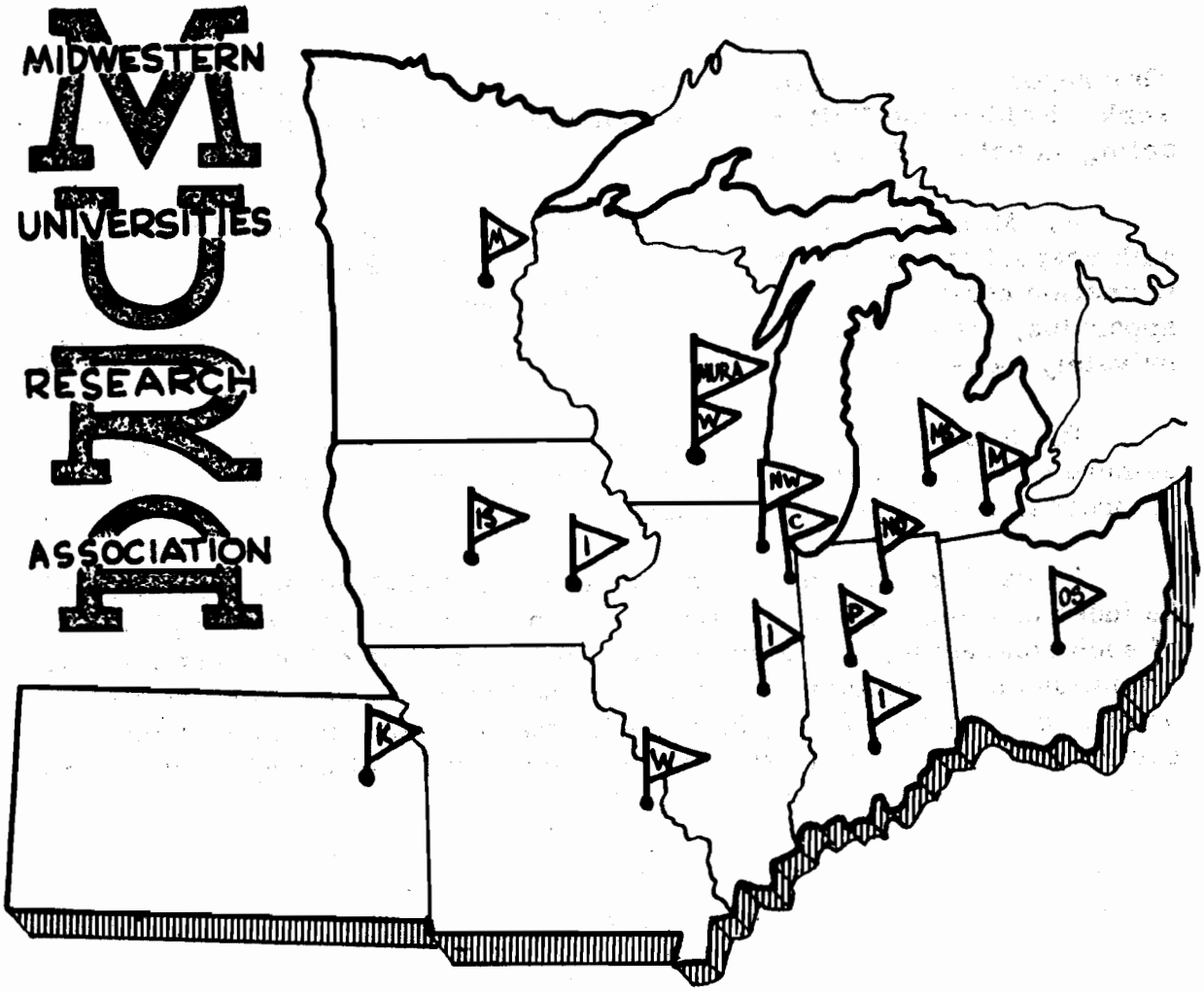

PARTICLE DYNAMICS IN LINEAR ACCELERATORS

Phil Lee Morton

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PARTICLE DYNAMICS IN

LINEAR ACCELERATORS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
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By

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ABSTRACT

In this dissertation problems of particle dynamics in a standing-wave Alvarez linear accelerator with quadrupole focusing magnets are treated. The previous single particle theories are extended to take into account the coupling between the longitudinal and transverse motion of a particle. The Hamiltonian governing the particle motion is derived and then used to obtain an adiabatic invariant for particles which travel along the axis. From this Hamiltonian the frequencies of small amplitude transverse and longitudinal oscillations are obtained as functions of the particle energy. The adiabatic invariant for on-axis particles is used to obtain the longitudinal oscillation amplitude. Next the Hamiltonian is transformed by the use of Moser theory to obtain a new Hamiltonian which is then used to treat the coupling between transverse and longitudinal particle motion. Specific results are illustrated in the case of the MURA linac for which the main coupling resonance is the one where the transverse oscillation frequency equals the longitudinal oscillation frequency. In this case, it is shown that transverse oscillations grow by less than 2 percent. Lastly, the influence of the space-charge forces on the longitudinal particle motion is studied, and a limit on the total number of particles which can be accelerated is obtained. The space-charge limit is calculated for the MURA linac to be 50 ma.

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CONTENTS

I. INTRODUCTION 1

A. Statement of Problem 1

B. Historical Review 1

C. Previous Investigations 5

D. Scope of Thesis 11

II. FORMULATION OF THE HAMILTONIAN 13

A. Description of the Field 13

B. Exact Hamiltonian 20

C. Approximate Hamiltonian 24

D. Carrier-Wave Approximation 32

III. TRANSFORMATION OF φ MOTION INTO ACTION-ANGLE VARIABLES 41

A. Relationship Between Old and New Variables 41

B. Adiabatic Invariance of the Action 54

IV. MOSER TRANSFORMATIONS 61

V. RAPID TRAVERSAL OF THE RESONANCE 72

VI. LONGITUDINAL SPACE-CHARGE EFFECTS 80

CONTENTS (Cont'd.)

APPENDIX

I 98

II 102

III 106

BIBLIOGRAPHY 112

AUTOBIOGRAPHY 114

ILLUSTRATIONS AND TABLES

Graph

1	Tune versus Energy for the MURA Linac	40
2	ℓ versus f_2	49
3	$d\ell/df_2$ versus f_2	50
4	$d_1(x)$ versus x	86
5	$d_2(x)$ versus x	87
6	g_n versus n	89
7	$K(\varphi-\varphi')$ versus $(\varphi-\varphi')$	90

Figure

1	The Wideröe Linear Accelerator	2

Table

1	Present-Day Linacs	6
2	Design Parameters for the MURA Linac	26
3	Approximation to Sin	42

I. INTRODUCTION

A. Statement of the Problem

In this dissertation we will treat problems of particle dynamics in a linear accelerator. We will concern ourselves with how the electromagnetic fields of the linear accelerator affect the motion of particles and not with how the electromagnetic fields are produced, except in the case where the fields are produced by the particles that are being accelerated.

B. Historical Review

The linear accelerator, often called linac, is an accelerator which uses a radio-frequency electric field to accelerate particles in a straight line. Any historical review of linear accelerators would start with a description of the Wideroe[1] accelerator. The Wideroe accelerator, the forerunner of all resonance accelerators, as shown in Fig. 1, consisted of two hollow cylindrical electrodes to which opposite periodic voltages were applied. The frequency of these voltages was such that the time required for a particle moving along the axis to transverse one electrode was just the time required for the electrodes to change polarity, and the particle was accelerated by receiving successive voltage kicks. Lawrence and Sloan[2] then extended the idea to ten or more accelerating electrodes and accelerated mercury ions to the 1 - 2 Mev range.

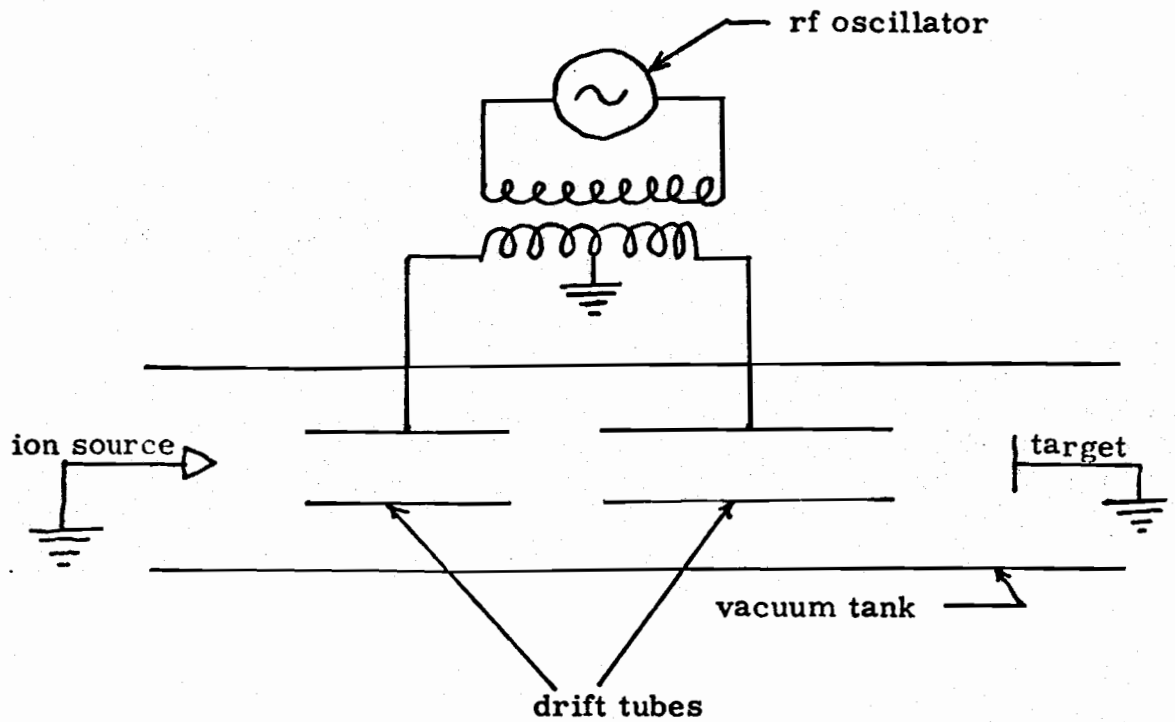


Figure 1

The Wideroe Linear Accelerator

Before World War II, when this early development work was being done, the cyclotron was coming into being. Because of the success of the cyclotron, this early type of linear accelerator was not exploited as a research tool.

After World War II, with the new strides made in the development of large rf power tubes, the interest in the linac was revived. At this time the Alvarez linear accelerator[3] was proposed. The Alvarez type of accelerator uses hollow cylindrical electrodes, called drift tubes, of increasing length as in earlier linacs; however, the method of obtaining the electric field is different. The electric field is obtained in this type of linac by placing drift tubes in a cavity resonant at the desired frequency. The ends of each drift tube are oppositely charged at any instant and all drift tubes are excited in phase. This mode of excitation does not require large driving currents in the drift-tube supporting stems and hence leads to lower power losses than were present in the earlier linacs. In the Alvarez structure the particle takes one full rf period to travel between gap centers of successive drift tubes instead of one-half of an rf period as in the Wideroe type.

There is at the present time another type of linac design which is better adapted to the acceleration of electrons than the Alvarez design. This type is called a traveling-wave linac, because the electric field is a wave traveling in a cylindrical pipe serving as a wave guide. The phase velocity of the traveling wave is slowed to the

velocity of the electron by loading the wave guide with disks of conducting material at regular intervals. The electrons remain in phase with the traveling electric wave and are accelerated by "riding on" the traveling wave. The fact that electrons approach the velocity of light at relatively low energies makes the traveling wave linac more useful for the acceleration of electrons, while the fact that in most proton linacs the protons can be considered as nonrelativistic makes the Alvarez linac more useful for accelerating protons. There are other differences in the linacs: the traveling wave linac is excited at S-band frequencies (1000 Mc/s to 3000 Mc/s) as compared to frequencies close to 200 Mc/s for the Alvarez linac; and it is not necessary to have special radial focusing devices in the traveling wave design. The velocity of the electrons is close to the velocity of light and it will be shown later that the radial defocusing force due to the rf field appraches zero as the velocity of the electrons approaches the velocity of light. The energy gain of the electrons appears almost entirely as an increase in the mass of the electron and for a given radial momentum the radial velocity decreases with energy so that the radial displacement increases very slowly.

The relative advantages of the two types of design have been analyzed by Slater[4] and by Ginzton, Hansen and Kennedy.[5] These papers show that for velocities low compared with the velocity of light the Alvarez design is more economical, while for velocities close to

the velocity of light the traveling wave design is more economical. This fact has led a group at Yale to study the problems of transition of particles from an Alvarez-type linac to a traveling wave linac at an energy which is at the upper end of the range of the Alvarez structures and at the lower end of the range of the traveling wave structure. Table 1 lists the linacs which have been built or are under construction. At present the only limit in energy is an economic one.

C. Previous Investigations

Most particles have either a slight error in energy or arrive at the gap when the voltage across the gap is at the wrong value, and we must therefore concern ourselves with their stability in energy and phase. We define a synchronous particle as one which has the proper energy and phase relationship to the radio-frequency electric field so that it travels longitudinally between successive gap centers in exactly one period of the electric field. There are three possibilities for a nonsynchronous particle. The first possibility is that the energy and phase oscillate about the synchronous values and the error in energy and phase is bounded. The second possibility is that error in energy and phase increases. The third possibility is the boundary between the first two; in this case the error in energy or phase neither oscillates nor increases but instead approaches some fixed value. For the first case we say that the motion is stable; for the second case we say that the motion is unstable, while for the third case we say that the motion

TABLE 1

PRESENT DAY LINACS

Location	Type of Particle	Energy (Mev)
Radiation Laboratory University of California	Proton	32
Radiation Laboratory University of California	Proton	10
Radiation Laboratory Livermore	Deuteron	7.5
University of Minnesota	Proton	68
U.S.S.R. (Moscow)	Proton	40
U.S.S.R. (Big Volga)	Proton	9
U.S.S.R. (Kharkov)	Proton	21
Radiation Laboratory University of California	C, N, O, Ne	10/nucleon
Yale University	C, N, O, Ne	10/nucleon
AERE (Harwell)	Proton	50
CERN (Geneva)	Proton	50
Brookhaven	Proton	50
Argonne	Proton	50

TABLE 1 (continued)

PRESENT DAY LINACS

Location	Type of Particle	Energy (Mev)
Stanford University	Electron	6
Stanford University	Electron	40
Stanford University	Electron	700
France (Orsay)	Electron	1,000
U.S.S.R. (Ukraine)	Electron	30
MIT	Electron	18

is at the threshold of instability. This longitudinal type of motion of the particle also is called phase motion, and this type of stability is called longitudinal stability or phase stability. The possibility of phase stability was discovered for circular accelerators by Veksler[6] and McMillan.[7]

The same three possibilities exist for the transverse motion of particles away from the axis of the linac. We say that motion of particles that oscillate in radius about the axis is stable motion. We call the motion of particles that have unbounded radial amplitudes unstable motion, and for particles which neither oscillate about the axis nor have unbounded radial amplitudes we call the motion on the threshold of instability. This type of motion is referred to as transverse motion and this type of stability is called transverse stability.

Panofsky was one of the first to study the problem of beam dynamics in the linear accelerator.[8] Panofsky showed that for phase stability the synchronous particle must be crossing the accelerating gap while the electric field is increasing in time. He also showed, neglecting second-order velocity focusing, that in order to have radial stability the particle must be crossing the gap while the electric field is decreasing with time. This incompatibility can be removed either by the introduction of grids on the entrances of the drift tubes, which introduce radial focusing electric fields by making electric field lines terminate on the grids instead of on the walls of the drift tubes, or by

the use of focusing magnets. Until the advent of strong focusing magnetic quadrupoles suggested by Blewett,[9] linacs were designed and built either with grids on the entrances of drift tubes, or with solenoids mounted inside the drift tubes. To be durable, the grids must be so thick that they substantially reduce the intensity of the beam. In addition, the maximum error in energy or phase that a particle can have and still execute both stable phase and stable radial oscillations is undesirably small in the grid focused linacs. The undesirable feature of solenoidal focusing is the large power necessary to produce the required focusing fields. This type of focusing is therefore unattractive because of the expense of supplying power and the difficulty of removing the heat generated. The most modern proton linacs are designed to obtain their radial focusing by the use of quadrupole magnets.

Smith and Gluckstern[10] have treated the problem of beam dynamics in an Alvarez linear accelerator with quadrupole focusing magnets in the linear approximation using matrix techniques. They showed that one can treat the radial motion as a response to focusing terms due to the quadrupoles and a defocusing term due to the rf field of the gap. Since the defocusing term is strongly dependent on the relative phase between the particle and the accelerating rf field, the radial motion of a particle is strongly coupled to its phase motion. Smith and Gluckstern treated the radial motion of a particle by

assuming that the phase between the particle and the accelerating rf field was constant.

It is useful to introduce a coordinate system which defines the z axis as the axis of the linac and the x and y axes as the axes of the quadrupoles. We define a quadrupole which is focusing in the x direction and defocusing in the y direction as a plus quadrupole, and a quadrupole which is defocusing in the x direction and focusing in the y direction as a minus quadrupole. If the types of quadrupoles are used together in some alternate fashion, as for example (+ - + -) or (+ + - - + + - -), then it is possible to obtain focusing in both the x and y direction. With one quadrupole magnet per drift tube, the linac parameters have a period in z which is dependent on the arrangement of the types of quadrupoles; for example, for the + - + - the period in z would be the length of two drift tubes. We will define the tunes ν_x, ν_y and ν_φ as the number of oscillations per period of the linac parameters of the x, y, and φ motion respectively. Smith and Gluckstern showed that if φ is held constant at various values, then the x and y motions which result will be stable (stability requires ν_x and ν_y non-integral) for certain types of quadrupole structures. When the x or y motion is unstable, the x or y amplitudes grow exponentially with z, and particles are lost on the walls of the drift tubes.

