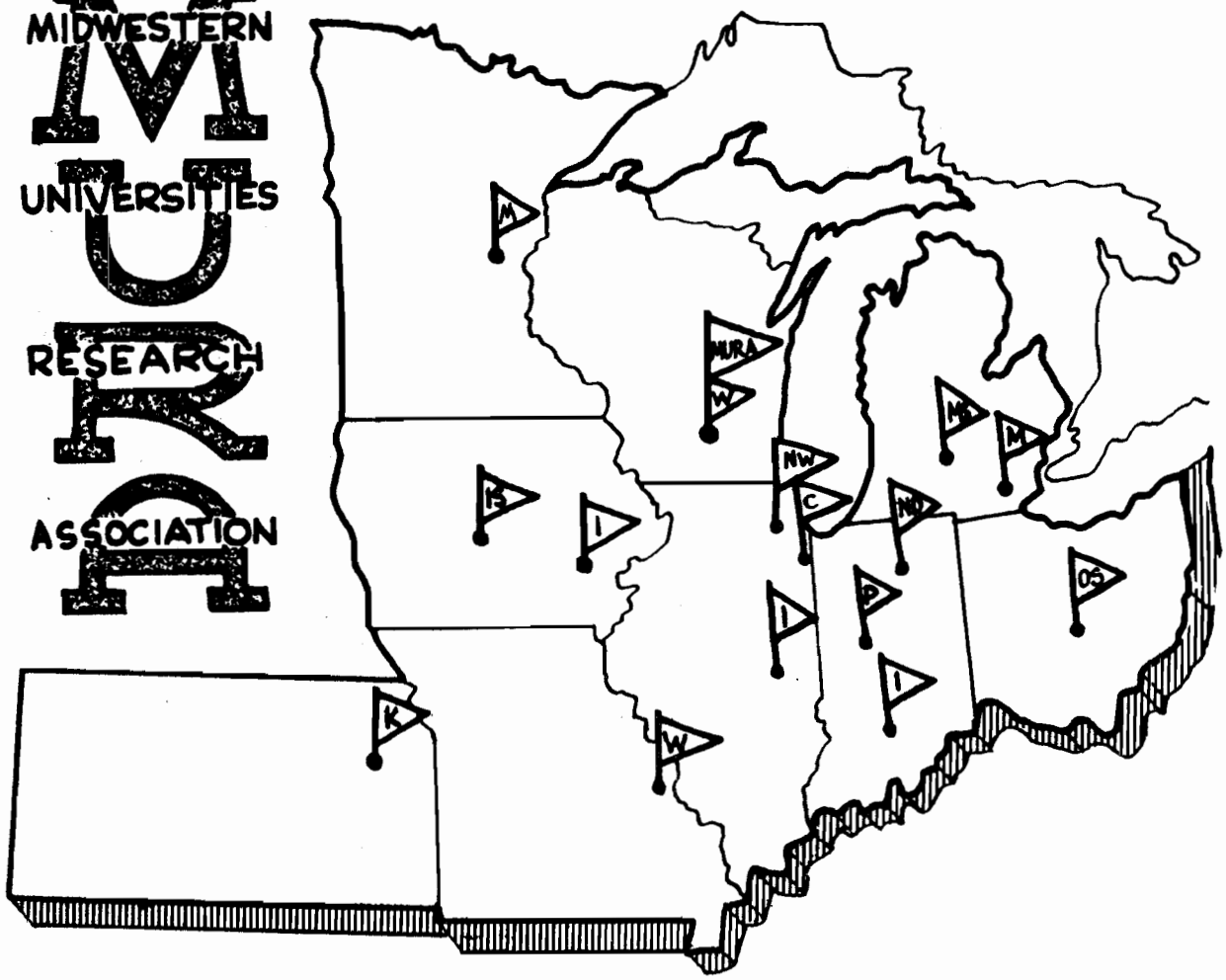
MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



ALGORISMS FOR RELAXATION CALCULATION OF FFAG MAGNETOSTATIC FIELDS

S. C. Snowdon

REPORT NUMBER 706

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Printed in USA. Price $1.75. Available from the Office of Technical Services, Department of Commerce, Washington 25, D. C.

MURA-706
UC-28: Particle Accelerators and
High Voltage Machines
TID-4500 (37th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

P. O. Box 6, Stoughton, Wisconsin

ALGORISMS FOR RELAXATION CALCULATION OF FFAG MAGNETOSTATIC FIELDS

S. C. Snowdon

February 10, 1965

ABSTRACT

A method of developing algorisms for a general second-order differential equation of the elliptic type is given. Application is made to the development of algorisms for an FFAG magnetostatic guide field. Algorism modifications are found for points near a two-dimensional edge of ideal iron that removes the analytical difficulty occasioned by the presence of infinite fields on the edge.

*AEC Research and Development Report. Supported by the U. S. Atomic Energy Commission through ANL by Subcontract 31-109-38-1707.

INTRODUCTION

This report contains a general description of the method of constructing algorisms suitable for the iterative calculation of a general three-dimensional Poisson equation. An application is made in finding the algorisms for a three-dimensional magnetostatic field in a coordinate system suitable for an FFAG guide field. The results for the FFAG field are available in various MURA reports but the general development is believed to be new.

II. ALGORISM FORMATION

The general problem of finding a suitable algorism to be used in the solution of a second-order partial differential equation by an iterative or relaxation process may be stated as follows. Suppose it is desired to find a solution of the linear partial differential equation

$$\mathcal{F}(U) = G, \tag{1}$$

where

$$\mathcal{F}(U) = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{3} B_i \frac{\partial U}{\partial x_i} + C\,U \,. \tag{2}$$

The functions A_{ij}, B_i, C, and G are given functions of position and A_{ij} is assumed to be symmetrical ($A_{ij} = A_{ji}$). In general a mesh is set up such that

$$\Delta x_i = h_i I_i \,, \tag{3}$$

where I_i are pure integer numbers. An algorism will have been found if it is possible to find a set of weights (ω) and a linear operator $\mathcal{O}$ such that

$$\sum \omega_{I_1 I_2 I_3} U_{I_1 I_2 I_3} = \mathcal{O}\mathcal{F}(U) \,. \tag{4}$$

In particular we shall restrict the weights to the neighboring points surrounding the standard point (0, 0, 0). Thus, in Eq. (4) I_1, I_2 and I_3 may

have the values ± 1, 0. In Eq. (3), however, since X_i may occur in the given coefficients, I_i may have any integer value.

