is associated to



$-\frac{i}{\hbar}g$

The number $\frac{1}{8}$ from Eq.(154) is called the symmetry factor of the corresponding graph.

Now we have all the ingredients of Feynman graphs of the theory we are dealing with: external points, vertex points with four lines coming out of each (in the case of ϕ^4 interaction), and lines connecting them associated to $i\hbar$ times the Feynman propagator. A line which ends in at least one external point is called an *external line*, while a line ending in vertex points only, is referred to as an *internal line*.

Explicit calculations may be synthesized in the following Feynman rules for the vacuum-to-vacuum amplitude:
In order to obtain $Z^{(n,p)}$, (n-even)[15]:

1. Draw all the topologically distinct diagrams with n external lines (ending in x-points) and p vertices (y-points), and sum all the contributions according to the following:

2. To each y-vertex attach a factor $-\frac{i}{\hbar}g$

3. To each external point, x, assign $\frac{i}{\hbar}J(x)$

4. To each line between two points (external or vertex), say z_1 and z_2, attach $i\hbar\Delta_F(z_1 - z_2)$

5. Multiply the contribution by the symmetry factor, $\frac{1}{S}$, of the graph. Its inverse, S, is obtained by multiplying the factors below:

    a) $m!$ if the graph is symmetric with respect to the interchange of m vertices or m external points

    b) $m!$ for m equivalent internal lines

    c) 2 for each closed line

    d) $m!$ if the (disconnected) diagram contains m identical connected diagrams

6. Integrate over x's and y's.

[15]Indeed, the terms with n-odd vanish because the number of $\frac{\delta}{\delta\phi}$'s is always even, and the number of ϕ's is $4p + n$-odd if n is odd.

The Feynman rules for the n-point Green's functions can be obtained by comparing the result of the above rules with the functional expansion of the generating functional, Eq.(65):

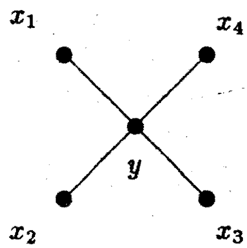
$$Z[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar}\right)^n \int dx_1 \cdots dx_n J(x_1) \cdots J(x_n) G(x_1, \ldots, x_n) \qquad (155)$$

In the remainder of this section we present the Feynman rules in the momentum space. To this end we start with the definition of the Fourier transform of a Green's function:

$$G(k_1, \ldots, k_n) = \int d^4x_1 \cdots d^4x_n e^{-i\sum_{j=1}^{n} k_j \cdot x_j} G(x_1, \ldots, x_n) \qquad (156)$$

The Feynman rules for $G(k_1, \ldots, k_n)$ are obtained by using the integral representation (126) of the Feynman propagators involved in $G(x_1, \ldots, x_n)$. We denote by p the external momenta, and by k the internal ones.

Let us study a simple example. Take the graph:



Its contribution to the four point Green's function, $G(x_1, x_2, x_3, x_4)$, reads

$$-\frac{i}{\hbar} g \int dy (i\hbar)^4 \Delta_F(x_1 - y) \Delta_F(x_2 - y) \Delta_F(x_3 - y) \Delta_F(x_4 - y) \qquad (157)$$

Using the momentum-space integral representation of Δ_F, the y-integration yields a δ-function which enforces momentum conservation. The Fourier transform of expression (157) is given by

$$(-\frac{i}{\hbar} g) \delta^4(p_1 + p_2 + p_3 + p_4) \prod_{j=1}^{4} \frac{i\hbar}{p_j^2 - m^2 + i\epsilon} \qquad (158)$$

The above calculation illustrates the way one derives the momentum-space Feynman rules from the coordinate-space rules. As a result of such computatios we find, that in order to obtain $G(p_1, \ldots, p_n)$, (n-even), one has to:

1. Draw all the topologically distinct diagrams with n external lines. Each line carries a momentum. Denote them by p's for external, and by k's for internal lines.
2. Assign the factor
$$\frac{i\hbar}{p^2 - m^2 + i\epsilon}$$
for an external line.
3. Assign the factor
$$\frac{d^4k}{(2\pi)^4} \frac{i\hbar}{k^2 - m^2 + i\epsilon}$$
for an internal line.
4. For each vertex assign $(-\frac{i}{\hbar})(2\pi)^4\delta^4(q)$, where q denotes the sum of the incoming momenta.
5. Multiply the contribution by the symmetry factor of the diagram. It is obtained as described above in the coordinate-space rules.
6. Integrate over the internal momenta (k's).
7. Sum the contributions of all the topologically distinct Feynman diagrams.

Adopting the point of view that a field theory is defined by its Feynman rules, we conclude, that the path integral method is an elegant way to define the quantum theory of a self-interacting scalar field.

Scalar electrodynamics requires the quantization of gauge fields, in particular the electromagnetic field, by similar methods. While it is just an alternative way to quantize electromagnetism, in the case of non-abelian gauge fields, the only generally accepted quantization procedure is the path integral method due to Fadeev and Popov.

In order to treat quantum electrodynamics, one must include Fermi fields. This is done by the use of anticommuting (Grassmann) variables. The components of the Dirac field will be Grassmann-valued space-time dependent functions, and the generating functional, a path integral over such variables. These topics will constitute the subject of a forthcoming volume on path integrals.

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