Smith and Gluckstern did not treat the problem on instability resulting from nonlinear periodic forces, and they neglected entirely

in the phase motion the fact that the accelerating rf fields are functions of the radial position. They considered only the wave component of the rf field which is synchronous with the particle.

When large numbers of particles are accelerated, the interaction between the particles can become important. The transverse motion of particles in a linac with space-charge forces has been treated by Kapchinskij and Vladimirskij,[11] and they have shown that the transverse space-charge limit is larger than the output of present ion sources. Nielsen and Sessler[12] have treated the problem of longitudinal space-charge forces in circular accelerators.

D. Scope of the Dissertation

In this dissertation we will treat only quadrupole-focused proton linacs of the Alvarez type. We will extend the single-particle theory of Smith and Gluckstern to take into account the coupling between the longitudinal and the radial motion of a particle, and investigate the validity of neglecting the wave components of the radio-frequency field which are not synchronous with the particle. We will derive an adiabatic invariant for particles with pure phase motion, i.e., for particles which travel along the axis. We will also use the techniques of Nielsen, Symon and Sessler to treat the problem of pure phase motion with longitudinal space-charge forces and obtain a limit for the beam intensity.

In Section II the Hamiltonian governing particle motion is derived, and the approximation of using only the synchronous wave is shown to be valid. In Section III the pure phase motion for on-axis particles is treated in detail with the derivation of an adiabatic invariant. In Section III the Hamiltonian is transformed into a form for use in Section IV. In Section IV the coupled motion is studied by the use of Moser theory. Moser theory is a technique of transforming the Hamiltonian so that terms of a particular order are transformed into higher order terms. The only terms that cannot be transformed by this technique are those nonlinear terms that lead to instability, terms for which $m\,\nu_x + n\,\nu_\varphi = p$, with m, n and p integers. In particular the term where $2\,\nu_x = 2\,\nu_\varphi$ is not transformed away. The instability at $2\,\nu_x = 2\,\nu_\varphi$ is treated in Section V, and as a numerical example results are obtained using the parameters of a 200 Mev linac designed at MURA. In Section VI the longitudinal space-charge problem is treated.

II. FORMULATION OF THE HAMILTONIAN

A. Description of the Field

We will concern ourselves first with finding the form of the radio-frequency electromagnetic field, and later discuss the field of the quadrupole focusing magnets. If one knows the field as a function of time at every point along the axis of the linac, then it is possible to determine from Maxwell's equations the field as a function of time at every point in the linac. We will use a rectangular coordinate system, in which z is the longitudinal dimension along the axis of the linac and the transverse x and y axes are the axes of the quadrupole fields.

Because of cylindrical symmetry, the relevant electromagnetic field is independent of azimuth. The linac is designed to resonate at a frequency $\omega/2\pi$. We may denote the longitudinal electromagnetic field on axis by $\mathcal{E}_3(0)$ and write $\mathcal{E}_3(0) = f_0(z) \cos \omega t$. The field at any radius r $(r = \sqrt{x^2 + y^2})$ may then be expressed as a power series of even and odd terms

$$\mathcal{E}_3 = \left[\sum_{n=0}^{\infty} f_n(z) r^{2n} + \sum_{n=0}^{\infty} F_n(z) r^{2n+1} \right] \cos \omega t. \qquad (2.A.1)$$

We use the Maxwellian condition in free space $\nabla^2 \vec{\mathcal{E}} - \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$ to derive the following recursion relationship:

$$f_{n+1} = -\frac{1}{4(n+1)^2}\left[\frac{d^2 f_n}{dz^2} + \frac{\omega^2}{c^2} f_n\right]. \tag{2.A.2}$$

It turns out that this condition requires that the coefficients of all odd power of r be zero, i.e., $F_n = 0$, for all n. The radial electric field $\mathcal{E}_r$ is obtained from the Maxwell equation $\nabla \cdot \vec{\mathcal{E}} = 0$, which gives the following expression for $\mathcal{E}_r$:

$$\mathcal{E}_r = -\left[\sum_{n=0}^{\infty} \frac{1}{2(n+1)} \frac{df_n}{dz} r^{2n+1}\right] \cos \omega t. \tag{2.A.3}$$

The magnetic field is derivable from the Maxwell equation $\nabla \times \vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$. Thus the azimuthal magnetic field B_θ is

$$B_\theta = -\frac{\omega}{c}\left[\sum_{n=0}^{\infty} \frac{1}{2(n+1)} f_n r^{2n+1}\right] \sin \omega t. \tag{2.A.4}$$

It can be shown that a cavity may resonate in two types of modes, one called a TE mode and the other a TM mode.[13] The standing wave-type linac that we are treating here is designed to resonate in a TM mode in which the magnetic field is entirely azimuthal. Hence we take $B_r = B_z = 0$ from which it follows that $\mathcal{E}_\theta = 0$. The transverse electric fields $\mathcal{E}_x$ and $\mathcal{E}_y$ are obtained from $\mathcal{E}_r$, and they are written below together with $\mathcal{E}_z$.

$$\mathcal{E}_x = -x \sum_{n=0}^{\infty} \frac{1}{2(n+1)} \frac{d f_n(z)}{dz} r^{2n} \cos \omega t .$$

$$\mathcal{E}_y = -y \sum_{n=0}^{\infty} \frac{1}{2(n+1)} \frac{d f_n(z)}{dz} r^{2n} \cos \omega t . \qquad (2.A.5)$$

$$\mathcal{E}_z = \sum_{n=0}^{\infty} f_n(z) \, r^{2n} \cos \omega t .$$

The magnetic field in a quadrupole magnet is produced by the use of currents flowing around four ferromagnetic poles. The faces of the poles are represented by the two hyperbolic surfaces $xy = a$ and $xy = -a$. The polarity of the pole faces is arranged so that magnetic field is antisymmetric with respect to reflection through both the $x = 0$ and $y = 0$ planes, and is symmetric with respect to reflection through both the $x = y$ and $x = -y$ planes. Then the magnetostatic potential U is given by $U = G x y$ where G is a constant. The transverse magnetic quadrupole fields B_x and B_y are given by

$$B_x = \frac{dU}{dx} = G y ,$$

$$B_y = \frac{dU}{dy} = G x . \qquad (2.A.6)$$

The fields given by equations (2.A.6) satisfy Maxwell's equations $\nabla \times \vec{B} = 0$ and $\nabla \cdot \vec{B} = 0$. The force on a particle of charge q moving parallel to the z axis with velocity v is given by

$$F_x = -qv B_y = -qv G x ,$$

$$F_y = qv B_x = qv G y . \qquad (2.A.7)$$

Thus for $G > 0$ the force is focusing in the x direction and defocusing in the y direction, while for $G < 0$ the force is focusing in the y direction and defocusing in the x direction. A quadrupole with $G > 0$ is a plus quadrupole, and a quadrupole with $G < 0$ is a minus quadrupole.

The Hamiltonian does not contain the fields but instead the potentials from which the fields are derivable. The total electromagnetic field, which is the sum of the radio-frequency field and the quadrupole field, can be derived from a vector potential $\vec{A}$ by the formulas

$$\vec{B} = \nabla \times \vec{A} \qquad \text{and} \qquad \vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

in which the gauge has been chosen to eliminate the scalar potential term. The three components of the vector $\vec{A}$ are given below, where $g(z) = G$ inside a quadrupole $g(z) = 0$ outside a quadrupole.

$$A_x = \frac{c}{\omega} x \sum_{n=0}^{\infty} \frac{1}{2(n+1)} \frac{d f_n(z)}{dz} r^{2n} \sin \omega t .$$

$$A_y = \frac{c}{\omega} y \sum_{n=0}^{\infty} \frac{1}{2(n+1)} \frac{d f_n(z)}{dz} r^{2n} \sin \omega t . \qquad (2.A.8)$$

$$A_z = -\frac{c}{\omega} \sum_{n=0}^{\infty} f_n(z) r^{2n} \sin \omega t - g(z) \frac{x^2 - y^2}{2} ,$$

with

$$r^2 = x^2 + y^2$$

and

$$f_{n+1} = -\frac{1}{4(n+1)^2} \left[\frac{d^2 f_n}{dz^2} + \frac{\omega^2}{c^2} f_n \right] .$$

To the approximation L may be considered constant, the radio-frequency fields in a linac are periodic in z with period L. We therefore expand $f_n(z)$ as a Fourier series in z, obtaining

$$f_n(z) = \sum_{m=0}^{\infty} C_{m,n} \cos\left(m \frac{2\pi}{L} z + \alpha_m\right) \quad , \text{ and}$$

$$\frac{d^2 f_n(z)}{dz^2} = -\sum_{m=0}^{\infty} m^2 K^2 C_{m,n} \cos\left(m \frac{2\pi}{L} z + \alpha_m\right) , \text{ in which } K = \frac{2\pi}{L}$$

and we have neglected terms of order $\Delta L / L$ (ΔL is the change in L in distance L). We let $C_{0,n}$ be the average value of $f_n(z)$ by choosing $\alpha_0 = 0$, and define the position where $z = 0$ by specifying $\alpha_1 = 0$.

We substitute for $d^2 f_n / dz^2$ and f_n in the recursion relationship for f_{n+1} and obtain

$$C_{m,n} = \frac{1}{2^{2n} n! n!} \left[\left(m^2 K^2 - \omega^2/c^2 \right)^{\frac{1}{2}} \right]^{2n} C_{m,0} .$$

We can therefore write the potential $\vec{A}$ as follows:

$$A_x = -\frac{c}{\omega} \frac{x}{r} \sum_{m=0}^{\infty} \frac{K^2 C_{m,0}}{q} I_1(qr) \sin(mKz + \alpha_m) \cos \omega t,$$

$$A_y = -\frac{c}{\omega} \frac{y}{r} \sum_{m=0}^{\infty} \frac{K^2 C_{m,0}}{q} I_1(qr) \sin(mKz + \alpha_m) \cos \omega t,$$

$$A_z = \frac{c}{\omega} \sum_{m=0}^{\infty} C_{m,0} I_0(qr) \cos(mKz + \alpha_m) \sin \omega t - g(z) \frac{x^2 - y^2}{2} \qquad (2.A.9)$$

$$q = \left(m^2 K^2 - \omega^2/c^2 \right)^{\frac{1}{2}}$$

in which $I_0(x) = J_0(ix)$. $I_1(x) = -i J_1(ix)$ and J_0 and J_1 are Bessel functions of the first kind.[14]

From the fact that

$$C_{m,0} = \frac{2}{L} \int f_0(z) \cos[mKz + \alpha_m] \, dz$$

and from the definitions $f_0 \equiv \frac{1}{L}\int_L f_0(z)\,dz$ (average value of the longitudinal electric field on axis),

$T_m \equiv \frac{1}{L f_0}\int_L f_0(z)\cos(mKz + \alpha_m)\,dz$ (transit time of the mth wave) and $\varphi \equiv Kz - \omega t$ (phase of the electromagnetic wave), we obtain

$$A_x = -\frac{c}{\omega}\frac{x}{r} f_0 \sum_{m=0}^{\infty}\left[S_m(z)\cos\varphi + U_m(z)\sin\varphi\right] I_1(\xi r),$$

$$A_y = -\frac{c}{\omega}\frac{y}{r} f_0 \sum_{m=0}^{\infty}\left[S_m(z)\cos\varphi + U_m(z)\sin\varphi\right] I_1(\xi r), \qquad (2.A.10)$$

$$A_z = \frac{c}{\omega} f_0 \sum_{m=0}^{\infty}\left[Q_m(z)\sin\varphi + R_m(z)\cos\varphi\right] I_0(\xi r) - g(z)\frac{x^2-y^2}{2}.$$

We have defined

$$Q(z) \equiv \sum_{m=0}^{\infty} Q_m(z) = \sum_{m=0}^{\infty} T_m\left\{\cos[(m+1)Kz + \alpha_m] + \cos[(m-1)Kz + \alpha_m]\right\},$$

$$R(z) \equiv \sum_{m=0}^{\infty} R_m(z) = \sum_{m=0}^{\infty} T_m\left\{\sin[(m+1)Kz + \alpha_m] - \sin[(m-1)Kz + \alpha_m]\right\},$$

$$S(z) \equiv \sum_{m=0}^{\infty} S_m(z) = \sum_{m=0}^{\infty} T_m\left\{-\cos[(m+1)Kz + \alpha_m] + \cos[(m-1)Kz + \alpha_m]\right\}, \qquad (2.A.11)$$

$$U(z) \equiv \sum_{m=0}^{\infty} U_m(z) = \sum_{m=0}^{\infty} T_m\left\{\sin[(m+1)Kz + \alpha_m] + \sin[(m-1)Kz + \alpha_m]\right\}.$$

For later use we also define

$$\bar{Q}(z) \equiv \sum_{m=0}^{\infty} m^2 T_m \left\{ \cos[(m+1)Kz + \alpha_m] + \cos[(m-1)Kz + \alpha_m] \right\},$$

$$\bar{R}(z) \equiv \sum_{m=0}^{\infty} m^2 T_m \left\{ \sin[(m+1)Kz + \alpha_m] - \sin[(m-1)Kz + \alpha_m] \right\}. \tag{2.A.12}$$

B. The Exact Hamiltonian

The relativistic Hamiltonian[15] $\mathcal{H}_0$ for a charged particle in an electromagnetic field is given below:

$$\mathcal{H}_0(x, P_x, y, P_y, z, P_z, t) =$$

$$c\sqrt{\left(P_x - \tfrac{e}{c}A_x\right)^2 + \left(P_y - \tfrac{e}{c}A_y\right)^2 + \left(P_z - \tfrac{e}{c}A_z\right)^2 + m_0^2 c^2}, \tag{2.B.1}$$

where P_x, P_y and P_z are the momenta canonically conjugate to the coordinates x, y and z, and $\mathcal{H}_0$ satisifies the modified Hamilton's principle,

$$\delta \int_{t_1}^{t_2} (P_x \dot{x} + P_y \dot{y} + P_z \dot{z} - \mathcal{H}_0)\, dt = 0 \tag{2.B.2}$$

The charge and rest mass of the proton are designated by e and m_0, and c is the velocity of light. Letting $\mathcal{H}_0 = E$, $x' = dx/dz$, $y' = dy/dz$ and $t' = dt/dt$, we can write Hamilton's principle as follows:

$$\delta \int_{z_1}^{z_2} [P_x x' + P_y y' + (-E)\, t' - (-P_z)]\, dz = 0 . \tag{2.B.3}$$

It follows that we can use for the Hamiltonian $-P_z(x, P_x, y, P_y, t, (-E), z)$ in which $(-E)$ plays the role of the momentum conjugate to t, and z is the independent variable. We then have as the expression for this new Hamiltonian

$$\mathcal{H}_1 = -P_z = -\left\{\left(\frac{E^2}{c^2} - m_0^2 c^2\right) - \left(P_x - \frac{e}{c} A_x(x, y, z, t)\right)^2 - \left(P_y - \frac{e}{c} A_y(x, y, z, t)\right)^2\right\}^{\frac{1}{2}} - \frac{e}{c} A_z(x, y, z, t) . \tag{2.B.4}$$

Next we can remove the linear part of the dependence of t upon the independent variable z by a canonical transformation from the variables $(-E, t)$ to new variables (η, φ). If we use the generating function[15]

$$G(t, \eta, z) = [Kz - \omega t]\, \eta , \tag{2.B.5}$$

we obtain

$$\frac{\partial G}{\partial \eta} = \varphi = (Kz - \omega t) , \quad \frac{\partial G}{\partial t} = -E = -\omega \eta . \tag{2.B.6}$$

We see that φ is the relative phase between the particle and the radio-frequency accelerating field. The Hamiltonian becomes

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial G}{\partial z} = \mathcal{H}_1 + K\eta \ ,$$

i.e.,

$$\mathcal{H}_2 = -\left\{\frac{\omega^2\eta^2}{c^2} - m_0^2c^2 - \left(P_x - \frac{e}{c}A_x(x,y,\varphi,z)\right)^2 - \left(P_y - \frac{e}{c}A_y(x,y,\varphi,z)\right)^2\right\}^{1/2} - \frac{e}{c}A_z(x,y,\varphi,z) + K\eta \qquad (2.B.7)$$

A synchronous particle is one for which $\varphi = \varphi_s$ where φ_s is a constant. In order to meet this condition, it is necessary to have $x \equiv P_x \equiv y \equiv P_y \equiv 0$. The synchronous particle will always have the same relative phase with respect to the accelerating field and hence will have an energy gain in distance L proportional to L, where L is the distance between the centers of two adjacent drift tubes. This definition determines both L and the energy of the synchronous particle as a function of z. We will use the subscript s to refer to quantities pertaining to the synchronous particles.

For the synchronous particle it follows that

$$\mathcal{H}_2 = -\left\{\frac{\omega_s^2\eta_s^2}{c^2} - m_0^2c^2\right\}^{1/2} - \frac{e}{\omega}f_0\left\{Q(z)\sin\varphi_s + R(z)\cos\varphi_s\right\} + K\eta_s \ , \qquad (2.B.8)$$

from which we obtain the following equations of motion:

$$\varphi_s' = \frac{\partial \mathcal{H}_2}{\partial \eta_s} = K - \frac{\omega^2}{c^2} \frac{\eta_s}{\left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}^{1/2}} = 0 , \qquad (2.B.9)$$

and

$$\eta_s' = -\frac{\partial \mathcal{H}_2}{\partial \varphi_s} = \frac{e}{\omega} f_0 \left\{ Q(z) \cos\varphi_s - R(z) \sin\varphi_s \right\}.$$

From the equation for φ_s' we obtain the synchronous condition

$$\frac{K c^2}{\omega^2} = \frac{\eta_s}{\left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}^{1/2}} . \qquad (2.B.10)$$

We then use the fact that

$$\eta = \frac{1}{\omega} E = \frac{m_0 c^2}{\omega (1-\beta_s^2)^{1/2}} \qquad \beta = \frac{\text{velocity of particle}}{\text{velocity of light}} \qquad (2.B.11)$$

and obtain the condition that

$$K = \frac{\omega}{\beta_s c} \qquad \text{or} \qquad L = \beta_s \lambda \qquad (2.B.12)$$

in which $\lambda = 2\pi c/\omega$ = wavelength of the rf field.