A Taylor series expansion of $U_{I_1 I_2 I_3}$ about the standard point (0, 0, 0) is carried out to the desired degree of approximation

$$U_{I_1 I_2 I_3} = U + \sum_{i=1}^{3} h_i I_i \frac{\partial U}{\partial x_i} + \frac{1}{2!} \sum_{i=1}^{3} \sum_{j=1}^{3} h_i h_j I_i I_j \frac{\partial^2 U}{\partial x_i \partial x_j}$$

$$+ \frac{1}{3!} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} h_i h_j h_k I_i I_j I_k \frac{\partial^3 U}{\partial x_i \partial x_j \partial x_k}$$

$$+ \frac{1}{4!} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} h_i h_j h_k h_l I_i I_j I_k I_l \frac{\partial^4 U}{\partial x_i \partial x_j \partial x_k \partial x_l} + \cdots \tag{5}$$

It is understood that U, $\frac{\partial U}{\partial x_i}$, etc. are to be calculated at the standard point (0, 0, 0).

Next, an operator $\mathcal{O}$ is formed as follows

$$\mathcal{O} = \lambda + \sum_{i=1}^{3} \lambda_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \cdots \tag{6}$$

where λ_{ij} is symmetrical ($\lambda_{ij} = \lambda_{ji}$). From Eqs. (2) and (6), it is found that

$$\mathcal{O}\mathcal{F}(U) = \left[\lambda C + \sum_i \lambda_i \frac{\partial C}{\partial x_i} + \sum_{ij} \lambda_{ij} \frac{\partial^2 C}{\partial x_i \partial x_j}\right] U$$

$$+ \sum_i \left\{ \lambda B_i + \lambda_i C + \sum_j \left(\lambda_j \frac{\partial B_i}{\partial x_j} + 2 \lambda_{ij} \frac{\partial C}{\partial x_j} \right) + \sum_{jk} \lambda_{jk} \frac{\partial^2 B_i}{\partial x_j \partial x_k} \right\} \frac{\partial U}{\partial x_i}$$

$$+ \sum_{ij} \left\{ \lambda A_{ij} + \lambda_i B_j + \lambda_{ij} C + \sum_k \left(\lambda_k \frac{\partial A_{ij}}{\partial x_k} + 2 \lambda_{ik} \frac{\partial B_j}{\partial x_k} \right) + \sum_{kl} \lambda_{kl} \frac{\partial^2 A_{ij}}{\partial x_k \partial x_l} \right\} \frac{\partial^2 U}{\partial x_i \partial x_j}$$

$$+ \sum_{ijk} \left(\lambda_i A_{jk} + \lambda_{ij} B_k + 2 \sum_l \lambda_{il} \frac{\partial A_{jk}}{\partial x_l} \right) \frac{\partial^3 U}{\partial x_i \partial x_j \partial x_k}$$

$$+ \sum_{ijkl} \lambda_{ij} A_{kl} \frac{\partial^4 U}{\partial x_i \partial x_j \partial x_k \partial x_l} + \cdots \tag{7}$$

Multiplication of Eq. (5) by $\omega_{I_1 I_2 I_3}$, summing over I_1, I_2, I_3, and using Eq. (4) yields the following equations for the moments of $\omega_{I_1 I_2 I_3}$.

$$\sum \omega_{I_1 I_2 I_3} = \lambda C + \sum_i \lambda_i \frac{\partial C}{\partial x_i} + \sum_{ij} \lambda_{ij} \frac{\partial^2 C}{\partial x_i \partial x_j} , \tag{8}$$

$$h_i \sum I_i \, \omega_{I_1 I_2 I_3} = \lambda B_i + \lambda_i C + \sum_j \left(\lambda_j \frac{\partial B_i}{\partial x_j} + 2 \lambda_{ij} \frac{\partial C}{\partial x_j} \right) + \sum_{jk} \lambda_{jk} \frac{\partial^2 B_i}{\partial x_j \partial x_k} , \tag{9}$$

$$\begin{aligned} \mathcal{P} \frac{1}{2!} h_i h_j \sum I_i I_j \, \omega_{I_1 I_2 I_3} = \mathcal{P} \Big\{ \lambda A_{ij} + \lambda_i B_j + \lambda_{ij} C + \sum_k \left(\lambda_k \frac{\partial^2 A_{ij}}{\partial x_k} + 2 \lambda_{ij} \frac{\partial B_i}{\partial x_k} \right) \\ + \sum_{kl} \lambda_{kl} \frac{\partial^2 A_{ij}}{\partial x_k \partial x_l} \Big\} , \end{aligned} \tag{10}$$

$$\mathcal{P} \frac{1}{3!} h_i h_j h_k \sum I_i I_j I_k \, \omega_{I_1 I_2 I_3} = \mathcal{P} \left\{ \lambda_i A_{jk} + \lambda_{ij} B_k + 2 \sum_l \lambda_{il} \frac{\partial A_{jk}}{\partial x_l} \right\} , \tag{11}$$

$$\mathcal{P} \frac{1}{4!} h_i h_j h_k h_l \sum I_i I_j I_k I_l \, \omega_{I_1 I_2 I_3} = \mathcal{P} \lambda_{ij} A_{kl} , \tag{12}$$

where $\mathcal{P}$ is the permutation operator. On the left-hand side, use of the permutation operator results in multiplication by the binomial coefficient corresponding to the complexion. For example, in Eq. (12) $\mathcal{P}$ gives 6 for the $h_1 h_1 h_2 h_2$ complexion. On the right-hand side $\mathcal{P}$ results in summing all terms that lead to the same derivative of $\mathcal{U}$.

There is one equation in Eq. (8), three equations in Eq. (9), six equations in Eq. (10), ten equations in Eq. (11) and fifteen equations in Eq. (12) yielding in all 35 independent moment relations up through the

fourth-order derivatives. For I_1, I_2 and I_3 ranging through ± 1, 0 there are 27 weights $\omega_{I_1 I_2 I_3}$. There is one λ, three λ_i, and six λ_{ii} required to produce a resultant operator $\mathcal{O}\mathcal{F}$ up through the fourth derivatives. Thus there are 37 unknowns - too many for a unique solution.