In order to study the deviation of general particle motion from synchronous particle motion, we define another transformation of the Hamiltonian from (η, φ) coordinates to new coordinates (P_φ, φ) by the transformation

$$G(\varphi, P_\varphi, z) = (P_\varphi + \eta_s)\varphi \tag{2.B.13}$$

From this transformation we obtain

$$\eta = \frac{\partial G}{\partial \varphi} = \eta_s + P_\varphi \quad , \quad \varphi = \frac{\partial G}{\partial \varphi} = \varphi \, , \tag{2.B.14}$$

and

$$\mathcal{H}_3 = \mathcal{H}_2 + \frac{\partial G}{\partial z} = \mathcal{H}_2 + \varphi \eta_s' \, ,$$

giving

$$\mathcal{H}_3 = \left\{ \frac{\omega^2}{c^2}\left(P_\varphi^2 + 2\eta_s P_\varphi + \eta_s^2\right) - m_0^2 c^2 \right.$$

$$\left. - \left[P_x - \tfrac{e}{c} A_x(x, y, \varphi, z)\right]^2 - \left[P_y - \tfrac{e}{c} A_y(x, y, \varphi, z)\right]^2 \right\}^{1/2} \tag{2.B.15}$$

$$- \tfrac{e}{c} A_z(x, y, \varphi, z) + K P_\varphi + K \eta_s' + \varphi \eta_s' \, .$$

A_x, A_y and A_z are given by equations (2.A.10).

C. The Approximate Hamiltonian

The exact Hamiltonian is too complicated to treat analytically, and it is therefore necessary to approximate the Hamiltonian by a simpler expression.

In order to approximate the above Hamiltonian, we need to examine the relative size of the terms that it contains. Since all

present linear designs are similar, we will use the parameters of the MURA linac, as are summarized in Table 2, to estimate the error of many of the approximations which will be made later. From the definition of $\eta = E/\omega$, it follows that $P_\varphi/(\eta_s - \frac{m_0 c^2}{\omega}) \leq \Delta E/(E - m_0 c^2)$ where ΔE is the maximum spread in energy of the particles in the linac. For the MURA linac parameters, $\Delta E/(E - m_0 c^2)$ is of the order of 1 percent so that we will be justified in expanding the Hamiltonian in a power series in $\frac{\omega^2}{c^2}\eta_s P_\varphi/(\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2)$.
Next we compare the sizes of the two quantities $(P_x - \frac{e}{c}A_x)^2$ and $(P_y - \frac{e}{c}A_y)^2$ with the size of the quanity $\left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}$.
We use again the definition $\eta = E/\omega$ to obtain

$$\left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}^{1/2} = \frac{m_0 \beta_s c}{\sqrt{1-\beta_s^2}} \quad . \tag{2.C.1}$$

It follows from the equations of motion, obtained from the original Hamiltonian, equation (2.B.1), that

$$(P_x - \frac{e}{c}A_x) = \frac{m_0 (\beta_s)_x c}{\sqrt{1-\beta_s^2}} \quad \text{and} \quad (P_y - \frac{e}{c}A_y) = \frac{m_0 (\beta_s)_y c}{\sqrt{1-\beta_s^2}} \tag{2.C.2}$$

$(\beta_s)_x = \frac{\dot{x}}{c}$ and $(\beta_s)_y = \frac{\dot{y}}{c}$. The motions of particles injected into linacs are such that the x and y components of the momentum are much smaller than the z component of the momentum, and therefore we can expand the Hamiltonian in powers of the quantities

TABLE 2

DESIGN PARAMETERS FOR THE MURA LINAC

Injection Energy (Mev)	0.75
Average Value of $f(z)$ (Mev/m)	2.4
Transit Time T_1	0.819
Final Energy (Mev)	200
Stable Phase Angle	26°
RF Frequency of Field (Cycles/sec)	2×10^{8}
Gradient of Quadrupole Field (Gauss/cm)	$160 \times \frac{\gamma_s}{\beta_s}$

$$\frac{\left(P_x - \frac{e}{c} A_x\right)^2}{\left(\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right)} \quad \text{and} \quad \frac{\left(P_y - e/c\, A_y\right)^2}{\left(\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right)}$$

We now expand the Hamiltonian, making use of the fact that $(P_x - e/c\, A_x)^2$, $(P_y - e/c\, A_y)^2$ and $\omega^2/c^2\ \eta_s P_\varphi$ are much less than $\left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}$. We define

$$W_s = \left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}^{1/2} = m_0 \beta_s \gamma_s c \ , \quad \gamma_s = \left(1 - \beta_s^2\right)^{-1/2} \ , \qquad (2.C.3)$$

and we neglect all terms which are of cubic or higher order in $(P_x - e/c\, A_x)/W_s$, $(P_y - e/c\, A_y)/W_s$ and $\frac{\omega^2}{c^2}\eta_s P_\varphi / W_s^2$. This still does not lead to linear equations of motion, because we do not neglect higher order terms in $(\varphi - \varphi_s)$. We thus obtain for the expanded Hamiltonian

$$\mathcal{H}_3 = -W_s \left\{ 1 + \frac{1}{2}\frac{\omega^2}{c^2}\frac{2\eta_s P_\varphi + P_\varphi^2}{W_s^2} - \frac{1}{8}\frac{\omega^4}{c^4}\frac{4\eta_s^2 P_\varphi^2}{W_s^4} \right.$$

$$\left. - \frac{1}{2}\frac{(P_x - e/c\, A_x)^2 + (P_y - e/c\, A_y)^2}{W_s^2} \right\}^{1/2} - \frac{e}{c} A_z + K P_\varphi + K \eta_s + \varphi \eta_s' \qquad (2.C.4)$$

Next we substitute for K and η_s' the expressions

$$K = \frac{\omega^2}{c^2} \frac{\eta_s}{\left\{\frac{\omega^2}{c^2}\eta_s^2 - m_0^2 c^2\right\}^{1/2}} = \frac{\omega^2}{c^2} \frac{\eta_s}{W_s} = \frac{1}{\beta_s} \frac{\omega}{c}$$

and

$$\eta_s' = \frac{e}{\omega} f_0 \left\{ Q(z) \cos \varphi_s - R(z) \sin \varphi_s \right\} ; \tag{2.C.5}$$

to obtain for the Hamiltonian

$$\begin{aligned} \mathcal{H}_3 = & - W_s + \frac{1}{2} \frac{p_\varphi^2}{W_s} \left(\frac{1}{\beta_s^2} - 1 \right) \frac{\omega^2}{c^2} + K \eta_s \\ & + \frac{e}{\omega} f_0 \left\{ Q(z) \cos \varphi_s - R(z) \sin \varphi_s \right\} \varphi - \frac{e}{c} A_z \\ & - \frac{1}{2} \frac{(p_x - \frac{e}{c} A_x)^2}{W_s} - \frac{1}{2} \frac{(p_y - \frac{e}{c} A_y)^2}{W_s} \end{aligned} \tag{2.C.6}$$

Because we can add or subtract an arbitrary function of z from the Hamiltonian and not affect the equations of motion, we will drop the terms W_s and $K \eta_s$. In addition, since $r/L \ll 1$, we will let $I_1(Kr) = \frac{Kr}{2}$ and $I_0(Kr) = 1 + \frac{K^2 r^2}{4}$ when we substitute for the vector potential $\vec{A}$. This gives an error in the Hamiltonian of the order $(r/L)^4$. We write out the expression for A_x, A_y and A_z in the Hamiltonian and use the fact that $W_s = m_0 \beta_s \gamma_s c$ to obtain for the Hamiltonian

$$\mathcal{H}_3 = \frac{\omega^2}{2E_0\beta_s^3\gamma_s^3 c} P_\varphi^2 + \frac{c}{2E_0\gamma_s\beta_s}(P_x^2 + P_y^2)$$

$$+ \frac{1}{E_0\gamma_s}\frac{ef_0}{2\beta_s^2}\left(S(z)\cos\varphi + U(z)\sin\varphi\right)(xP_x + yP_y)$$

$$+ \frac{e^2 f_0^2}{8E_0\beta_s^3\gamma_s c}\left(S(z)\cos\varphi + U(z)\sin\varphi\right)^2 (x^2+y^2)$$

(2.C.7)

$$- \frac{e}{\omega} f_0\left(R(z)\cos\varphi + Q(z)\sin\varphi\right)\left[1 - \frac{K^2\beta_s^2}{4}(x^2+y^2)\right]$$

$$- \frac{e}{\omega} f_0\left(\bar{R}(z)\cos\varphi + \bar{Q}(z)\sin\varphi\right)\left[\frac{K^2(x^2+y^2)}{4}\right]$$

$$- \frac{e\,g(z)}{c}\left(\frac{x^2-y^2}{2}\right) + \frac{ef_0}{\omega}\varphi\left(Q(z)\cos\varphi_s - R(z)\sin\varphi_s\right),$$

where $E_0 = m_0 c^2$ and the quantities $Q(z)$, $\bar{Q}(z)$, $R(z)$ $\bar{R}(z)$, $S(z)$ and $U(z)$ are defined by equations (2.A.11) and (2.A.12).

Because the period of the field in the variable z is not a constant, it is desirable to change from the independent variable z

to θ , where $d\theta = K/N \; dz$ and N is the number of quadrupoles per period. Again we return to Hamilton's principle

$$\delta \int_{z_0}^{z_1} \left(\sum P_i q_i' - \mathcal{H}_3(z) \right) dz = \delta \int_{\theta_0}^{\theta_1} \left(\sum P_i q_i' \frac{dz}{d\theta} - \mathcal{H}_3(z) \frac{dz}{d\theta} \right) d\theta$$

$$= \delta \int_{\theta_0}^{\theta_1} \left(\sum P_i \frac{dq_i}{d\theta} - \mathcal{H}_4(\theta) \right) d\theta = 0 . \tag{2.C.8}$$

Thus $\mathcal{H}_4(\theta)$ is the new Hamiltonian, where

$$\mathcal{H}_4(\theta) = \frac{dz}{d\theta} \mathcal{H}_3(z) = \frac{N \beta_s c}{\omega} \mathcal{H}_3(z(\theta)). \tag{2.C.9}$$

Next we remove the cross term between x and P_x , and y and P_y by transformation to x_1 , P_{x_1} , y_1 , P_{y_1} , φ_1 , and P_{φ_1} , with the generating function

$$G(x, P_{x_1}, y, P_{y_1}, \varphi, P_{\varphi}, \theta) = \left(\frac{E_0 \gamma_s \omega}{N c^2} \right)^{1/2} (x P_{x_1} + y P_{y_1})$$

$$+ \varphi P_{\varphi_1} - \frac{e f_0}{2 \beta_s c} \left[S(\theta) \cos\varphi + U(\theta) \sin\varphi \right] \left(\frac{x^2 + y^2}{2} \right) \tag{2.C.10}$$

From the generating function, we obtain

$$P_x = \frac{\partial G}{\partial x} = \left(\frac{E_0 \gamma_s \omega}{N c^2}\right)^{1/2} P_{x_1} - \frac{e f_0}{2 \beta_s c}\left[S(\theta)\cos\varphi + U(\theta)\sin\varphi\right] x ,$$

$$x_1 = \frac{\partial G}{\partial P_{x_1}} = \left(\frac{E_0 \gamma_s \omega}{N c^2}\right) x ,$$

$$P_y = \frac{\partial G}{\partial y} = \left(\frac{E_0 \gamma_s \omega}{N c^2}\right)^{1/2} P_{y_1} - \frac{e f_0}{2 \beta_s c}\left[S(\theta)\cos\varphi + U(\theta)\sin\varphi\right] y ,$$

$$y_1 = \frac{\partial G}{\partial P_{y_1}} = \left(\frac{E_0 \gamma_s \omega}{N c^2}\right)^{1/2} y ,$$

$$P_\varphi = \frac{\partial G}{\partial \varphi} = P_{\varphi_1} + \frac{e f_0}{2 \beta_s c}\left[S(\theta)\sin\varphi - U(\theta)\cos\varphi\right]\frac{x^2+y^2}{2} \simeq P_{\varphi_1} , \qquad (2.C.11)$$

$$\varphi_1 = \frac{\partial G}{\partial P_{\varphi_1}} = \varphi ,$$

$$\mathcal{H}_s = \mathcal{H}_1 + \frac{\partial G}{\partial \theta} = \mathcal{H}_1 - \frac{e f_0}{2 \beta_s c}\left[\frac{dS}{d\theta}\cos\varphi + \frac{dU}{d\theta}\sin\varphi\right]\left(\frac{x^2+y^2}{2}\right)$$

$$+ \frac{1}{\gamma_s}\frac{d\gamma_s}{d\theta}\left(\frac{E_0 \gamma_s \omega}{N c^2}\right)^{1/2}\left(P_{x_1} x + P_{y_1} y\right)$$

$$+ \frac{1}{\beta_s}\frac{d\beta_s}{d\theta}\frac{e f_0}{2 \beta_s c}\left[S(\theta)\cos\varphi + U(\theta)\sin\varphi\right]\left(\frac{x^2+y^2}{2}\right)$$

The third term in the new Hamiltonian can be neglected because the change in γ_s over one period of the x or y motion is less than

1 percent of γ_s . The last term will be neglected because $\frac{1}{\beta_s}\frac{d\beta_s}{d\theta} = \frac{1}{2\pi}\frac{\Delta L}{L} < .01 << 1$ where ΔL is the change in L in the distance L. We also neglect the new term of order $x^2 P_\varphi$ that arises from the transformation, because we have neglected terms of this order before. The new Hamiltonian which results is

$$\mathcal{H}_s = \frac{N\omega P_\varphi^2}{2E_0\beta_s^2\gamma_s^3} + \frac{P_{x_1}^2 + P_{y_1}^2}{2} - \frac{eN^2 g(\theta)\beta_s c^2}{E_0\gamma_s\omega^2}\left(\frac{x_1^2 - y_1^2}{2}\right)$$

$$+ \frac{Nef_0\beta_s c}{\omega^2}\left[Q(\theta)(\varphi\cos\varphi_s - \sin\varphi) - R(\theta)(\varphi\sin\varphi_s + \cos\varphi)\right] \tag{2.C.12}$$

$$- \frac{ef_0N^2c}{2E_0\beta_s\gamma_s\omega}\left[\left\{\left(\bar{R}(\theta) - \beta_s^2 R(\theta)\right) + \frac{dS}{d\theta}\right\}\cos\varphi\right.$$

$$\left. + \left\{\left(\bar{Q}(\theta) - \beta_s^2 Q(\theta)\right) + \frac{dU}{d\theta}\right\}\sin\varphi\right]\left(\frac{x_1^2 + y_1^2}{2}\right).$$

D. Carrier Wave Approximation

In this section we shall justify neglecting the θ or z dependence of the functions Q , $\bar{Q}$, R , $\bar{R}$, U and S . This neglect means using the average values of these quantities in the Hamiltonian.

First we will treat the φ motion. The equations of motion are

$$\frac{d\varphi}{d\theta} = \frac{N\omega}{E_0 \beta_s^2 \gamma_s^3} P_\varphi = (\varphi - \varphi_s)' , \tag{2.D.1}$$

$$\frac{dP_\varphi}{d\theta} = -\frac{Ne f_0 \beta_s c}{\omega^2}\left[Q\cdot(\varphi-\varphi_s)\sin\varphi_s + R\cdot(\varphi-\varphi_s)\cos\varphi_s\right]$$

$+$ higher order terms in X_1, P_{X_1} and $(\varphi - \varphi_s)$.

Combining the above equations, we obtain

$$\left\{\left[\frac{E_0 \beta_s^2 \gamma_s^3}{N\omega}\right](\varphi-\varphi_s')\right\}' + \frac{Ne f_0 \beta_s c}{\omega^2}\left[Q\sin\varphi_s + R\cos\varphi_s\right](\varphi-\varphi_s) \tag{2.D.2}$$

$=$ higher order terms.

Neglecting for the moment the z dependence of β_s and γ_s, we can write the φ motion equation as

$$(\varphi-\varphi_s)'' + \left[\sum_{n=0}^{\infty} a_n \cos(nN\theta + \alpha_n)\right](\varphi-\varphi_s) \tag{2.D.3}$$

$=$ higher order terms.

There are two reasons to keep terms in the Fourier series other than the constant term. The first is that the linear tune may depend strongly on the higher harmonics. The second is that one of

the harmonics might drive a resonance (either linear or nonlinear). Because the T_m's are of the order of unity or smaller, then

$$a_n \sim \frac{N^2 e f_0 c}{E_0 \beta_s \gamma_s^3 \omega} .$$

We use the smooth approximation[16] to calculate the tune

$$\nu_\varphi^2 \simeq a_0 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n^2}{n^2 N^2} . \qquad (2.D.4)$$

Now if

$$\left(e f_0 c \,/\, 2 E_0 \beta_s \gamma_s^3 \omega \right) \ll 1 ,$$

we can let

$$\nu_\varphi^2 \simeq a_0 \qquad (2.D.5)$$

We use the design parameters for the MURA linac from Table 1 and calculate that

$$\left(e f_0 c \,/\, 2 E_0 \beta_s \gamma_s^3 \omega \right) \sim 10^{-2} \quad \text{at injection energy}$$

and smaller than 10^{-2} at higher energies. Thus the linear tune is given very accurately by the assumption $a_n = 0$ for $n \neq 0$. Because $\nu_\varphi \simeq a_0^{1/2} \simeq .24$, and because the first linear resonance occurs at $\nu_\varphi \simeq 2$, then none of the harmonics can drive a linear resonance. It will be shown later, when we derive an expression for ν_x and ν_y , the linear x and y tunes, that only the constant term will drive non-linear resonances

Next we treat the X motion to show that we can neglect the periodic θ dependence of the functions Q, $\bar{Q}$, R, $\bar{R}$, S, and U. It will not, however, be possible to replace $g(\theta)$ by its average value (the average value of $g(\theta) = 0$). The x_1 equations of motion are

$$\frac{dx_1}{d\theta} = P_{x_1} ,$$

$$\frac{dP_x}{d\theta} = \frac{e N^2 g(\theta) \beta_s c^2}{E_0 \gamma_s \omega^2} x_1 \tag{2.D.6}$$

$$+ x_1 \frac{e f_0 c N^2}{2 E_0 \beta_s \gamma_s \omega} \left\{ \left(\bar{R} - \beta_s R + \frac{dS}{d\theta}\right) \cos\varphi_s + \left(\bar{Q} - \beta_s Q + \frac{dU}{d\theta}\right) \sin\varphi_s \right\}$$

$+$ higher order terms in $x_1 (\varphi - \varphi_s)$.

Thus the x equation is of the form

$$x'' + \left[\sum_{n=0}^{\infty} b_n \cos(nN\theta + \delta_n) - \frac{e N^2 \beta_s c^2}{E_0 \gamma_s \omega^2} \sum_{n=0}^{\infty} g_n \cos(n\theta + \gamma_n) \right. \tag{2.D.7}$$

$=$ higher order terms,

with $$g(\theta) = \sum_{n=0}^{\infty} g_n \cos(n\theta + \gamma_n).$$

From the smooth approximation we obtain

$$\nu_x^2 = + b_0 + \frac{1}{2}\left(\frac{eN^2c^2\beta_s}{E_0\gamma_s\omega^2}\right)^2 \sum_{\substack{n=1 \\ n \neq mN}}^{\infty} \frac{g_n^2}{n^2} \tag{2.D.8}$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m^2N^2}\left[\left(\frac{eN^2c^2\beta_s}{E_0\gamma_s\omega^2}\right) g_{mN} + b_m\right]^2$$

It is seen that contribution to the value of ν_x^2 of g_1^2 is N^2 times the contribution of $g_N b_1$, thus if

$$\frac{eN^2c^2\beta_s}{E_0\gamma_s\omega^2}\,|g_1| \gg \left|\frac{g_N}{g_1}\right| \frac{1}{N^2}|b_1|$$

we can neglect all 's for .

We again use the MURA design parameters to obtain

$$\frac{eN^2c^2\beta_s}{E_0\gamma_s\omega^2}\,|g_1| \sim .3$$

$$\left|\frac{g_N}{g_1}\right| \frac{1}{N^2}|b_R| < \frac{1}{N^2} b_R \sim \frac{ef_0c}{2E_0\gamma_s\beta_s\omega} \sim 5\times10^{-3}$$ at injection.

Therefore, for calculating the tune, we can ignore b_m for $m \neq 0$. It is also a fact that, if the b_m for $m \neq 0$ term drives a resonance, that the g_{mN} term will drive the same resonance and since $|g_{mN}| \gg |b_m|$ we still can neglect b_m for $m \neq 0$ In calculating the tune

$$\nu_x^2 \simeq b_0 + \frac{1}{2}\left(\frac{e N^2 c^2 \beta_s}{E_0 \gamma_s \omega^2}\right)^2 \sum \frac{g_n^2}{n^2} \quad , \tag{2.D.9}$$

we obtain for the MURA linac at injection

$$\nu_x \simeq .18.$$

Thus nonlinear resonances of the type $m\,\nu_x + n\,\nu_y = p$, where m , n , and p are integers, are important only for $p = 0$.

We have shown above that we can neglect the θ dependence of the functions Q , $\bar{Q}$, R , $\bar{R}$, S , and U , and we can use the average values of these functions in the Hamiltonian. We denote the average value by the subscript a , and we note that in terms of the transit time of the fundamental wave , T_1 ,

$$\begin{aligned} Q_a &= T_1 & R_a &= 0 \\ \bar{Q}_a &= T_1 & \bar{R}_a &= 0 \\ S_a &= T_1 & U_a &= 0 \end{aligned} \tag{2.D.10}$$

Thus we now can write the Hamiltonian as follows

$$\mathcal{H}_4 = \frac{N\omega}{2E_0\beta_s^2\gamma_s^3} P_\varphi^2 + \frac{N e f_0 T \beta_s c}{\omega^2}(\varphi \cos\varphi_s - \sin\varphi)$$

$$+ \frac{P_{x_1}^2 + P_{y_1}^2}{2} - \frac{e N^2 g(\theta)\beta_s c^2}{E_0 \gamma_s \omega}\left(\frac{X_1^2 - Y_1^2}{2}\right) \qquad (2.D.11)$$

$$- \frac{e f_0 c N^2 T}{2 E_0 \beta_s \gamma_s^3 \omega} \sin\varphi \left(\frac{X_1^2 + Y_1^2}{2}\right),$$

where we have dropped the subscript from T_1 .

From this approximation we obtain the linearized equations of motion:

$$\left(\frac{E_0 \beta_s^2 \gamma_s^3}{N\omega}(\varphi - \varphi_s)'\right)' + \frac{N e f_0 T \beta_s c \sin\varphi_s}{\omega^2}(\varphi - \varphi_s) = 0$$

(2.D.12)

and

$$X_1'' - \left[\frac{e f_0 N^2 c T \sin\varphi_s}{2 E_0 \beta_s \gamma_s^3 \omega} + \frac{e N^2 g(\theta)\beta_s c^2}{E_0 \gamma_s \omega^2}\right] X_1 = 0$$

We use the smooth approximation to obtain the following expressions for the linear tunes:

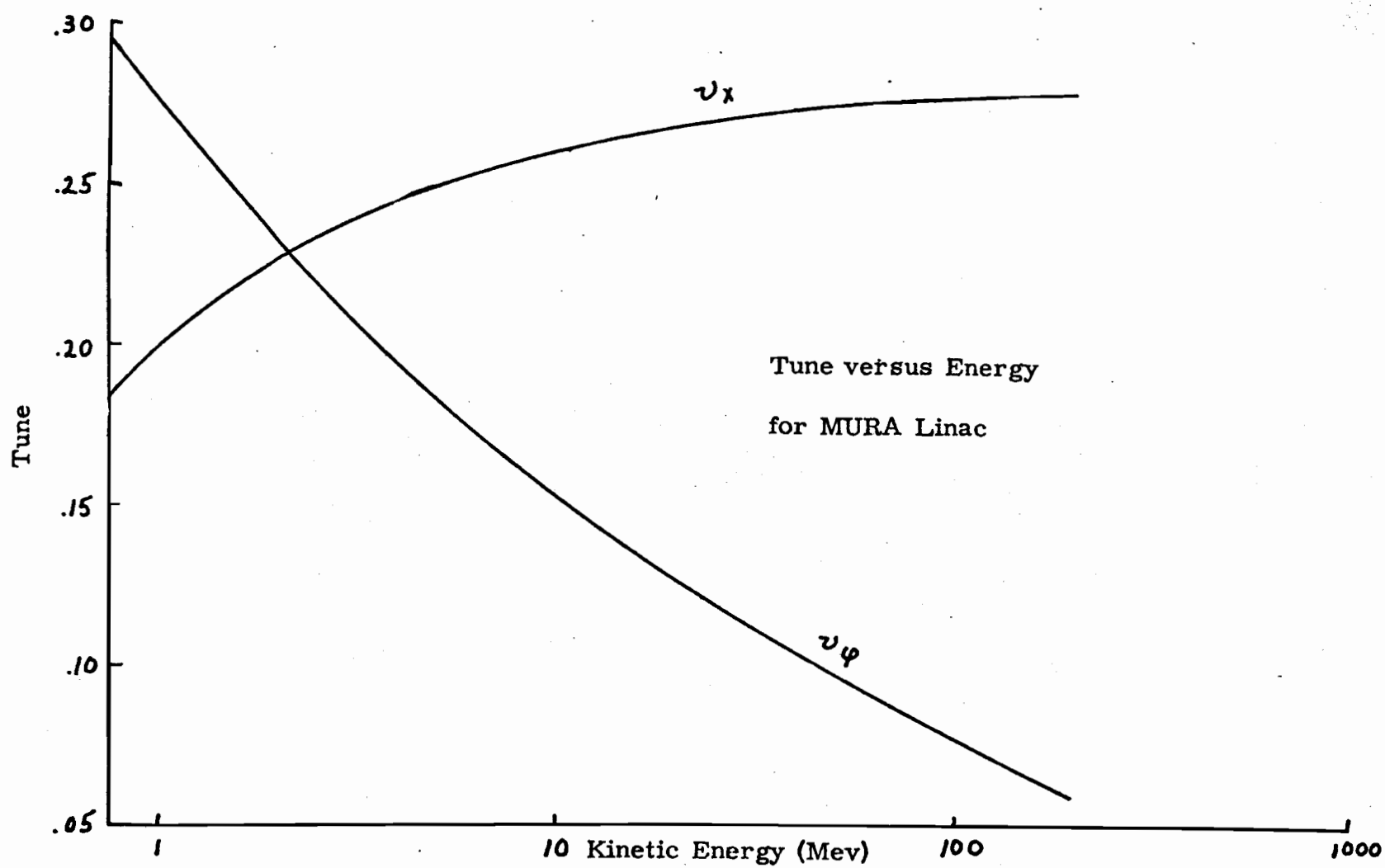
$$\nu_\varphi^2 = \frac{N^2 e f_0 T c \sin\varphi_s}{E_0 \beta_s \gamma_s^3 \omega} , \tag{2.D.13}$$

$$\nu_x^2 = - \frac{N^2 e f_0 T c \sin\varphi_s}{2 E_0 \beta_s \gamma_s^3 \omega} + \frac{e N^2 c^2}{2 E_0 \omega} \frac{\beta_s}{\gamma_s} \sum_{n=1}^{\infty} \frac{g_n^2}{n^2} \tag{2.D.14}$$

where g_n is defined by the equation

$$g(\theta) = \sum_{n=0}^{\infty} g_n \cos(n\theta + \tau_n) . \tag{2.D.15}$$

Because the expressions for the tunes contain the quantities β_s and γ_s, the tunes are functions of energy. Graph 1 shows the variation of ν_φ and ν_x with energy for the MURA linac.


.30
.25
.20
.15
.10
.05
Tune
ν_x
ν_φ
Tune versus Energy
for MURA Linac
1
10 Kinetic Energy (Mev)
100
1000

Graph 1

III. TRANSFORMATION OF φ MOTION INTO ACTION-ANGLE VARIABLES

A. Relationships Between Old and New Variables

From the expression for $\mathcal{H}_5$ (equation (2.D.11)), it is seen that the Hamiltonian is not a constant of the motion for two reasons. The first reason is that β_s and γ_s are not constant but instead vary slowly with θ. The second type of variation arises from the term $g(\theta)$. For treatment of pure φ motion and also for later work it will be advantageous to obtain a constant of the motion for

$x_1 \equiv p_{x_1} \equiv y_1 \equiv p_{y_1} \equiv 0$, the case of pure phase motion. For this case, only the θ dependence of β_s and γ_s is important.

Because the trigonometric function in the Hamiltonian is difficult to treat analytically, it is necessary to approximate it with another function. A Taylor series expansion is, however, not satisfactory, because φ varies over a range of approximately 3 φ_s, which in most linacs is of order $\pi/2$. If we expand $\sin\varphi$ in Chebyshev polynomials and then expand the result in terms of a power series, we can obtain results with adequate accuracy. Expanding about $\varphi = 0$ and keeping only terms cubic or lower, we obtain

$$\sin\varphi = \frac{J_1(2\varphi_s) + 3J_3(2\varphi_s)}{\varphi_s}\varphi - \frac{J_3(2\varphi_s)}{\varphi_s^3}\varphi^3, \quad (3.A.1)$$

TABLE 3

APPROXIMATION TO SIN φ

φ (rad.)	$\left[\begin{array}{c}(J_1(2\varphi_s)+3J_3(2\varphi_s))\frac{\varphi}{\varphi_s} \\ -J_3(2\varphi_s)\left(\frac{\varphi}{\varphi_s}\right)^3\end{array}\right]$	$\sin\varphi$
0	0	0
0.1	0.099670	0.099833
.2	.198391	.198669
.3	.295213	.295520
.4	.389186	.389418
.5	.479361	.479426
.6	.564712	.564642
.7	.644517	.644218
.8	.717600	.717356
.9	.783086	.783327

where φ is limited by $-\varphi_s \leq \varphi \leq \approx 2\varphi_s$ and J_n is the nth order Bessle function of the first kind.

Table 3 lists both the value of sin φ and the value obtained from the above approximation for various values of φ, where φ_s is equal to 0.4537856 rad or 26°. From this table it follows that the maximum error in the approximation over the interval $-\varphi_s \leq \varphi \leq 2\varphi_s$ is 0.0003.

We substitute for sin φ and obtain the following expression for the Hamiltonian:

$$\mathcal{H}_s = A\left\{\frac{P_\varphi^2}{B} - a^2\varphi + \varphi^3\right\} + \frac{P_{x_1}^2 + P_{y_1}^2}{2} - C\left(b^2\varphi - \varphi^3\right)\left(\frac{x_1^2 + y_1^2}{2}\right) - D\left(\frac{x_1^2 - y_1^2}{2}\right). \tag{3.A.2}$$

in which

$$A = A(\theta) = \frac{N e f_0 T \beta_s c}{\omega} \frac{J_3(2\varphi_s)}{\varphi_s^3},$$

$$B = B(\theta) = \frac{2 e f_0 T E_0 c \beta_s^3 \gamma_s^3}{\omega^3} \frac{J_3(2\varphi_s)}{\varphi_s^3},$$

$$C = C(\theta) = \frac{e f_0 T c N^2}{2 E_0 \beta_s \gamma_s^3 \omega} \frac{J_3(2\varphi_s)}{\varphi_s^3}, \tag{3.A.3}$$

$$a^2 = \frac{\varphi_s^3}{J_3(2\varphi_s)} \left(\frac{J_1(2\varphi_s) + 3 J_3(2\varphi_s)}{\varphi_s} - \cos\varphi_s \right),$$

$$b^2 = \frac{\varphi_s^3}{J_3(2\varphi_s)} \left(\frac{J_1(2\varphi_s) + 3 J_3(2\varphi_s)}{\varphi_s} \right),$$

and $$D = D(\theta) = \frac{e N^2 \beta_s c^2}{E_0 \gamma_s \omega} g(\theta).$$

We now transform this Hamiltonian from (φ, P_φ) variables to w, J, variables in order to obtain a constant of the motion for particles on the axis. We use the generating function

$$G(\varphi, J, \theta) = B^{\frac{1}{2}} \int_0^{\varphi} \sqrt{\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\, a^3 \ell(J) + a^2\varphi - \varphi^3}\, d\varphi, \tag{3.A.4}$$

where $\ell(J)$ is a function of J yet to be determined. We restrict ℓ to the limits $-1 \leq \ell \leq 1$ so that there are three real roots, α, β and γ, of the equation $\frac{2}{3}\sqrt{\frac{1}{3}}\, a^3 \ell + a^2 \varphi - \varphi^3 = 0$, ordered so that $\gamma \leq \beta \leq \alpha$. From this transformation we obtain the expression for P_φ:

$$P_\varphi = \frac{\partial G}{\partial \varphi} = B^{\frac{1}{2}} \sqrt{\frac{2}{3}\sqrt{\frac{1}{3}}\, a^3 \ell + a^2 \varphi - \varphi^3} \quad . \tag{3.A.5}$$

The turning points for the motion occur for values of φ such that $P_\varphi = 0$. There are thus three turning points, α, β and γ. For the range $\beta \leq \varphi \leq \alpha$, P_φ is a real quantity, while for the range $\gamma \leq \varphi \leq \beta$, P_φ is an imaginary quantity. Thus the particle oscillates between $\varphi = \beta$ and $\varphi = \alpha$ with a period in θ equal to $2\pi / \nu_\varphi$. The roots α, β and γ, together with the period τ are functions of ℓ.