Since the weights are relative, a solution may be obtained if 36 unknowns are to be determined by 35 equations. Thus one weight or λ -coefficient must be chosen arbitrarily. The question as to the best choice for this auxiliary relation is not investigated, but presumably one corner point weight might be set equal to zero or one of the λ_{ij} coefficients representing a derivative that has a small A_{ij} coefficient might be set equal to zero. The best practical procedure can only be determined when the actual operator $\mathcal{F}$ is known since many symmetries may then be in evidence.

III. ALGORISMS FOR FFAG MAGNETOSTATICS

A. EXAMPLE

As an example of the general procedure outlined in Section II, consider the potential problem that results from the use of the following differential equation.[1]

$$\mathcal{F}(U) = \frac{\partial^2 U}{\partial x^2} + (a + by^2)\frac{\partial^2 U}{\partial y^2} + c\frac{\partial^2 U}{\partial z^2} + e\,y\frac{\partial^2 U}{\partial x \partial y}$$

$$+ f\frac{\partial^2 U}{\partial x \partial z} + g\,y\frac{\partial U}{\partial y} = G\,, \tag{13}$$

where a, b, c, e, f, and g are constants.

B. SECOND-ORDER ALGORISMS

Initially, consider only terms up through the second-order derivatives in Eq. (5). To be consistent with this approximation, the operator $\mathcal{O}$ must

be taken as

$$\mathcal{J} = \lambda \tag{14}$$

Comparing Eq. (13) with Eq. (2) one finds

$$\begin{aligned} &A_{11} = 1 && A_{12} = \tfrac{1}{2} e y && A_{13} = \tfrac{1}{2} f \\ & && A_{22} = a + b y^2 && A_{23} = 0 \\ & && && A_{33} = c \\ &B_1 = 0 && B_2 = g y && B_3 = 0 \qquad C = 0 \end{aligned} \tag{15}$$

Letting

$$I_1 = I \quad I_2 = J \quad I_3 = K \qquad h_1 = h \quad h_2 = \ell \quad h_3 = p \tag{16}$$

one has for Eqs. (8), (9), and (10)

$$\sum \omega_{IJK} = 0 \tag{17}$$

$$h \sum I \omega_{IJK} = 0 \tag{18}$$

$$\ell \sum J \omega_{IJK} = \lambda g y \tag{19}$$

$$p \sum K \omega_{IJK} = 0 \tag{20}$$

$$\tfrac{1}{2} h^2 \sum I^2 \omega_{IJK} = \lambda \tag{21}$$

$$\tfrac{1}{2} \ell^2 \sum J^2 \omega_{IJK} = \lambda (a + b y^2) \tag{22}$$

$$\tfrac{1}{2} p^2 \sum K^2 \omega_{IJK} = \lambda c \tag{23}$$

$$h \ell \sum IJ \omega_{IJK} = \lambda e y \tag{24}$$

$$\ell p \sum JK \omega_{IJK} = 0 \tag{25}$$

$$h p \sum IK \omega_{IJK} = \lambda f \tag{26}$$

At this stage there are 27 independent ω_{IJK} and one λ or 28 unknowns with only 10 moment equations given by Eqs. (17) - (26). To obtain a solution one notices that the following relations are consistent with the symmetries in Eq. (13).

$$\omega_{100} = \omega_{-100} \tag{27}$$

$$\omega_{001} = \omega_{00-1} \tag{28}$$

$$\omega_{110} = \omega_{-1-10} = -\omega_{1-10} = -\omega_{-110} \tag{29}$$

$$\omega_{101} = \omega_{-10-1} = -\omega_{10-1} = -\omega_{-101} \tag{30}$$

$$\omega_{011} = \omega_{0-1-1} = -\omega_{0-11} = -\omega_{01-1} = 0 \tag{31}$$

$$\omega_{111} = \omega_{11-1} = \omega_{1-11} = \omega_{1-1-1} = 0 \tag{32}$$

$$\omega_{-111} = \omega_{-11-1} = \omega_{-1-11} = \omega_{-1-1-1} = 0 \tag{33}$$

These restrictions on ω_{IJK} reduce Eqs. (18), (20), and (25) to identities. Thus, there are seven independent moment relations and eight independent unknowns (ω_{000} ω_{100} ω_{010} ω_{0-10} ω_{001} ω_{110} ω_{101} and λ). This permits a solution for the weights in terms of λ . Thus

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & \ell & -\ell & 0 & 0 & 0 \\ 0 & h^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\ell^2 & \frac{1}{2}\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2h\ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2hp \end{bmatrix} \cdot \begin{bmatrix} \omega_{000} \\ \omega_{100} \\ \omega_{010} \\ \omega_{0-10} \\ \omega_{001} \\ \omega_{110} \\ \omega_{101} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda g y \\ \lambda \\ \lambda(a+by^2) \\ \lambda c \\ \frac{1}{2}\lambda e y \\ \frac{1}{2}\lambda f \end{bmatrix} \tag{34}$$

Equation (34) may be solved to obtain

$$\omega_{101} = \frac{\lambda f}{4hp} \qquad \omega_{110} = \frac{\lambda e y}{4h\ell} \qquad \omega_{001} = \frac{\lambda c}{p^2}$$

$$\omega_{100} = \frac{\lambda}{h^2} \qquad \omega_{010} = \frac{\lambda}{\ell^2}(a+by^2) + \frac{\lambda g y}{2\ell}$$

$$\omega_{0-10} = \frac{\lambda}{\ell^2}(a+by^2) - \frac{\lambda g y}{2\ell} \qquad \omega_{000} = -\frac{2\lambda}{h^2} - \frac{2\lambda c}{p^2} - \frac{2\lambda}{\ell^2}(a+by^2) \tag{35}$$

These results are seen to be identical with that presented in MURA-591 after identifying a, b, c, etc.