We now define ℓ as a function of J. From the generating function we obtain the expression for w:

$$w = \frac{\partial G}{\partial \ell} = \frac{1}{3}\sqrt{\frac{1}{3}}\, \frac{d\ell}{dJ}\, a^3 B^{\frac{1}{2}} \int_\beta^\varphi \frac{d\varphi}{\sqrt{\frac{2}{3}\sqrt{\frac{1}{3}}\, a^3 \ell + a^2 \varphi - \varphi^3}} \quad . \tag{3.A.6}$$

We now require that Δw over one period of the φ oscillation be equal to unity, thus fixing the function $d\ell/dJ$.

$$\Delta w = 1 = \tfrac{2}{3}\sqrt{\tfrac{1}{3}}\,\frac{d\ell}{dJ}\,a^3 B^{1/3} \int_\beta^\alpha \frac{d\varphi}{\sqrt{\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\,a^3\ell + a^2\varphi - \varphi^3}}\,, \qquad (3.A.7)$$

or

$$\frac{dJ}{d\ell} = 2B^{1/3}\,\frac{d}{d\ell}\int_\beta^\alpha \sqrt{\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\,a^3\ell + a^2\varphi - \varphi^3}\;d\varphi\,. \qquad (3.A.8)$$

We integrate $dJ/d\ell$ and, because the equations of motion are independent of the integration constant, we set the integration constant equal to zero. This now gives us J as a function of ℓ .

$$J = 2B^{1/2}\int_\beta^\alpha \sqrt{\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\,a^3\ell + a^2\varphi - \varphi^3}\;d\varphi\,. \qquad (3.A.9)$$

Note that it follows from equation (3.A.5) that J is the action or phase area integral:

$$J = 2\int_{\beta}^{\alpha} p_\varphi \, d\varphi = \oint p_\varphi \, d\varphi \tag{3.A.10}$$

and hence is an adiabatic constant of the motion.[17]

In Appendix I we evaluate this integral and show that

$$J = \frac{4}{15} B^{\frac{1}{2}} a^{\frac{5}{2}} f_2(\ell) \tag{3.A.11}$$

in which

$$f_2(\ell) = \left(\frac{\alpha-\gamma}{a}\right)^{\frac{5}{2}} \left\{ (2k^4 - 2k^2 + 2)\, E(k^2) - (k^4 - 3k^2 + 2)\, K(k^2) \right\}, \tag{3.A.12}$$

$$k^2 = \left(\frac{\alpha-\beta}{\alpha-\gamma}\right).$$

$K(k^2)$ is the complete elliptic integral of the first kind[18] and $E(k^2)$ is the complete elliptic integral of the second kind. The roots α, β, and γ are linear functions of a and hence $f_2(\ell)$ is independent of a. Thus f_2 is independent of any of the linac parameters and, once it is determined as a function of ℓ, it will apply to any linac. If φ_s is small compared to 2π, we can expand a as a function of φ_s to obtain $a \approx \sqrt{3}\,\varphi_s$, and substituting

this into the expression for J we note that J, which for $\ell = 1$ is equal to the maximum stable longitudinal phase space J_m, varies as $\varphi_s^{5/2}$.

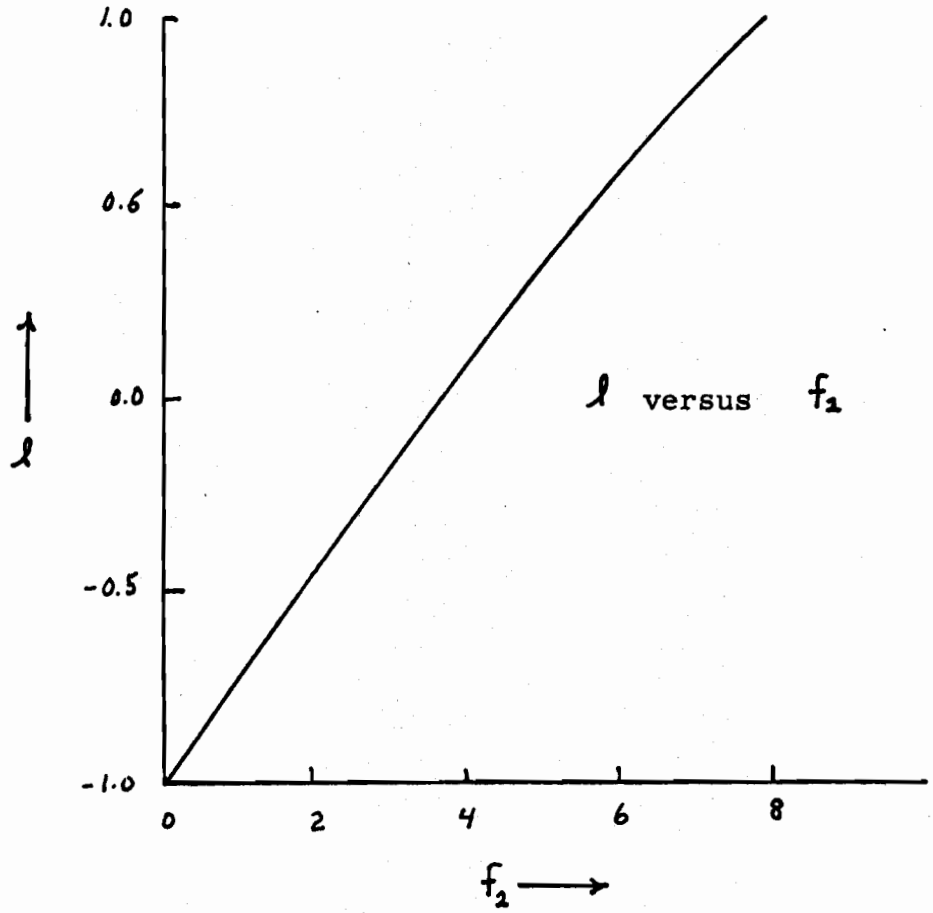
We want ℓ as a function of J, or the equivalent ℓ as a function of f_2, in order to obtain the Hamiltonian as a function of J and w. The expression for $f_2(\ell)$ is rather complicated to be inverted as a function $\ell(f_2)$, because the roots α, β, and γ are not simple functions of ℓ, and therefore we will obtain an approximate expression for $\ell(f_2)$. Since the equations of motion will contain derivatives of ℓ with respect to f_2, it really is the function $d\ell/df_2$ which we will approximate and then integrate to obtain $\ell(f_2)$. In Appendix I we have evaluated $d\ell/df_2$ which is given below:

$$\frac{d\ell}{df_2} = \frac{\sqrt{3}}{5}\left(\frac{\alpha-\gamma}{a}\right)^{1/2}\frac{1}{K(k^2)} \qquad (3.A.13)$$

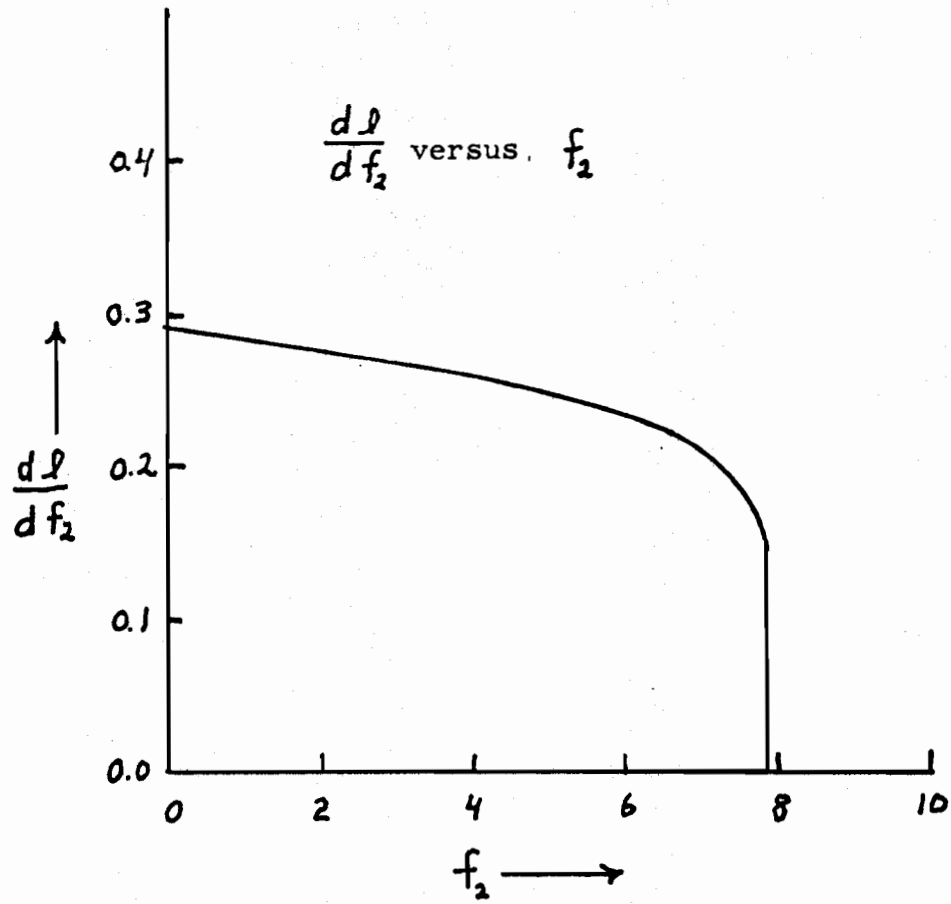
Graphs 2 and 3 show ℓ and $d\ell/df_2$ as functions of f_2. It can easily be shown that within an error of 0.5 percent,

$$\frac{d\ell}{df_2} = \frac{1.725}{5.94 - \ln\left(1 - f/f_m\right)}, \qquad (3.A.14)$$

with $f_m = f_m(\ell = 1) = 7.89644$



Graph 2



Graph 3

Upon integrating this function, we obtain

$$\ell = 1 - 5194 \cdot \Gamma\left(0,\ 5.94 - \ln\left(1 - f_2/f_m\right)\right), \qquad (3.A.15)$$

in which

$$\Gamma(0, x) = \int_x^\infty \frac{e^{-t}}{t}\, dt = E_i(-x).$$ [19]

We could use the asumptotic form for $\Gamma(0, x)$,

$$\Gamma(0, x) = \frac{e^{-x}}{x}\left\{1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right\},$$

and then expand $\ln\left(1 - f_2/f_m\right)$ as a series in f_2. Instead we approximate $d\ell/d f_2$ by the function

$$\frac{d\ell}{df_2} = .293 - .00884\, f_2 \qquad (3.A.16)$$

which gives

$$\ell = -1 + .293\, f_2 - .00442\, f_2^2 \qquad (3.A.17)$$

This approximation for ℓ has a maximum error of about 3 percent; however, for $f_2/f_m = .9$, the error in $d\ell/df_2$ is about 10 percent, while for $f_2/f_m > .9$ the error in $d\ell/d f_2$ becomes much larger. Thus for particles near the

stability boundary $(f_2 = f_m)$, we have to use the approximation for ℓ which contains the incomplete gamma function, while for particles away from the stability boundary $(f_2 \leq .9 f_m)$ we can use the power series expansion for ℓ. From the plot of $d\ell / d f_2$ vs. f_2, it is seen that for particles away from the stability boundary the linear φ tune, which is proportional to $d\ell / d f_2$, varies very slowly with f_2, but for particles near the stability boundary ν_φ is very strongly dependent on f_2.

We combine equations (3.A.11) and (3.A.17) to obtain for particles with small enough φ amplitudes

$$\ell(J) = \left\{ -1 + .293 \left(\frac{15}{4}\right) \frac{J}{a^{5/3} B^{1/2}} - .00442 \left(\frac{15}{4}\right)^2 \frac{J^2}{a^5 B} \right\}. \tag{3.A.18}$$

In addition, by inverting equation (3.A.5), we obtain

$$\ell = \tfrac{3}{2}\sqrt{3}\ \frac{1}{a^3} \left\{ \frac{P_\varphi^2}{B} - a^2\varphi + \varphi^3 \right\}. \tag{3.A.19}$$

Thus for particles with small enough φ amplitudes we write for the first term in the Hamiltonian the following function of J:

$$A \left\{ \frac{P_\varphi^2}{B} - a^2\varphi + \varphi^3 \right\} =$$

$$\frac{2}{3} \frac{a^3}{\sqrt{3}} A \left[-1 + .293 \left(\frac{15}{4}\right) \frac{J}{a^{5/2} B^{1/2}} - .00442 \left(\frac{15}{4}\right)^2 \frac{J^2}{a^5 B} \right\}. \tag{3.A.20}$$

The next step in the transformation is to write φ as a function of w and J. We rewrite the expression for w as

$$3\sqrt{3}\,\frac{dJ}{d\ell}\,\frac{1}{a^3 B^{1/2}}\,w = \int_{\beta}^{\varphi} \frac{d\varphi}{\sqrt{(\alpha-\varphi)(\varphi-\beta)(\varphi-\gamma)}} \,. \tag{3.A.21}$$

With the aid of the substitution

$$\varphi = \frac{\beta - \gamma k^2 \sin^2 u}{1 - k^2 \sin^2 u} \,, \tag{3.A.22}$$

the integral can be performed and the w equation becomes

$$\frac{4\,K(k^2)}{(\alpha-\gamma)^{1/2}}\,w = \frac{2}{(\alpha-\gamma)^{1/2}}\,sn^{-1}\left[\frac{1}{k^2}\left(\frac{\varphi-\beta}{\varphi-\gamma}\right), k\right] , \tag{3.A.23}$$

in which sn^{-1} is an inverse Jacobian elliptic function.[18] Solving for φ gives

$$\varphi/a = \frac{\beta/a - \gamma/a\, k^2\, sn^2(2Kw, k)}{dn^2(2Kw, k)} \,, \tag{3.A.24}$$

with

$$dn^2 = 1 - k^2 sn^2 \,.$$

The expansion for $sn\,u$ and $dn\,u$ are of the following form:[18]

$$\operatorname{sn} u = \sum_{s=0}^{\infty} a_s(k^2) \sin\left(\frac{(2s+1)\pi u}{2K}\right),$$

$$\operatorname{dn} u = \sum_{s=1}^{\infty} b_s(k^2) \cos\left(\frac{s\pi u}{2K}\right). \tag{3.A.25}$$

Hence we can write φ/a, which is an even function of w, as

$$\varphi/a = \sum_{n=m=0}^{m=\infty,\ n=\infty} a_{m,n} f_2^{m/2} \cos 2\pi n w. \tag{3.A.26}$$

Since the coefficients $a_s(k^2)$ and $b_s(k^2)$ decrease very rapidly with s except for $k^2 = 1$, we again restrict ourselves to amplitudes away from the stability boundary. The expansion for φ/a is given below:

$$\begin{aligned}\varphi/a = (.577 - .0157 f_2) - .265 f_2^{3/2} \cos 2\pi w \\ + .0473 f_2 \cos 4\pi w + \text{Terms}(f_2^{3/2}, f_2^2, f_2^{5/2}, \ldots).\end{aligned} \tag{3.A.27}$$

B. Adiabatic Invariance of the Action

The final step in the transformation is to examine the term $\partial G/\partial\theta$. We will show that if β varies slowly enough with θ we can neglect the term $\partial G/\partial\theta$ in the transformation and therefore the new Hamiltonian will be equal to the old Hamiltonian with φ and p_φ written in terms of the new variables w and J. Because J is the action integral, the neglect of $\partial G/\partial\theta$ corresponds to the neglect

of the change in J with θ . The approximation which neglects the charge in J with θ is called the adiabatic approximation. This approximation can be proved in general[17] to be exact, for a finite change in B , provided that the change in B occurs over an infinite interval in θ and in such a way that $dB/d\theta$ is zero.

Because $\mathcal{L}$ is a function only of f_2 and because the only explicit θ dependence in f_2 arises from the θ dependence of B , it follows that the only explicit θ dependence of G arises from the θ dependence of B . Hence we write

$$\frac{\partial G}{\partial \theta} = \frac{dB}{d\theta}\frac{\partial G}{\partial B} = B' \frac{\partial G}{\partial B} . \tag{3.B.1}$$

We assume that for a constant B , $\nu_\varphi \neq 0$ so that we can define a period of the motion equal to $2\pi/\nu_\varphi$. This proof will not be valid for the case where ν_φ goes to zero.

From the expression for φ , equation (3.A.24), it follows that φ is a periodic function of w , and from the expression for G the same must be true for $\frac{\partial G}{\partial B}$ when we substitute for φ its expression in terms of w and J . We therefore write the term $\partial G/\partial B$ as a Fourier series.

$$\frac{\partial G}{\partial B} = \sum \bar{A}_n(J, B)\, e^{2\pi i n w} . \tag{3.B.2}$$

If we write w as a function of θ, $\frac{\partial G}{\partial \theta}$ becomes

$$\frac{\partial G}{\partial B} = \sum A_n (J, B)\, e^{in\nu_\varphi(J,B)\theta} . \tag{3.B.3}$$

In order to estimate the error in neglecting the term $\partial G/\partial \theta$ in the new Hamiltonian, we must investigate the effect of this term on the equations of motion. The equation for $dJ/d\theta$ and $dw/d\theta$ are

$$\frac{dJ}{d\theta} = -\frac{\partial}{\partial w}\mathcal{H}_s(P_\varphi(J,w), \varphi(J,w)) - B'\frac{\partial}{\partial w}\left(\frac{\partial G}{\partial B}\right), \tag{3.B.4}$$

$$\frac{dw}{d\theta} = \frac{\partial}{\partial J}\mathcal{H}_s(P_\varphi(J,w), \varphi(J,w)) + B'\frac{\partial}{\partial w}\left(\frac{\partial G}{\partial B}\right).$$

It is interesting to note that for on-axis particles $\frac{\partial}{\partial w}\mathcal{H}_s = 0$, and for this case the total change in J comes from the term $\partial G/\partial B$. We now denote the change in J due to the $\partial G/\partial B$ term as $(J_2 - J_1)$ and the change in θ as $(\theta_2 - \theta_1)$. Thus

$$J_2 - J_1 = \int_{\theta_1}^{\theta_2} B'\frac{\partial}{\partial w}\left(\frac{\partial G}{\partial B}\right) d\theta . \tag{3.B.5}$$

Since the variation of B with θ is slow and not connected with the period of the motion, we can bring B' outside of the integral sign. We now use the fact that $\partial G/\partial B$ can be written as a Fourier series in w to obtain

$$\frac{J_2 - J_1}{B'} = - \int_{\theta_1}^{\theta_2} \left(\sum_{n=1}^{\infty} 2\pi i n \bar{A}(J,B)\, e^{2\pi i n w} \right) d\theta, \tag{3.B.6}$$

where prime designates derivative with respect to θ . We now write w as a function of θ to give

$$\frac{J_2 - J_1}{B'} = - \int_{\theta_1}^{\theta_2} \left(\sum_{n=1}^{\infty} 2\pi i n\, A_n(J,B)\, e^{i n \nu_\varphi (J,B)\, \theta} \right) d\theta. \tag{3.B.7}$$

This expression is no longer exactly periodic in θ because the A_n's and ν depend on $B(\theta)$. We now expand these quantities in powers of θ , where the expansion will be used only for values of θ smaller than $2\pi / \nu_\varphi$. Letting the subscript o designate the value of the expanded quantities at $\theta = \theta_1$, we obtain for the integrand:

$$\begin{aligned} &\sum_{n=1}^{\infty} 2\pi i n\, A_{n0}\, e^{i n \nu_{\varphi_0} \theta} \\ &+ B'\theta \sum_{n=1}^{\infty} 2\pi i n \left\{ \left(\frac{\partial A_n}{\partial B} \right)_0 + i n A_{n0} \left(\frac{\partial \nu_\varphi}{\partial B} \right)_0 \theta \right\} e^{i n \nu_{\varphi_0} \theta} . \end{aligned} \tag{3.B.8}$$

We now do the above expansion at the beginning of the period (θ_1, θ_2) and integrate over one period of the first term; then we

expand again at the end of the first period and integrate over the second period of this first term. We continue doing this until we have integrated over the total interval (θ_1, θ_2) . The last integral will, in general, not be taken over a complete period; however, the integral of the first term will be finite. The integral of the second term will be of the order of $B'(\theta_2 - \theta_1)$.

We have shown that it is possible to have a finite change in B , i.e., $B'(\theta_2 - \theta_1) \neq 0$, and if B varies slowly enough, still have a zero variation in J .

We now show that we can neglect the error caused by neglecting the change in ω' due to the $\partial G/\partial B$ term. In this case the term $\frac{\partial \mathcal{H}_s}{\partial J}$ is not equal to zero; we must show that

$$\frac{\partial}{\partial J} \mathcal{H}_s \gg B' \frac{\partial}{\partial J}\left(\frac{\partial G}{\partial B}\right), \tag{3.B.9}$$

in the case where B' is small. We use the fact that $\omega = \frac{\partial G}{\partial J}$ and obtain

$$\omega' = \frac{\partial \mathcal{H}_s}{\partial J} + B' \frac{\partial \omega}{\partial B} \tag{3.B.10}$$

If we write ω as a function of θ , we obtain

$$\omega' = \frac{\partial \mathcal{H}_s}{\partial J} + \frac{B'}{2\pi} \sum \frac{\partial}{\partial B} \left[\nu_\varphi(J, B)\right] \theta . \tag{3.B.11}$$

We integrate the above equation over one period of $w = 2\pi/\nu_\varphi$ and obtain

$$1 = \frac{2\pi}{\nu_\varphi}\frac{\partial \mathcal{H}_5}{\partial J} + \frac{B'}{2\pi}\left\{\frac{1}{2}\left(\frac{2\pi}{\nu_\varphi}\right)^2 \frac{\partial \nu_\varphi}{\partial B}\right\} \tag{3.B.12}$$

Because of the variation of ν_φ with B is finite and of the order of magnitude of ν_φ / B, it follows that for the case where $(B'/B)\left(\frac{2\pi}{\nu_\varphi}\right) << 1$ we can neglect the second term in the w' equation.

We have now shown that, if the change in B over one oscillation is small, that we can neglect the term $\partial G/\partial \theta$ in the new Hamiltonian.

Substituting for φ and P_φ their previous expressions in terms of J, and w, we obtain for the Hamiltonian

$$\begin{aligned}\mathcal{H}_6 = {} & \tfrac{2}{3}\frac{a^3}{\sqrt{3}} A\left[-1 + C_{002}\frac{J}{2\pi} + C_{004}\frac{J^2}{4\pi^2}\right] \\ & + \frac{P_{x_1}^2 + P_{y_1}^2}{2} - C_{200}\left(\frac{x_1^2+y_1^2}{2}\right) - D(\zeta)\left(\frac{x_1^2-y_1^2}{2}\right) \\ & + C_{202}\frac{J}{2\pi}(x_1^2+y_1^2) + C_{211}\left(\frac{J}{2\pi}\right)^{\frac{1}{2}}(x_1^2+y_1^2)\cos 2\pi w \\ & + C_{422}\frac{J}{2\pi}(x_1^2+y_1^2)\cos 4\pi w,\end{aligned} \tag{3.B.13}$$

in which

$$C_{002} = \left(\frac{15}{4}\right)(.293)\,\frac{2\pi}{a^{5/2}B^{1/2}} ,$$

$$C_{004} = -.00442\left(\frac{15}{4}\right)^2 \frac{4\pi^2}{a^5 B} ,$$

$$C_{202} = \frac{aC}{2}\left(\frac{15}{4}\right)(.0157b^2 + .0451a^2)\left(\frac{2\pi}{a^{5/2}B^{1/2}}\right), \tag{3.B.14}$$

$$C_{211} = \frac{aC}{2}\left(\frac{15}{4}\right)(.265b^2 + .265a^2)\left(\frac{2\pi}{a^{5/2}B^{1/2}}\right),$$

$$C_{222} = -\frac{aC}{2}\left(\frac{15}{4}\right)(.0473b^2 - .108a^2)\left(\frac{2\pi}{a^{5/2}B^{1/2}}\right),$$

and

$$C_{200} = aC\,(.577b^2 - .192\,a^2) .$$

IV. MOSER TRANSFORMATIONS

Since the x_1 and y_1 motion equations are similar and uncoupled, differing only by a constant phase of π in the quadrupole term, we will treat only the x_1 and φ motion in all further work. The conclusions for the y_1 motion will be the same as those drawn for the x_1 motion.

Moser theory[20] is a technique that is used to study the instabilities of particle motion because of the coupling between the longitudinal and transverse particle motion. The motion of a particle can be unstable whenever $m\nu_x + n\nu_\varphi = p$ for m, n and p integers. We say that an instability of this type is driven by the $m\nu_x + n\nu_\varphi = p$ resonance. Moser theory employs a series of transformations, called Moser transformations designed to transform all low order terms, except the resonance terms which drive an instability, in the Hamiltonian into terms of higher order. We decide which resonance to study by determining for what values of m, n and p, $m\nu_x + n\nu_\varphi = p$. In order to use Moser theory we will remove the periodic θ dependence from the quadrupole term by the transformation from (x_1, P_{x_1}) variables to the $(\xi, -i\bar{\xi})$ variables, where ξ and $\bar{\xi}$ are complex conjugate variables. This transformation is due to K. R. Symon and is described in detail in Appendix II. It is shown in Appendix II that

$$x_1 = \tfrac{1}{2}\left(\psi\xi + \bar{\psi}\bar{\xi}\right),$$

$$P_{x_1} = \frac{i}{2}\left(\eta\xi - \bar{\eta}\bar{\xi}\right), \tag{4.A.1}$$

where ψ and $\bar{\psi}$ are the usual Floquet functions and η and $\bar{\eta}$ are defined by

$$\eta = \nu_x \psi - i\frac{\partial\psi}{\partial\theta},$$

$$\bar{\eta} = \nu_x \bar{\psi} + i\frac{\partial\psi}{\partial\theta}. \tag{4.A.2}$$

The transformed Hamiltonian is still not in a convenient form for applying Moser theory. In order to bring it into such a form we transform from (w, J) variables to $(\rho, -i\bar{\rho})$ variables, where again the bar above the variables stands for complex conjugate, by use of the following generating function:

$$G = \frac{i}{2}\rho^2 e^{-4\pi i w} \tag{4.A.3}$$

This gives

$$\rho = \left(\frac{J}{2\pi}\right)^{1/2} e^{2\pi i w}, \qquad \bar{\rho} = \left(\frac{J}{2\pi}\right)^{1/2} e^{-2\pi i w} \tag{4.A.4}$$

and

$$\mathcal{H}_7 = \frac{2}{3}\frac{a^3}{\sqrt{3}} A\left[C_{002}\, \varsigma\bar{\varsigma} + C_{004}\, \varsigma^2\bar{\varsigma}^2\right] + \nu_x \xi\bar{\xi}$$

$$+ \frac{1}{4}\Big\{ C_{211}\big(\psi^2\xi^2\varsigma + 2\psi\bar{\psi}\xi\bar{\xi}\varsigma + \bar{\psi}^2\bar{\xi}^2\varsigma + \psi^2\xi^2\bar{\varsigma}$$

$$+ 2\psi\bar{\psi}\xi\bar{\xi}\bar{\varsigma} + \bar{\psi}^2\bar{\xi}^2\bar{\varsigma}\big) + C_{202}\big(\psi^2\xi^2\varsigma\bar{\varsigma} + 2\psi\bar{\psi}\xi\bar{\xi}\varsigma\bar{\varsigma}$$

$$+ \bar{\psi}^2\bar{\xi}^2\bar{\varsigma}^2\big) + C_{222}\big(\psi^2\xi^2\varsigma^2 + 2\psi\bar{\psi}\xi\bar{\xi}\varsigma^2 \qquad (4.A.5)$$

$$+ \bar{\psi}^2\xi^2\varsigma^2 + \psi^2\xi^2\bar{\varsigma}^2 + 2\psi\bar{\psi}\xi\bar{\xi}\bar{\varsigma}^2 + \bar{\psi}^2\xi^2\varsigma^2\big)\Big\}.$$

The Hamiltonian has the general form

$$\mathcal{H}_7 = \nu_\varphi(\varsigma\bar{\varsigma}) + \nu_x(\xi\bar{\xi}) + \Omega_0^{(3)} + \Omega_0^{(4)} \qquad (4.A.6)$$

with

$$\Omega_0^{(3)} = \sum_{\substack{k,l,k',l' \\ k+l+k'+l'=3}} \Omega^{(3)}_{0,k,l,k',l',m}\, \xi^k\bar{\xi}^l\varsigma^{k'}\bar{\varsigma}^{l'} e^{im\theta} \qquad (4.A.7)$$

$$\Omega_0^{(4)} = \sum_{\substack{k,l,k',l' \\ k+l+k'+l'}} \Omega^{(4)}_{0,k,l,k',l',m}\, \xi^k\bar{\xi}^l\varsigma^{k'}\bar{\varsigma}^{l'} e^{im\theta}$$

and

$$\nu_\varphi = \frac{2}{3}\frac{a^3}{\sqrt{3}} A\, C_{002} \qquad (4.A.8)$$

The Hamiltonian is now of the proper form for applying Moser theory. The first transformation will remove all cubic terms from the Hamiltonian except those driving resonances under study. This transformation will have no effect on the quadratic terms or the cubic terms which are retained. However, the higher power terms will be altered by this transformation. Next all the fourth and fifth power terms which do not contribute to the resonances under consideration will be transformed away. Again terms higher than fifth power are introduced, while those of fifth or lower are not affected.

This process can be repeated any number of times, although the transformations become very complicated. Eventually the process is stopped, and the terms of all orders above the last transformed are dropped. The justification of dropping all higher order terms is that in present-day linacs these higher order terms are so small, such as to have only a negligible effect on the oscillation frequency of a particle and these higher order terms cannot contribute to the resonances under study, i.e., these higher order terms cannot affect the amplitude of a particle's oscillation.

In order to transform the cubic terms away, we use the generating function

$$S_3 = \rho\bar{\rho}_1 + \xi\bar{\xi}_1 + \sum_{\substack{k,l,k',l',m \\ k+l+k'+l'=3}} \frac{\Omega^{(3)}_{0,k,l,k',l',m}\, e^{im\theta}}{-m+(k-l)\nu_x+(k'-l')\nu_\varphi}\, \xi^k \bar{\xi}_1^{\,l} \rho^{k'} \bar{\rho}_1^{\,l'}, \tag{4.A.9}$$

where the subscript 1 is used to designate the new variables. Since the resonant terms, for which $m = (k-l)\nu_x + (k'-l')\nu_\varphi$ would have a term in the generating function going to infinity, we exclude this term from the sum. This transformation will leave the resonant term $\Omega^{(3)}_{0,k,l,k',l',m}$ as a cubic term in the new Hamiltonian.

From the generating function we obtain the equations relating the new variables to the old variables. For example,

$$\xi_1 = \xi + \sum_{k,l,k',l',m} \frac{l\,\Omega^{(3)}_{0,k,l,k',l',m}\, e^{im\theta}}{-m+(k-l)\nu_x+(k'-l')\nu_\varphi}\, \xi^k \bar{\xi}_1^{\,l-1} \rho^{k'} \bar{\rho}_1^{\,l'}, \tag{4.A.10}$$

We solve the above for ξ_1 by an iterative procedure. To first order in variables ρ, $\bar{\rho}$, ξ and $\bar{\xi}$, $\xi_1 = \xi$, $\bar{\xi}_1 = \bar{\xi}$, etc. To second order in ξ, $\bar{\xi}$, ρ and $\bar{\rho}$

$$\xi_1 = \xi + \sum_{k',l,k',l',m} \frac{l\,\Omega^{(3)}_{0,k,l,k',l',m}\, e^{im\theta}}{-m+(k-l)\nu_x+(k'-l')\nu_\varphi}\, \xi^k \bar{\xi}^{\,l-1} \rho^{k'} \bar{\rho}^{\,l'}. \tag{4.A.11}$$

etc. This iterative procedure is repeated until the order of the approximation in the variables ξ, $\bar{\xi}$, ζ and $\bar{\zeta}$ is one less than the order of the terms in the Hamiltonian to be considered. The highest order terms we will consider here are fourth order terms; we will therefore need the expressions for the new variables through third order.

Since most linacs are designed to miss the third order resonances (resonances driven by third order terms in the Hamiltonian), it is possible to transform all the third order terms to higher order terms. Thus, the new Hamiltonian, neglecting fifth order terms and higher, is

$$\mathcal{H}_8 = \nu_\varphi \zeta_1 \bar{\zeta}_1 + \nu_x \xi \bar{\xi} + \Omega_1^{(4)} , \tag{4.A.12}$$

with

$$\Omega_1^{(4)} = \Omega_0^{(4)} + \text{terms from transforming } \Omega_0^{(3)} \tag{4.A.13}$$

From Graph 1 we see that the only fourth order resonance present is the $2\nu_x - 2\nu_\varphi = 0$ resonance. We therefore next transform all fourth order terms away except for those driving this $2\nu_x - 2\nu_\varphi = 0$ resonance. That is, we keep $\Omega^{(4)}_{1,k,l,k',l',0}$ term when $(k-l) = -(k'-l')$ and transform all other terms to higher order. For this we use the generating function

$$S_4 = \rho_1 \bar{\rho}_2 + \xi_1 \bar{\xi}_2 + \sum_{\substack{k,l,k',l',m \\ k+l+k'+l'=4}} S_{4,k,l,k',l',m}\, \xi_1^k \bar{\xi}_2^l \rho_1^{k'} \bar{\rho}_2^{l'} , \qquad (4.A.14)$$

in which

$$S_{4,k,l,k',l',m} = \frac{\Omega^{(4)}_{1,k,l,k',l',m}\, e^{im\theta}}{-m + (k-l)\nu_x + (k'-l')\nu_\varphi} , \qquad (4.A.15)$$

unless $(k-l) + (k'-l') = 0$ and $m = 0$, in which case $S_{4,k,l,k',l',m} = 0$. The subscript 2 refers to the new variables.