C. IMPROVED ACCURACY FOR ALGORISMS

If it is true, as it is in the case of the FFAG magnetostatic problem, that the principal contribution to Eq. (13) consists of the diagonal second-order derivatives, then an improved algorism for Eq. (13) may be obtained by first obtaining an algorism for the operator

$$\mathcal{J} = \frac{\partial^2}{\partial x^2} + a\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial z^2} \tag{36}$$

where the quadratic term in y in the coefficients is also dropped. The improvement will consist in finding an algorism for Eq. (36) good to the fourth order in the Taylor series of Eq. (5). Comparison of this improved algorism with that obtained in Eq. (35) will then permit the effect of the smaller terms that were dropped to be again added to second order.

For Eq. (36)

$$
\begin{aligned}
&A_{11} = 1 && A_{12} = 0 && A_{13} = 0 \\
& && A_{22} = a && A_{23} = 0 \\
& && && A_{33} = c \\
&B_1 = 0 && B_2 = 0 && B_3 = 0 \\
& && && C = 0 \,.
\end{aligned}
\tag{37}
$$

To be consistent with the operator $\mathcal{F}$, let the operator

$$\mathcal{O} = \lambda + \lambda_{11}\frac{\partial^2}{\partial x^2} + \lambda_{22}\frac{\partial^2}{\partial y^2} + \lambda_{33}\frac{\partial^2}{\partial z^2} \tag{38}$$

Again making the substitutions of Eq. (16), one has for Eqs. (8) - (12)

$$\sum \omega_{IJK} = 0 \tag{39}$$

$$\tfrac{1}{2} h^2 \sum I^2 \omega_{IJK} = \lambda \tag{40}$$

$$\tfrac{1}{2} \ell^2 \sum J^2 \omega_{IJK} = \lambda a \tag{41}$$

$$\tfrac{1}{2} p^2 \sum K^2 \omega_{IJK} = \lambda c \tag{42}$$

$$\tfrac{1}{24} h^4 \sum I^4 \omega_{IJK} = \lambda_{11} \tag{43}$$

$$\tfrac{1}{24} \ell^4 \sum J^4 \omega_{IJK} = \lambda_{22} a \tag{44}$$

$$\tfrac{1}{24} p^4 \sum K^4 \omega_{IJK} = \lambda_{33} c \tag{45}$$

$$\tfrac{1}{4} h^2 \ell^2 \sum I^2 J^2 \omega_{IJK} = \lambda_{11} a + \lambda_{22} \tag{46}$$

$$\tfrac{1}{4} \ell^2 p^2 \sum J^2 K^2 \omega_{IJK} = \lambda_{22} c + \lambda_{33} a \tag{47}$$

$$\tfrac{1}{4} h^2 p^2 \sum I^2 K^2 \omega_{IJK} = \lambda_{11} c + \lambda_{33} \tag{48}$$

The equations that reduce to identities under the following relations have been omitted.

$$\omega_{100} = \omega_{-100} \qquad \omega_{010} = \omega_{0-10} \qquad \omega_{001} = \omega_{00-1} \tag{49}$$

$$\omega_{110} = \omega_{-1-10} = \omega_{1-10} = \omega_{-110} \tag{50}$$

$$\omega_{101} = \omega_{-10-1} = \omega_{10-1} = \omega_{-101} \tag{51}$$

$$\omega_{011} = \omega_{0-1-1} = \omega_{01-1} = \omega_{0-11} \tag{52}$$

$$\omega_{111} = \omega_{11-1} = \omega_{1-11} = \omega_{1-1-1} = 0 \tag{53}$$

$$\omega_{-111} = \omega_{-11-1} = \omega_{-1-11} = \omega_{-1-1-1} = 0 \quad . \tag{54}$$

These relations are consistent with the symmetries permitted by the operator in Eq. (36). There are thus seven independent weights ω and four λ -values. Thus the ten equations, Eqs. (39) - (48) admit of a solution for the weights in terms of, say, λ . Thus

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 4 & 4 & 4 & 0 & 0 & 0 \\ 0 & h^2 & 0 & 0 & 2h^2 & 0 & 2h^2 & 0 & 0 & 0 \\ 0 & 0 & \ell^2 & 0 & 2\ell^2 & 2\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^2 & 0 & 2p^2 & 2p^2 & 0 & 0 & 0 \\ 0 & \frac{1}{12}h^4 & 0 & 0 & \frac{1}{6}h^4 & 0 & \frac{1}{6}h^4 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{12}\ell^4 & 0 & \frac{1}{6}\ell^4 & \frac{1}{6}\ell^4 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & \frac{1}{12}p^4 & 0 & \frac{1}{6}p^4 & \frac{1}{6}p^4 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & h^2\ell^2 & 0 & 0 & -a & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ell^2p^2 & 0 & 0 & -c & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & h^2p^2 & -c & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \omega_{000} \\ \omega_{100} \\ \omega_{010} \\ \omega_{001} \\ \omega_{110} \\ \omega_{011} \\ \omega_{101} \\ \lambda_{11} \\ \lambda_{22} \\ \lambda_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ \lambda a \\ \lambda c \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{55}$$

The solution of Eq. (55) yields

$$\lambda_{11} = \frac{1}{12} h^2 \lambda \tag{56}$$

$$\lambda_{22} = \frac{1}{12} \ell^2 \lambda \tag{57}$$

$$\lambda_{33} = \frac{1}{12} p^2 \lambda \tag{58}$$

$$\omega_{110} = \omega_{-1-10} = \omega_{1-10} = \omega_{-110} = \frac{1}{12}\left(\frac{a}{\ell^2} + \frac{1}{h^2}\right)\lambda \tag{59}$$

$$\omega_{011} = \omega_{0-1-1} = \omega_{01-1} = \omega_{0-11} = \frac{1}{12}\left(\frac{c}{p^2} + \frac{a}{\ell^2}\right)\lambda \tag{60}$$

$$\omega_{101} = \omega_{-10-1} = \omega_{10-1} = \omega_{-101} = \frac{1}{12}\left(\frac{c}{p^2} + \frac{1}{h^2}\right)\lambda \tag{61}$$

$$\omega_{001} = \omega_{00-1} = \left(\frac{2}{3}\frac{c}{p^2} - \frac{a}{6\ell^2} - \frac{1}{6h^2}\right)\lambda \tag{62}$$

$$\omega_{010} = \omega_{0-10} = \left(\frac{2}{3}\frac{a}{\ell^2} - \frac{1}{6h^2} - \frac{c}{6p^2}\right)\lambda \tag{63}$$

$$\omega_{100} = \omega_{-100} = \left(\frac{2}{3}\frac{1}{h^2} - \frac{a}{6\ell^2} - \frac{c}{6p^2}\right)\lambda \tag{64}$$

$$\omega_{000} = -\frac{4}{3}\left(\frac{1}{h^2} + \frac{a}{\ell^2} + \frac{c}{p^2}\right)\lambda \tag{65}$$

The operator of Eq. (36) treated to the approximation of Section II B gives

$$\omega_{110} = \omega_{-1-10} = \omega_{1-10} = \omega_{-110} = 0 \tag{66}$$

$$\omega_{011} = \omega_{0-1-1} = \omega_{01-1} = \omega_{0-11} = 0 \tag{67}$$

$$\omega_{101} = \omega_{-10-1} = \omega_{10-1} = \omega_{-101} = 0 \tag{68}$$

$$\omega_{001} = \omega_{00-1} = \frac{\lambda c}{p^2} \tag{69}$$

$$\omega_{010} = \omega_{0-10} = \frac{\lambda a}{\ell^2} \tag{70}$$

$$\omega_{100} = \omega_{-100} = \frac{\lambda}{h^2} \tag{71}$$

$$\omega_{000} = -2\left(\frac{1}{h^2} + \frac{a}{\ell^2} + \frac{c}{p^2}\right)\lambda \tag{72}$$

Subtraction of the weights in Eqs. (66) - (72) from the weights in Eq. (35) gives the second-order correction weights.

$$\Delta\omega_{110} = \Delta\omega_{-1-10} = -\Delta\omega_{1-10} = -\Delta\omega_{-110} = \frac{\lambda e g}{4h\ell} \tag{73}$$

$$\Delta\omega_{011} = \Delta\omega_{0-1-1} = -\Delta\omega_{01-1} = -\Delta\omega_{0-11} = 0 \tag{74}$$

$$\Delta\omega_{101} = \Delta\omega_{-10-1} = -\Delta\omega_{10-1} = -\Delta\omega_{-101} = \frac{\lambda f}{4hp} \tag{75}$$

$$\Delta\omega_{001} = \Delta\omega_{00-1} = 0 \tag{76}$$

$$\Delta\omega_{010} = \frac{\lambda b y^2}{\ell^2} + \frac{\lambda g y}{2\ell} \tag{77}$$

$$\Delta\omega_{0-10} = \frac{\lambda b y^2}{\ell^2} - \frac{\lambda g y}{2\ell} \tag{78}$$

$$\Delta\omega_{100} = \Delta\omega_{-100} = 0 \tag{79}$$

$$\Delta\omega_{000} = -\frac{2\lambda b y^2}{\ell^2} \tag{80}$$

D. COMPOSITE SYSTEM OF ALGORISMS

Adding the corrections of Eqs. (73) - (80) to the weights in Eqs. (59) - (65) gives, after letting,

$$\lambda = \frac{3}{4}\ell^2 \tag{81}$$

$$r = \frac{\ell}{h} \tag{82}$$

$$s = \frac{\ell}{p} \tag{83}$$

$$\omega_{100} = \omega_{-100} = \frac{1}{2}r^2 - \frac{a}{8} - \frac{cs^2}{8} \tag{84}$$

$$\omega_{010} = \frac{1}{2}\left(a - \frac{1}{4}r^2 - \frac{cs^2}{4} + \frac{3}{2}by^2\right) + \frac{3\ell g}{8}y \tag{85}$$

$$\omega_{0-10} = \frac{1}{2}\left(a - \frac{1}{4}r^2 - \frac{cs^2}{4} + \frac{3}{2}by^2\right) - \frac{3\ell g}{8}y \tag{86}$$

$$\omega_{001} = \omega_{00-1} = \frac{1}{2}cs^2 - \frac{a}{8} - \frac{1}{8}r^2 \tag{87}$$

$$\omega_{110} = \omega_{-1-10} = \frac{1}{16}(a + r^2) + \frac{3re}{16}y \tag{88}$$

$$\omega_{1-10} = \omega_{-110} = \frac{1}{16}(a + r^2) - \frac{3re}{16}y \tag{89}$$

$$\omega_{011} = \omega_{0-1-1} = \omega_{01-1} = \omega_{0-11} = \frac{1}{16}(a + cs^2) \tag{90}$$

$$\omega_{101} = \omega_{-10-1} = \frac{1}{16}(r^2 + cs^2) + \frac{3rsf}{16} \tag{91}$$

$$\omega_{10-1} = \omega_{-101} = \frac{1}{16}(r^2 + cs^2) - \frac{3rsf}{16} \tag{92}$$

$$\omega_{111} = \omega_{11-1} = \omega_{1-11} = \omega_{1-1-1} = 0 \tag{93}$$

$$\omega_{-111} = \omega_{-11-1} = \omega_{-1-11} = \omega_{-1-1-1} = 0 \tag{94}$$

$$-\omega_{000} = a + r^2 + cs^2 + \frac{3}{2}by^2 \tag{95}$$

Since convergence difficulties are experienced when any of the weights except ω_{000} take on negative values, it is important to determine the allowed values of r and s for positive weights. To estimate the range the small effects due to the presence of y and y^2 terms may be neglected. Thus for positive weights

$$\frac{1}{2} r^2 - \frac{a}{8} - \frac{c s^2}{8} > 0 \tag{96}$$

$$\frac{1}{2} a - \frac{1}{8} r^2 - \frac{1}{8} c s^2 > 0 \tag{97}$$

$$\frac{1}{2} c s^2 - \frac{a}{8} - \frac{1}{2} r^2 > 0 \tag{98}$$

In particular, if

$$r = 1 \qquad a = 1 \qquad c = 1 \tag{99}$$

then

$$\frac{1}{2} < s^2 < 3 \tag{100}$$

is the condition on s.

If s is chosen to be unity indicating that one expects the z-variations of the potential to be as strong as the x- and y-variations, the above set of weights is an appropriate set.