The new Hamiltonian is now void of all fourth order terms except those driving the $2\nu_x - 2\nu_\varphi = 0$ resonance. It has the form

$$\begin{aligned} \mathcal{H}_9 = {} & \nu_\varphi \rho_2 \bar{\rho}_2 + \nu_x \xi_2 \bar{\xi}_2 \\ & + \Omega^{(4)}_{1,0,0,2,2,0} \rho_2^2 \bar{\rho}_2^2 + \Omega^{(4)}_{1,2,2,0,0,0} \xi_2^2 \bar{\xi}_2^2 \\ & + \Omega^{(4)}_{1,2,0,0,2,0} \xi_2^2 \bar{\rho}_2^2 + \Omega^{(4)}_{1,0,2,2,0,0} \bar{\xi}_2 \rho_2^2 \\ & + \Omega^{(4)}_{1,1,1,1,1,0} \xi_2 \bar{\xi}_2 \rho_2 \bar{\rho}_2 . \end{aligned} \qquad (4.A.16)$$

It is shown in Appendix II that we can approximate the Floquet functions ψ and $\bar{\psi}$ by $\sqrt{2/\nu_x}$. With this approximation incorporated the coefficients of the fourth order terms in Hamiltonian are

$$\Omega^{(4)}_{0,0,2,2,0} = \frac{2}{3}\frac{a^3}{\sqrt{3}} A C_{004} ,$$

$$\Omega^{(4)}_{1,2,2,0,0,0} = \frac{C_{211}^2}{16}\left\{\frac{16}{\nu_x^2\nu_\varphi} + \frac{4}{\nu_x^2(2\nu_x+\nu_\varphi)} - \frac{4}{\nu_x^2(2\nu_x-\nu_\varphi)}\right\},$$

$$\Omega^{(4)}_{1,0,2,3,0,0} = \frac{C_{222}}{2\nu_x} + \frac{C_{211}^2}{8}\left\{\frac{4}{\nu_x^2\nu_\varphi} + \frac{4}{\nu_x^2(2\nu_x-\nu_\varphi)}\right\}, \qquad (4.A.17)$$

$$\Omega^{(4)}_{1,2,0,0,2,0} = \Omega^{(4)}_{1,0,2,3,0,0}$$

$$\Omega^{(4)}_{1,1,1,1,1,0} = \frac{C_{202}}{2\nu_x} + \frac{C_{211}^2}{4}\left\{\frac{4}{\nu_x^2(2\nu_x+\nu_\varphi)} + \frac{4}{\nu_x^2(2\nu_x-\nu_\varphi)}\right\},$$

$$\nu_\varphi = \frac{2}{3}\frac{a^3}{\sqrt{3}} A\, C_{202} .$$

Now that we have completed all of the Moser transformations, we return to action angle variables by means of the generating function

$$G = -\frac{i}{2}\,\rho_2^2\, e^{-4\pi i w_\varphi} - \frac{i}{2}\,\xi_2^2\, e^{-4\pi i w_x} \tag{4.A.18}$$

which gives

$$\rho_2 = \left(\frac{J_\varphi}{2\pi}\right)^{\frac{1}{2}} e^{i2\pi w_\varphi} \qquad \xi_2 = \left(\frac{J_x}{2\pi}\right)^{\frac{1}{2}} e^{i2\pi w_x} \tag{4.A.19}$$

$$\bar{\rho}_2 = \left(\frac{J_\varphi}{2\pi}\right)^{\frac{1}{2}} e^{-i2\pi w_\varphi} \qquad \bar{\xi}_2 = \left(\frac{J_x}{2\pi}\right)^{\frac{1}{2}} e^{-i2\pi w_x}$$

and

$$\begin{aligned} \mathcal{H}_{10} = {} & \frac{\nu_\varphi}{2\pi} J_\varphi + \frac{\nu_x}{2\pi} J_x + \frac{\Omega^{(4)}_{1,0,0,2,2,0}}{(2\pi)^2} J_\varphi^2 \\ & + \frac{\Omega^{(4)}_{1,1,1,1,1,0}}{(2\pi)^2} J_x J_\varphi + \frac{\Omega^{(4)}_{1,2,2,0,0,0}}{(2\pi)^2} J_x^2 \\ & + \frac{\Omega_{1,2,0,0,2,0}}{(2\pi)^2} J_x J_\varphi \left[e^{i4\pi(w_x - w_\varphi)} + e^{-i4\pi(w_x - w_\varphi)} \right] . \end{aligned} \tag{4.A.20}$$

In order to obtain a constant of the motion, we transform from these variables to new variables (K_1 , α and K_2 , β)by means of the generating function

$$G = K_1(2w_x + 2w_\varphi) + K_2(2w_x - 2w_\varphi) , \tag{4.A.21}$$

which gives the equations

$$J_x = 2(K_1 + K_2) \quad , \quad J_\varphi = 2(K_1 - K_2) ,$$
$$\alpha = (2w_x + 2w_\varphi) \quad , \quad \beta = (2w_x - 2w_\varphi) \tag{4.A.22}$$

and

$$\mathcal{H}_{11} = \mathcal{H}_{10}$$

The new Hamiltonian can now be written as

$$\begin{aligned}\mathcal{H}_{11} = {} & \frac{2\nu_x + 2\nu_\varphi}{2\pi} K_1 + \frac{2\nu_x - 2\nu_\varphi}{2\pi} K_2 \\ & + \frac{\Omega^{(4)}_{1,0,0,2,2,0}}{\pi^2}(K_1^2 - K_2^2) + \frac{\Omega^{4}_{1,2,2,0,0,0}}{\pi^2}(K_1^2 + K_2^2) \\ & + \frac{\Omega^{(4)}_{1,1,1,1,1,0}}{\pi^2}(K_1^2 - K_2^2) + \frac{2\Omega^{(4)}_{1,2,0,0,2,0}}{\pi^2}(K_1^2 - K_2^2)\cos 2\pi\beta .\end{aligned} \tag{4.A.23}$$

The new Hamiltonian is independent of the variable α and thus $K_1 = \partial \mathcal{H}_{11}/\partial \alpha = 0$ and K_1 is a constant of the motion. Since it is possible to add or subtract an arbitrary function of θ from the Hamiltonian without affecting the equations of motion, we will subtract the term

$$\frac{2\nu_x + 2\nu_\varphi}{2\pi} K_1 + 2\left(\Omega^{(4)}_{1,0,0,2,2,0} + \Omega^{(4)}_{1,2,2,0,0,0}\right)\frac{K_1^2}{\pi^2}$$

from $\mathcal{H}_{11}$. This gives

$$\mathcal{H}_{11} = (\epsilon + pK_1)K_2 + (q + r\cos 2\pi\beta)(K_1^2 - K_2^2), \quad (4.A.24)$$

with

$$\epsilon = \frac{2\nu_x - 2\nu_\varphi}{2\pi}, \quad p = \frac{2}{\pi^2}\left(\Omega^{(4)}_{1,2,3,0,0,0} - \Omega^{(4)}_{1,0,0,2,2,0}\right), \quad (4.A.25)$$

$$q = \frac{1}{\pi^2}\left(\Omega^{(4)}_{1,1,1,1,1,0} - \Omega^{(4)}_{1,0,0,2,2,0} - \Omega^{(4)}_{1,2,2,0,0,0}\right),$$

and

$$r = 2/\pi^2 \; \Omega^{(4)}_{1,2,0,0,2,0}.$$

V. RAPID TRAVERSAL OF THE RESONANCE

The Hamiltonian can be written

$$\mathcal{H}_{11} = (\epsilon + pK_1)K_2 + (q + r\cos 2\pi\beta)(K_1^2 - K_2^2) \tag{5.A.1}$$

where K_1 is a constant, ϵ , p , q and r are slowly varying functions of the independent variable θ , and K_2 and β are canonically conjugate variables. The equations of motion are

$$(K_1 + K_2)' = (K_1^2 - K_2^2)\, r \sin 2\pi\beta , \tag{5.A.2}$$

and

$$\beta' = (\epsilon + pK_1) - 2K_2\,(q + r\cos 2\pi\beta).$$

We use the relationships

$$J_x = 2(K_1 + K_2) \quad \text{and} \quad J_\varphi = 2(K_1 - K_2), \tag{5.A.3}$$

to obtain the quations of motion

$$J_x' = \tfrac{1}{2} J_x J_\varphi\, r \sin 2\pi\beta ,$$

$$\beta' = (\epsilon + pK_1) - 2(J_x - J_\varphi)(q + r\cos 2\pi\beta)$$

For small values of J_x , it follows from equations (5.A.3) that

$$K_2 \approx -K_1 \;,\; J_\varphi = 4K_1 \quad \text{and} \quad (J_x - J_\varphi) = -2K_1$$

The equations of motion for small values of J_x become

$$\frac{d}{d\theta}(\ln J_x) = 2K_1 r \sin 2\pi\beta,$$

$$\beta' = (\epsilon + pK_1) + 4K_1(q + r\cos 2\pi\beta) \tag{5.A.4}$$

For the case where $|\epsilon| >> 4K_1 r$ and $\beta' \neq$, it follows from equation (5.A.4) that we can approximate β by

$\beta = \beta_0 + \gamma\theta$ and obtain for J_x the approximation

$$J_x = J_{x_0} e^{2K_1 r \cos 2\pi(\beta_0 + \gamma\theta)} \tag{5.A.5}$$

Thus for $\beta' \neq 0$ the quantity J_x and hence the x motion is bounded. But the parameters ϵ , p , q and r are slowly varying functions of θ . For the MURA linac there is a value of θ such that $\beta' = 0$. When $\beta' = 0$, the equation for J_x is

$$J_x = J_{x_0} e^{(2\pi K_i r_i \sin 2\pi\beta_i)(\theta - \theta_i)} \tag{5.A.6}$$

where the subscript i designates the value of the various quantities when $\beta' = 0$. Thus for $0 < \beta_i < 1/2$, the quantity grows exponentially. But the parameters ϵ , p , q and r are functions of θ . Thus $\beta'' \neq 0$ when $\beta' = 0$ and we will pass through the resonance. We assume that we can approximate β by

$$\beta = \beta_i + \tfrac{1}{2}\beta_i''(\theta - \theta_i)^2. \tag{5.A.7}$$

Then for small J_x we can write

$$J_{xf} = J_{x_0} = e^{2K_1 r \int_{\theta_0}^{\theta_f} \sin 2\pi\left(\beta_i + \frac{1}{2}\beta_i''(\theta-\theta_i)^2\right) d\theta}, \tag{5.A.8}$$

where the subscript f designates the final value of a quantity and the subscript o designates the initial value of a quantity. The integral in the exponential is a Fresnel integral[21] and is bounded even for an infinite interval of integration. Thus the upper limit of the final value of J_x is

$$J_{xf} \leq J_{x_0}\, e^{2K_1 r \sqrt{\pi/\beta_i''}\,(\cos 2\pi\beta_i + \sin 2\pi\beta_i)}. \tag{5.A.9}$$

In the case of small J_x, we differentiate the equation for β' and obtain

$$\beta_i'' = (\epsilon_i' + P_i K_1) + 4K_1(q_i' + r_i' \cos 2\pi\beta_i). \tag{5.A.10}$$

As an example, we now use the MURA linac design values to obtain an estimate for the growth in the x amplitude caused by this resonance. For small values of J_x we find

$$\frac{1}{4} K_1 = J_\varphi = J = \frac{4}{15} B^{1/2} a^{3/2} f_2. \tag{5.A.11}$$

From the graph of ℓ versus f_2, we obtain the maximum value of f_2, $f_2 = 7.896$. Since K_1 is a constant of the motion for any particle, it is always equal to its value at injection energy. Thus the maximum value of K_1 at any energy must equal the maximum value of K_1 at injection energy. From MURA design parameters the maximum value of $K_1 \approx 2 \times 10^{11}$, and $\beta' = 0$ for a particle with kinetic energy of 23 Mev. At this energy $r = 1.5 \times 10^7$ and $\beta'' = 1.6 \times 10^{-3}$. Using these values in the formula for J_{xf}, we obtain

$$J_{xf} \leq J_{x_0} e^{.026}, \tag{5.A.12}$$

or

$$J_{xf} \leq 1.026 J_{x_0}. \tag{5.A.13}$$

We now relate J_x to the original variable x by using the approximation that J_x is small,

$$\frac{x_f}{x_0} = \frac{x_{1f}}{x_{10}} = \frac{|\xi|_f}{|\xi|_0} \approx \frac{|\xi_2|_f}{|\xi_2|_0} = \frac{(J_{xf})^{1/2}}{(J_{x_0})^{1/2}} \tag{5.A.14}$$

The growth in J_x for a particle with the largest possible value of J_φ in the MURA linac is less than 2.6 percent. For small J_x the quantity x, which is the actual displacement, is proportional to $(J_x)^{\frac{1}{2}}$. Thus the maximum growth in x is less than 1.3 percent.

Unfortunately, at this time the MURA linac has not been built so that it is not possible to experimentally check these theoretical results. In the CERN linac values for the tunes are chosen such that $\nu_x = 0.75\ \nu_\varphi$ at the input end of the linac and ν_x/ν_φ increases up to $\nu_x = 1.5\ \nu_\varphi$ at the output end. Thus particles accelerated in the CERN linac cross through the $2\ \nu_x = 2\ \nu_\varphi$ resonance. The experimental results for the CERN linac do not demonstrate any growth in the transverse motion. The fact that a particle can cross the $2\ \nu_x = 2\ \nu_\varphi$ resonance without experiencing x growth is a very fortunate result. It would be necessary to have $\nu_x > \nu_\varphi$ at the injection energy (since ν_φ decreases with energy) if the transverse motion experienced a large growth, and this requirement would demand higher quadrupole fields in the first few drift tubes. However, the quadrupole field requirements for the first few drift tubes are already very difficult to meet and it is doubtful that higher fields would be possible.

The results we have obtained can be applied to any linac that is designed to pass through the $2\ \nu_x - 2\ \nu_\varphi = 0$ resonance by the

following procedure. First, the following formulas are used to calculate ν_x and ν_φ as a function of energy, and the energy for which $\nu_x = \nu_\varphi$ is obtained.

$$\nu_x^2 = -\frac{N^2 e f_0 T c \sin\varphi_s}{2E_0 \beta_s \gamma_s^3 \omega} + \frac{e N^2 c^2 \beta_s}{2E_0 \omega \gamma_s} \sum \frac{g_n^2}{n^2} , \tag{5.A.15}$$

$$\nu_\varphi^2 = \frac{N^2 e f_0 T c \sin\varphi_s}{E_0 \beta_s \gamma_s^3 \omega}$$

This energy is then used to obtain a value for β'' and r from

$$\beta'' = \frac{N \beta_s c}{2\pi \omega} \frac{d}{dz} \left(2\nu_x - 2\nu_\varphi \right) , \tag{5.A.16}$$

$$r = \frac{2}{\pi^2} \left\{ \frac{1}{2\nu_x} C_{222} + \frac{1}{\nu_x^3} C_{211}^2 \right\} ,$$

for which

$$C_{222} = -\frac{aC}{2}\left(.0473\, b^2 - .108\, a^2\right)\frac{15\pi}{2a^{5/2}B^{1/2}} ,$$

$$a^2 = \frac{\varphi_s^3}{J_3(2\varphi_s)}\left(\frac{J_1(2\varphi_s) + 3J_3(2\varphi_s)}{\varphi_s} - \cos\varphi_s\right),$$

$$b^2 = \frac{\varphi_s^3}{J_3(2\varphi_s)}\left(\frac{J_1(2\varphi_s) + 3J_3(2\varphi_s)}{\varphi_s}\right), \tag{5.A.17}$$

$$B = \frac{2e f_0 T E_0 c \beta_s^3 \gamma_s^3}{\omega^3}\,\frac{J_3(2\varphi_s)}{\varphi_s^3} ,$$

and

$$C = \frac{e f_0 T c N^2}{2E_0 \beta_s \gamma_s^3 \omega}\,\frac{J_3(2\varphi_s)}{\varphi_s^3}$$

Since K_1 is a constant of the motion, it must be evaluated at the injection energy of the linac. The expression for K_1 is

$$K_1 = \frac{4}{15} B^{1/2} a^{5/2} f_2 \tag{5.A.18}$$

where f_2 has a maximum value of 7.896. These values for β_i'', r_i and K_1 can be substituted into the expression for x_f / x_0

$$\frac{X_f}{X_o} = e^{K_1 r_i \sqrt{\pi/\beta_i''}} \tag{5.A.19}$$

to obtain a value for the growth in the transverse oscillation amplitude due to passage through the $2\ \nu_x = 2\ \nu_\varphi$ resonance.

VI. LONGITUDINAL SPACE CHARGE EFFECTS

If a large number of particles is accelerated simultaneously, then the electromagnetic forces between particles will affect the particle motion. These forces will be divided into two categories. The first which influence the transverse motion will be referred to as transverse space-charge forces. The second which influence the phase motion will be called longitudinal space-charge forces. The transverse space-charge problem[11] has been discussed in detail for linacs, while the longitudinal space-charge problem has been discussed for circular-type accelerators.[12, 22, 23] We will adapt the techniques used in the treatment of the longitudinal space-charge problem in circular accelerators to the treatment of the longitudinal space-charge problem in a linac.

The repulsive longitudinal space-charge forces tend to weaken the total phase-focusing force and for a sufficient particle density result in a complete loss of stability. We will define the maximum number of particles that can be accelerated per unit time and still be longitudinally stable as the longitudinal space-charge limit. We will first obtain a general expression for the longitudinal space-charge limit, and as an example the longitudinal space-charge limit will be calculated for the MURA proposal linac. The results for this example will show that the space-charge limit is greater than the expected beam intensity.

It has been shown in Section II-B that we can write the Hamiltonian which governs the pure phase motion as follows:

$$\mathcal{H}_2(\eta, \varphi, z) = -\left\{\frac{\omega^2}{c^2}\eta^2 - m^2c^2\right\}^{\frac{1}{2}} - \frac{e}{c}A_z(\varphi, z) + K\eta, \tag{6.A.1}$$

where $\eta = E/\omega$ and $\varphi = \frac{2\pi}{L}z - \omega t$. In Section II-D the carrier-wave approximation was discussed in detail and it was shown that only one component of the standing wave had an important effect on the particle motion. If we neglect all but one component of the standing wave, we obtain

$$A_z = \frac{c}{\omega} f_0 T \sin\varphi - \frac{2\pi}{L}\frac{c}{\omega} U(\varphi), \tag{6.A.2}$$

in which $U(\varphi)$ is the space-charge potential.

We make the same definition of the synchronous particle as in Section II-B, namely $\eta = \eta_s$ and $\varphi = \varphi_s$. We then make the same transformation as in Section II-C to new variables P_φ and φ. This results in the following expression for the Hamiltonian

$$\mathcal{H}_3(P_\varphi, \varphi, z) = \frac{P_\varphi^2}{2B} + \frac{e f_0 T}{\omega}(\varphi\cos\varphi_s - \sin\varphi) + \frac{2\pi}{L}\frac{e}{\omega}U(\varphi), \tag{6.A.3}$$

with

$$B = \frac{E_0 \beta_s^3 \gamma_s^3 c}{\omega^2} \quad , \quad E_0 = m_0 c^2$$

(6.A.4)

$$\beta_s = v_s / c \qquad \text{and} \qquad \gamma_s = (1 - \beta_s^2)^{-1/2}$$

In the following discussion we will be applying proofs and techniques which are discussed in detail by Nielsen and Sessler[12] to the problem of space charge in linacs. We now define the quantity $\psi(P_\varphi, \varphi, z)$ as the density distribution function in longitudinal phase space. For a "stationary" solution, i.e., a solution ψ such that $\partial\psi/\partial z = 0$, the function $\psi(P_\varphi, \varphi)$ must satisfy the Boltzmann equation

$$\frac{\partial \psi}{\partial P_\varphi} P_\varphi' + \frac{\partial \psi}{\partial \varphi} \varphi' = 0, \tag{6.A.5}$$

i.e., $\nabla \psi \cdot \vec{v}_p = 0$, where $\vec{v}_p$ is the phase velocity vector directed along a phase trajectory. Thus any distribution function which has only a variation in a direction perpendicular to the phase trajectories is a stationary distribution. We will assume that the linac fills a region bounded by a phase trajectory with a constant density of particles. We thus write $\psi(P_\varphi, \varphi) = \sigma$ for $P_\varphi < P_b(\varphi)$ and $\psi(P_\varphi, \varphi) = 0$, for $P_\varphi > P_b(\varphi)$ with $P_b(\varphi)$ as a

curve representing a phase trajectory. We now can write $\lambda(\varphi)$ which is equal to the charge density per unit angle of φ.

$$\lambda(\varphi) = e \int_{-\infty}^{\infty} \psi(P_\varphi, \varphi)\, dP_\varphi = 2e\sigma\, |P_b(\varphi)|, \tag{6.A.6}$$

and we have used the fact that $P_b(\varphi)$ is symmetrical about $P_b = 0$. Since $P_b(\varphi)$ is a phase curve which corresponds to the motion of a particle, we can write

$$H_b = \frac{P_b^2(\varphi)}{2B} + \frac{e f_0 T}{\omega}(\varphi \cos\varphi_s - \sin\varphi) + \frac{2\pi e}{\omega L} U(\varphi) \tag{6.A.7}$$

as the equation which defines $P_b(\varphi)$ as a function of φ for a value of $\mathcal{H}_3 = H_b$.