On the other hand, as sometimes occurs, if one expects a weak z-variation, the mesh spacing p may be chosen larger. If, for example, $s = \frac{1}{3}$, then the condition of Eq. (100) is not met. In this case one takes advantage of the weak z-variation to reduce the problem locally to a two-dimensional problem. This may be done by summing the weights over the z-index. Thus

$$\omega_{10} = \omega_{10-1} + \omega_{100} + \omega_{101} = \frac{1}{8}(5r^2 - a) \tag{101}$$

$$\omega_{-10} = \omega_{-10-1} + \omega_{-100} + \omega_{-101} = \frac{1}{8}(5r^2 - a) \tag{102}$$

$$\omega_{01} = \omega_{01-1} + \omega_{010} + \omega_{011} = \frac{1}{8}(5a - r^2) + \frac{3}{4}by^2 + \frac{3\ell g}{8}y \tag{103}$$

$$\omega_{0-1} = \omega_{0-1-1} + \omega_{0-10} + \omega_{0-11} = \frac{1}{8}(5a - r^2) + \frac{3}{4}by^2 - \frac{3\ell g}{8}y \tag{104}$$

$$\omega_{11} = \omega_{11-1} + \omega_{110} + \omega_{111} = \frac{1}{16}(a + r^2) + \frac{3re}{16}y \tag{105}$$

$$\omega_{-1-1} = \omega_{-1-1-1} + \omega_{-1-10} + \omega_{-1-11} = \frac{1}{16}(a + r^2) + \frac{3re}{16}y \tag{106}$$

$$\omega_{1-1} = \omega_{1-1-1} + \omega_{1-10} + \omega_{1-11} = \frac{1}{16}(a + r^2) - \frac{3re}{16}y \tag{107}$$

$$\omega_{-11} = \omega_{-11-1} + \omega_{-110} + \omega_{-111} = \frac{1}{16}(a + r^2) - \frac{3re}{16}y \tag{108}$$

$$-\omega_{00} = -\omega_{00-1} - \omega_{000} - \omega_{001} = \frac{5}{4}(a + r^2) + \frac{3}{2}by^2 \tag{109}$$

For positive weights the condition now becomes

$$\frac{1}{5}a < r^2 < 5a \tag{110}$$

which is easily met for $r \sim a \sim 1$.

The algorisms are thus used as follows. It is assumed that the x- and y-variations of the potential U are equally strong. This suggests that the unit meshes in these directions be chosen about equal, i.e., $h \sim \ell$ or $r \sim 1$. If one expects that the z-variation of the potential U is about as strong, then one must use Eqs. (84) - (95) for the algorisms with $p \sim \ell$ or $s \sim 1$. On the other hand, if a region of weak variation of the potential

U with z is expected, the iteration time can be decreased without appreciable loss of accuracy by increasing the unit mesh distance in the z-direction, say for example $p \sim 3\ell$ or $s \sim \frac{1}{3}$. In this instance, the use of the algorisms in Eqs. (84) - (95) leads to some negative weights. This seems to lead to non-convergent iterative processes.[2] However, the use of algorisms in Eqs. (101) - (109) is consistent with a weak z-variation of the potential .

E. INHOMOGENEOUS ALGORISMS

The final weight system which may be utilized in an iterative solution of Eq. (1) is given by Eq. (5). After substituting for $\mathcal{J}(U)$ from Eq. (1), one has

$$\sum \omega_{I_1 I_2 I_3} U_{I_1 I_2 I_3} = \mathcal{O} G . \tag{111}$$

In the example of Section III A, the weights ω_{IJK} have been determined to the desired approximation. In addition, the λ -coefficients of the operator $\mathcal{O}$ have been determined. Thus

$$\mathcal{O} = \frac{3}{4}\ell^2\left[1 + \frac{1}{12}h^2\frac{\partial^2}{\partial x^2} + \frac{1}{12}\ell^2\frac{\partial^2}{\partial y^2} + \frac{1}{12}p^2\frac{\partial^2}{\partial z^2}\right] . \tag{112}$$

Notice that the term in $\frac{\partial}{\partial y}$ is missing. This is due to the assumption that it was more important to improve the accuracy of the second-order terms. Since G is a given function at the (0, 0, 0) or standard point, the inhomogeneous algorism becomes

$$\sum \omega_{IJK} U_{IJK} = \mathcal{O} G . \tag{113}$$

For magnetostatics, the inhomogeneous contribution is generally a small effect since practical current distributions tend to have a uniform density.

F. SURFACE ALGORISMS

Magnetostatic[3] fields differ from electrostatic fields in that a supplementary but known term must be added to the gradient of the potential U. The supplementary term exists in the regions of current. Therefore, as one passes from air to current regions, surface layer discontinuities in the potential U are required to take into account the discontinuity in the supplementary term. These discontinuities, which are of both the single layer and double layer type, are responsible for the major contribution to the source of the potential U. The inhomogeneous term represented by G leads to a much smaller source contribution.

As an example, suppose the interface between the air region and the current carrying region occurs on a y = constant surface. Continuity of the normal component of $\vec{H}$ and the tangential components of $\vec{H}$ (excluding the case of current sheets) yields, via the discontinuity of the supplementary term, the discontinuity in the potential (double layer) and the discontinuity in the normal derivative of the potential (single layer). Both the single layer and double layer densities thus are known as functions of the surface coordinates x and z.

If Δ is the potential discontinuity on the surface

$$\Delta = W - U, \tag{113}$$

where W is the potential function within the current carrying region and U is the potential function in the air region. In particular, for the example chosen $\Delta(x,z)$ and $\Delta_y(x,z)$ are known functions. It then follows that all derivatives are known because either they can be found by differentiation of $\Delta(x,z)$ and $\Delta_y(x,z)$ or by differentiation and subsequent use of the differential equation holding in each region.

If all derivatives are known, then a Taylor series expansion can be carried out to yield $W - U$ for the points (I_1, I_2, I_3)

$$[W-U]_{I_1 I_2 I_3} = \Delta + \sum_i h_i I_i \Delta_{x_i} + \frac{1}{2!}\sum_{ij} h_i h_j I_i I_j \Delta_{x_i} \Delta_{x_j}$$

$$+ \frac{1}{3!} \sum_{ijk} h_i h_j h_k I_i I_j I_k \Delta_{x_i x_j x_k}$$

$$+ \frac{1}{4!} \sum_{ijkl} h_i h_j h_k h_l I_i I_j I_k I_l \Delta_{x_i x_j x_k x_l} \quad . \tag{115}$$

The central point (0, 0, 0) is understood to reside on the interface.

After multiplying Eq. (115) by the weights to give

$$\sum \omega_{I_1 I_2 I_3} [W-U]_{I_1 I_2 I_3} \quad ,$$

a term is obtained which must be added to the left-hand side of Eq. (4) in order to subtract out $U_{I_1 I_2 I_3}$ in the region of the current and replace it by $W_{I_1 I_2 I_3}$ in this region. Thus if " $U_{I_1 I_2 I_3}$" designates the potential $U_{I_1 I_2 I_3}$ for the air points and $W_{I_1 I_2 I_3}$ for the points within the current, Eq. (4) becomes

$$\sum \omega_{I_1 I_2 I_3} \left[\text{“}U\text{”}_{I_1 I_2 I_3} + C V_{I_1 I_2 I_3} \right] = \partial G. \tag{116}$$

The supplementary terms are called the current values[5] and are given by Eq. (115) on any surface for which Δ and its normal derivative exists.

There is at least one other surface on which current values must be introduced. This surface is any surface in the air that may be used to cap a current carrying coil thereby preventing more than one passage completely

through the coil. On this surface a constant potential discontinuity or double layer is introduced whose magnitude equals the potential rise in passing once around the coil through the interior region.

The specification of the problem is now complete in the sense that a unique solution may be obtained if boundary conditions on the potential are given. If a coil does not touch the iron, its potential may be taken as zero ideally. If a coil touches the iron, the variation of potential on the iron may be found from the corresponding tangential derviatives of Δ .

G. ALGORISMS AT EDGES OF IDEAL IRON

It is not sufficiently accurate to expand a potential in a Taylor series in the neighborhood of a point for which the field becomes infinite such as at the edge or corner of the ideal iron. In order to understand the nature of this difficulty, only a two-dimensional Laplace equation for which a mesh having $h = \ell$ or $r = 1$ will be used. Also only expansions of the potential up through second-order terms will be considered.

For ideal iron corners the expansion of the magnetostatic potential in powers of the radial distance ρ from the corner is

$$U = A_0 + A_1 \rho^{2/3} \sin \tfrac{2}{3}\theta + A_2 \rho^{4/3} \sin \tfrac{4}{3}\theta + A_3 \rho^{2} \sin 2\theta + \cdots \tag{117}$$

Let (β) designate the central point and (i, j) the neighboring points. Then

$$U_{ij}^{(\beta)} = \sum B_{ij,n}^{(\beta)} A_n \qquad \left(B_{ij,n}^{(\beta)} = \rho^{\frac{2}{3}n} \sin \tfrac{2}{3} n \theta_{ij} \right) \tag{118}$$

The problem is to find an five-point algorism for which

$$\sum_{ij} \omega_{ij}^{(\beta)} V_{ij}^{(\beta)} = 0 \quad . \tag{119}$$

The weights are then determined by demanding that Eq. (119) be independent of A_n. Since a five-point algorism is desired, four conditions are needed plus, say, the arbitrary choice that $\omega_{00}^{(\beta)} = -4$. Thus, the conditions are

$$\sum_{ij} \omega_{ij}^{(\beta)} B_{ij,0}^{(\beta)} = 0 \qquad \left(B_{ij,0}^{(\beta)} = 1 \right) \tag{120}$$

$$\sum_{ij} \omega_{ij}^{(\beta)} B_{ij1}^{(\beta)} = 0 \tag{121}$$

$$\sum_{ij} \omega_{ij}^{(\beta)} B_{ij2}^{(\beta)} = 0 \tag{122}$$

$$\sum_{ij} \omega_{ij}^{(\beta)} B_{ij3}^{(\beta)} = 0 \quad . \tag{123}$$

For a typical corner shown in Fig. 1,

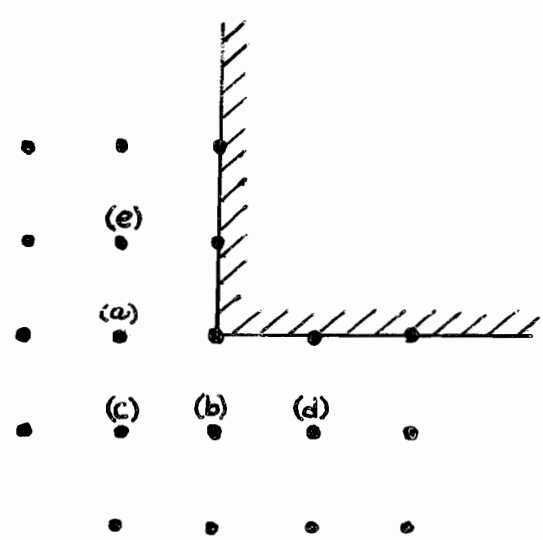


Figure 1. Mesh Near Ideal Iron Corner

the $B_{ij,n}^{(\beta)}$ given by Eq. (118) for central points $\beta = a, b, c, d, e$ are shown in Table I.

TABLE I. TABLE OF $B_{ij,n}^{(\beta)}$ COEFFICIENTS

	a	b	c	d	e
$B_{10,1}$	0.0000	-0.6300	-0.8660	-0.5207	0.0000
$B_{00,1}$	-0.8660	-0.8660	-1.2599	-0.6300	-0.6300
$B_{-10,1}$	-1.3747	-1.2599	-1.6708	-0.8660	-1.1503
$B_{01,1}$	-0.6300	0.0000	-0.8660	0.0000	-0.5207
$B_{0-1,1}$	-1.2599	-1.3747	-1.6708	-1.1503	-0.8660
$B_{10,2}$	0.0000	-0.6873	-0.4330	-0.8513	0.0000
$B_{00,2}$	0.4330	-0.4330	0.0000	-0.6873	0.6873
$B_{-10,2}$	1.0911	0.0000	0.6076	-0.4330	1.4620
$B_{01,2}$	0.6873	0.0000	0.4330	0.0000	0.8513
$B_{0-1,2}$	0.0000	-1.0911	-0.6076	-1.4620	0.4330
$B_{10,3}$	0.0000	-2.0000	0.0000	-4.0020	0.0000
$B_{00,3}$	0.0000	0.0000	+2.0000	-2.0000	-2.0000
$B_{-10,3}$	0.0000	2.0000	4.0020	0.0000	-4.0020
$B_{01,3}$	-2.0000	0.0000	0.0000	0.0000	-4.0020
$B_{0-1,3}$	2.0000	0.0000	4.0020	-4.0020	0.0000

Point a:

$$\omega_{10} + \omega_{-10} + \omega_{01} + \omega_{0-1} = 4$$

$$-1.3747\,\omega_{-10} - .6300\,\omega_{01} - 1.2599\,\omega_{0-1} = -3.4641$$

$$1.0911\,\omega_{-10} + .6873\,\omega_{01} = 1.7320$$

$$-2\,\omega_{01} + 2\,\omega_{0-1} = 0$$

Solution (Point a):

$$\omega_{10} = .6974 \qquad \omega_{01} = 1.2519$$
$$\omega_{-10} = .7988 \qquad \omega_{0-1} = 1.2519 \tag{123}$$
$$\omega_{00} = -4$$

Point b:

$$\omega_{10} + \omega_{-10} + \omega_{01} + \omega_{0-1} = 4$$
$$-.6300\,\omega_{10} - 1.2599\,\omega_{-10} - 1.3747\,\omega_{0-1} = -3.4641$$
$$-.6873\,\omega_{10} - 1.0911\,\omega_{0-1} = -1.7320$$
$$-2\,\omega_{10} + 2\,\omega_{-10} = 0$$

Solution (Point b):

$$\omega_{10} = 1.2519 \qquad \omega_{01} = .6974$$
$$\omega_{-10} = 1.2519 \qquad \omega_{0-1} = .7988 \tag{124}$$
$$\omega_{00} = -4$$

Point c: Determinant of coefficients is zero, hence homogeneous form for algorism is not suitable. See later.

Point d:

$$\omega_{10} + \omega_{-10} + \omega_{01} + \omega_{0-1} = 4$$
$$-.5207\,\omega_{10} - .8660\,\omega_{-10} - 1.1503\,\omega_{0-1} = -2.5198$$
$$-.8513\,\omega_{10} - .4330\,\omega_{-10} - 1.4620\,\omega_{0-1} = -2.7492$$
$$-4.0020\,\omega_{10} - 4.0020\,\omega_{0-1} = -8.0000$$

Solution (Point d):

$$\omega_{10} = .9580 \qquad \omega_{01} = 1.0501$$

$$\omega_{-10} = .9509 \qquad \omega_{0-1} = 1.0410 \tag{125}$$

$$\omega_{00} = -4$$

Point e:

$$\begin{aligned} \omega_{10} + \omega_{-10} + \omega_{01} + \omega_{0-1} &= 4 \\ -1.1503\,\omega_{-10} - .5207\,\omega_{01} - .8660\,\omega_{0-1} &= -2.5198 \\ 1.4620\,\omega_{-10} + .8513\,\omega_{01} + .4330\,\omega_{0-1} &= 2.7492 \\ -4.0020\,\omega_{-10} - 4.0020\,\omega_{01} &= -8.0000 \end{aligned}$$

Solution (Point e)

$$\omega_{10} = 1.0501 \qquad \omega_{01} = .9580$$

$$\omega_{-10} = 1.0410 \qquad \omega_{0-1} = .9509 \tag{126}$$

$$\omega_{00} = -4$$

Point c:

Since Eq. (119) cannot be used, try to find weights such that

$$\sum_{ij} \omega_{ij}^{(c)} U_{ij}^{(c)} = \alpha A_0 . \tag{127}$$

The conditions then are that

$$\sum_{ij} \omega_{ij}^{(c)} B_{ij0}^{(c)} = \alpha \tag{128}$$

$$\sum_{ij} \omega_{ij}^{(c)} B_{ij1}^{(c)} = 0 \tag{129}$$

$$\sum_{ij} \omega_{ij}^{(c)} B_{ij2}^{(c)} = 0 \tag{130}$$

$$\sum_{ij} \omega_{ij}^{(c)} B_{ij3}^{(c)} = 0 \tag{131}$$

From the symmetry of this point, one expects that $\omega_{0i} = \omega_{10}$ and $\omega_{0-1} = \omega_{-10}$. The equations then become

$$
\begin{aligned}
2\,\omega_{10} + 2\,\omega_{-10} &= 4 + \alpha \\
1.7320\,\omega_{10} + 3.3416\,\omega_{-10} &= 5.0397 \\
8.0040\,\omega_{-10} &= 8.0000
\end{aligned}
$$

Solution (Point c):

$$
\begin{aligned}
\omega_{10} &= .9814 & \qquad \omega_{01} &= .9814 \\
\omega_{-10} &= .9995 & \qquad \omega_{0-1} &= .9995 \\
\omega_{00} &= -4 & \qquad \alpha &= -.0392
\end{aligned}
\tag{132}
$$

The results in Eqs. (123), (124), (125), (126), and (132) provide an improvement in the weight system that would have been obtained using a Taylor series expansion ($\omega_{10} = \omega_{01} = \omega_{-10} = \omega_{0-1} = 1$, $\omega_{00} = -4$). This leaves the weights in the neighborhood of a three-dimensional corner to be investigated. This problem seems too complicated to handle for the moment. Notice however that in practice there are many more points associated with a two-dimensional edge than with a three-dimensional corner. Presumably the use of two-dimensional weights near a corner will give some correction. The weights as determined in this report have been used by R. S. Christian in the iterative calculation of FFAG fields using the program UNICYL.[6]

REFERENCES

1. S. C. Snowdon, "Relaxation Calculation of Integral Scaling Magnetic Fields Produced by Distributed Currents", MURA Report No. 591, October 17, 1960 (unpublished).

2. Private Communication (R. S. Christian).

3. S. C. Snowdon, "Current Sources for Scaling Magnetic Fields," MURA Report No. 535, December 2, 1959 (unpublished).

4. R. S. Christian and S. C. Snowdon, "Alternative Formulations of Magnetostatic Problems," MURA Report No. 568, April 1, 1960 (unpublished).

5. L. Jackson Laslett, "Development of Algorisms for FOROCYL Potential Program, MURA Report No. 205, November 10, 1956 (unpublished).

6. Private Communication (R. S. Christian).