We now have reduced the problem of longitudinal space-charge effects to a problem of determining the function $P_b(\varphi)$ for various values of the constant H_b. From the function $P_b(\varphi)$ we can find the values of α and β such that $P_b(\alpha) = P_b(\beta) = 0$ and $P_b^2(\varphi) \geq 0$ for $\beta \leq \varphi \leq \alpha$, where we have defined $\beta \leq \alpha$. Then the total number of particles accelerated per period of the radio-frequency electric field is equal to N, and

$$N = 2\sigma \int_{\beta}^{\alpha} |P_b(\varphi)|\, d\varphi. \tag{6.A.8}$$

Thus by knowing the function $P_b(\varphi)$, we can find the number of particles accelerated per unit time for various values of H_b.

The solution of $P_b(\varphi)$ is dependent upon the function $u(\varphi)$ which in turn is dependent upon the function $\lambda(\varphi)$ by means of the following equation:

$$u(\varphi) = \int K(\varphi - \varphi')\,\lambda(\varphi')\,d\varphi' \tag{6.A.9}$$

in which $K(\varphi - \varphi')$ is a potential kernel which takes into account the shielding effect of the drift tubes.

The potential kernel $K(\varphi - \varphi')$ is derived in Appendix III for the case of a beam with a periodic structure enclosed in a long cylindrical pipe. The pipe is assumed to have walls of infinite conductivity. The potential kernel can be written as follows:

$$K(\varphi - \varphi') = \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} g_n e^{in(\varphi - \varphi')} \tag{6.A.10}$$

with

$$g_n = g_{-n},$$

$$g_n = -\frac{1}{2\pi}\left\{ d_2\left(\frac{2\pi n b}{L}\right) + d_1\left(\frac{2\pi n b}{L}\right) \frac{K_0\left(\frac{2\pi n a}{L}\right)}{I_0\left(\frac{2\pi n a}{L}\right)} \right\},$$

$$g_0 = \frac{1}{2\pi}\left\{ 1 + 2 \ln \frac{a}{b} \right\},$$

a is the radius of the cylindrical pipe, (6.A.12)

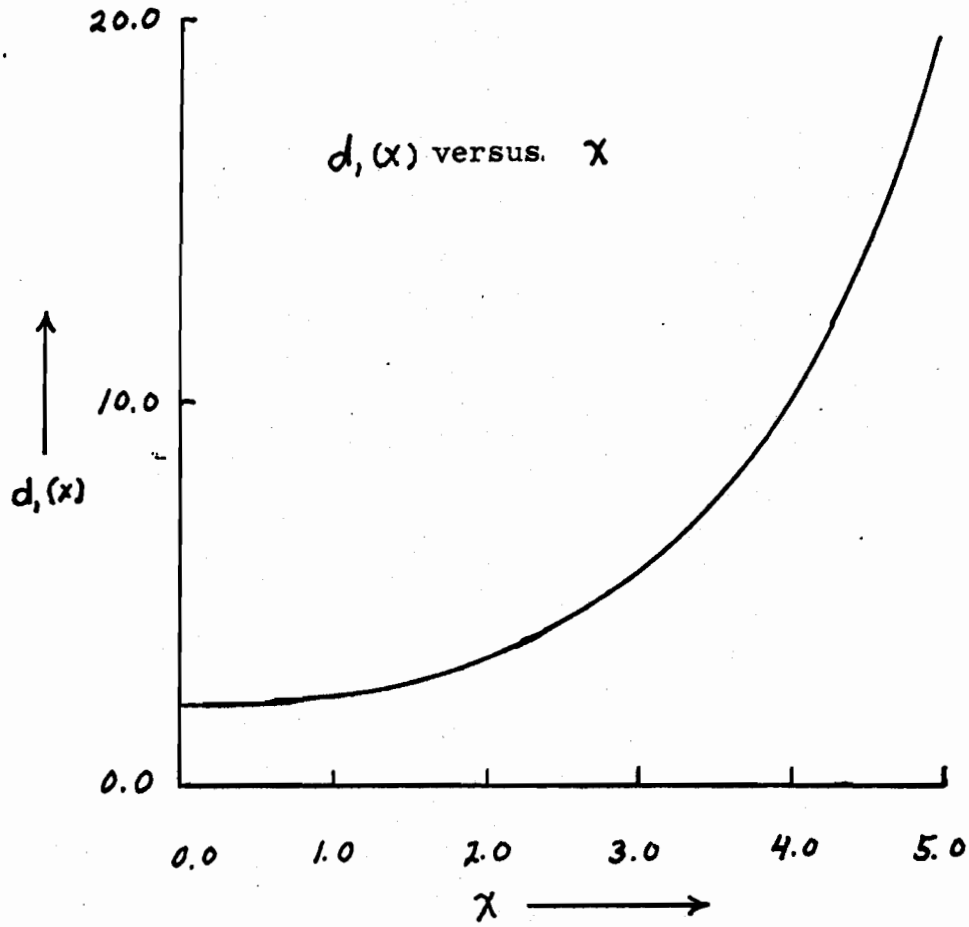
b is the radius of the beam,

$$d_1(x) = \frac{4}{x^2} \int_0^x t\, I_0(t)\, dt,$$

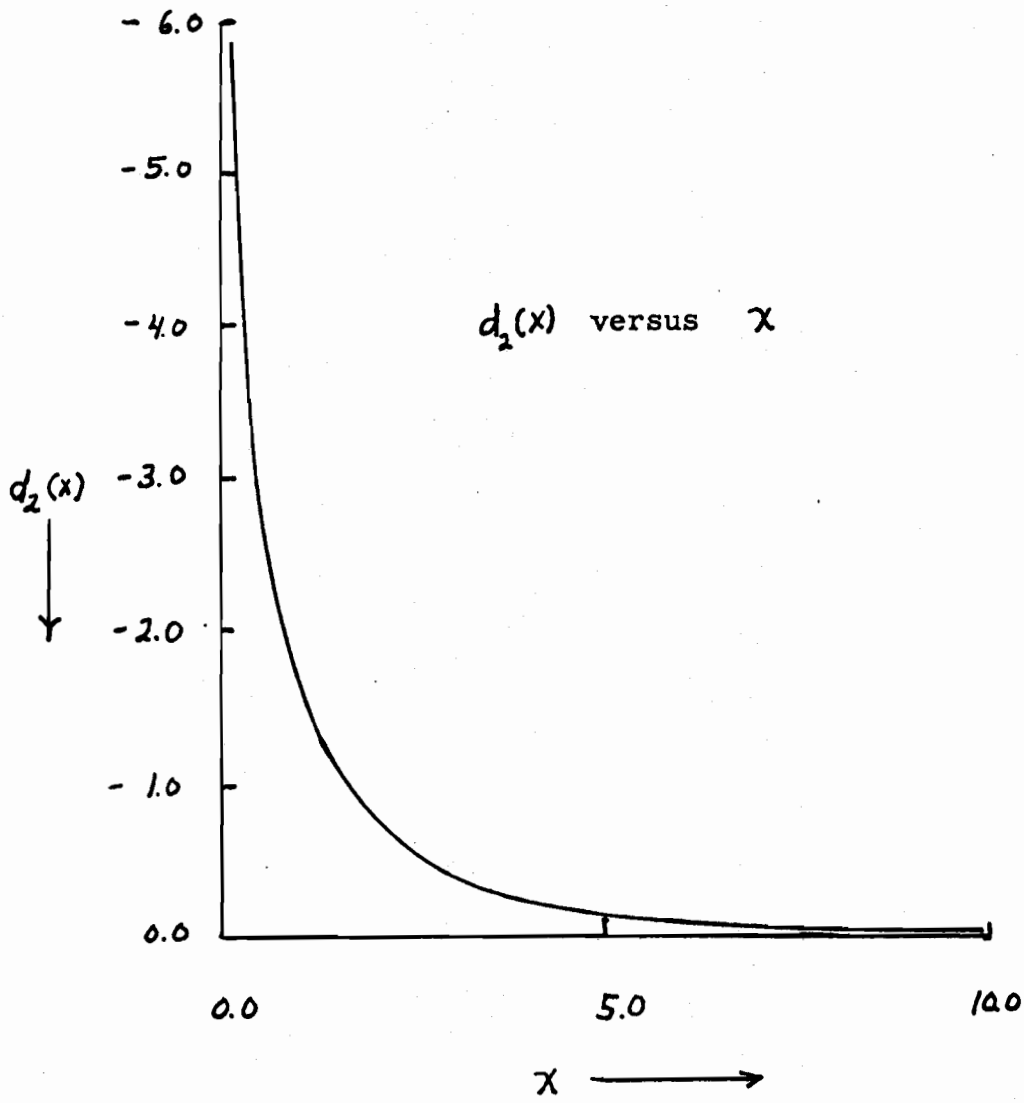
$$d_2(x) = -\frac{4}{x^2} \int_0^x t\, K_0(t)\, dt,$$

and I_0 and K_0 are zeroth order modified Bessel functions.[14]

The quantities $d_1(x)$ and $d_2(x)$ are plotted on Graphs 4 and 5 as functions of x. Since L is a function of energy, the values of the g_n's and hence the form of the potential kernel are energy dependent. The physical distance, Δz, between particles separated by an angle $\Delta\varphi$ increases with the energy of the particles, so that the space-charge force between particles separated by an angle $\Delta\varphi$ is largest at injection energy. For this reason the space-charge



Graph 4

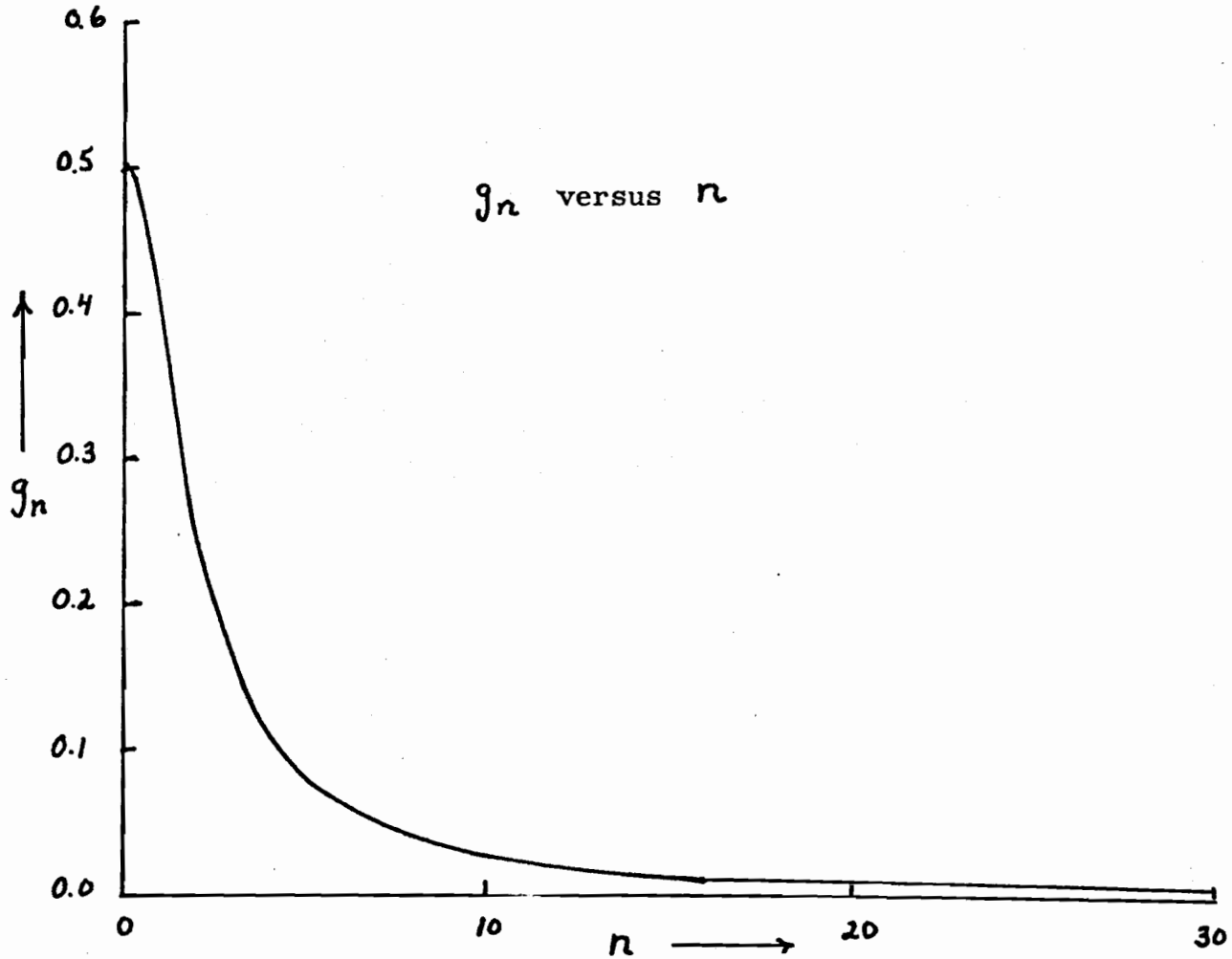


Graph 5

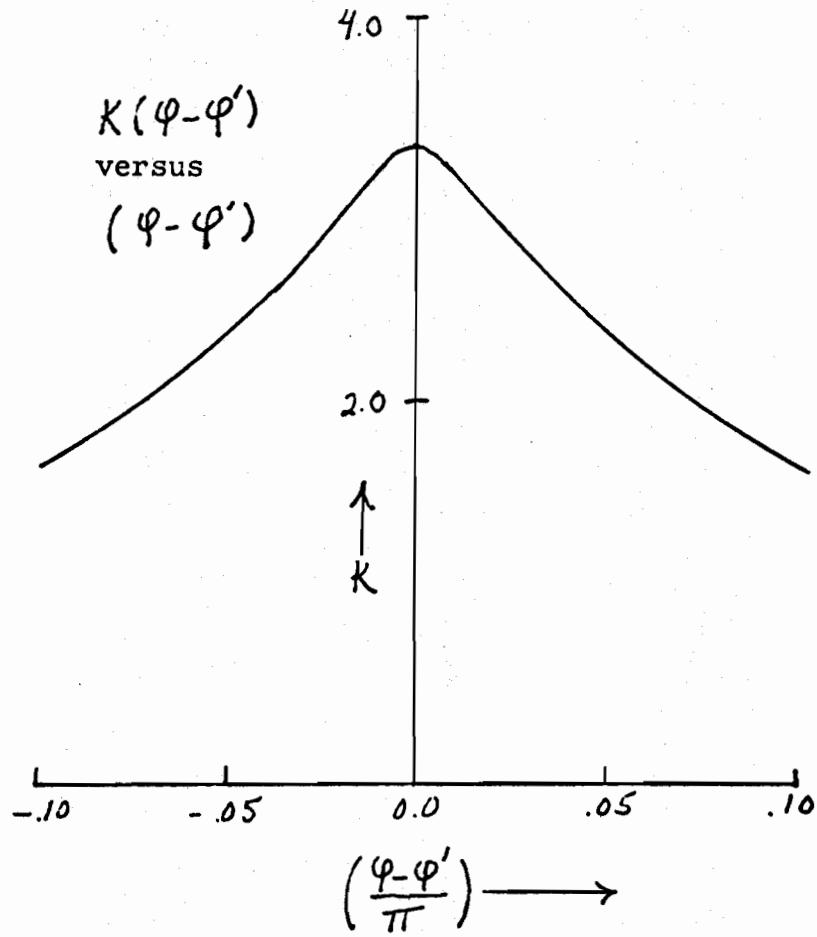
limit occurs at injection energy and we will restrict ourselves to the problem of space-charge forces at injection energy. In the following calculations we will assume for the size of the beam a radius of approximately 0.5 cm and for the size of drift tubes a radius of about 1.5 cm. We use these values along with Graphs 4 and 5 to obtain the values of g_n. Graph 6 shows g_n as a function of n where it is to be understood that n has only integer values and that $g_{-n} = g_n$. From Graph 6 it is seen that $g_n \to 0$ for $n \to \infty$, and therefore we will approximate the potential kernel by letting $g_n = 0$ for $n > n_{max}$. This approximation will be valid if the variation $\lambda(\varphi)$ is small over the angle $\Delta\varphi = \pi / n_{max}$. The potential kernel $K(\varphi - \varphi')$ is plotted as a function of $(\varphi - \varphi')$ on Graph 7 where we have used the first 50 harmonics to calculate $K(\varphi - \varphi')$.

We have approximated the potential kernel by a finite sum of trigonometric functions. We now proceed to further approximate the potential kernel by approximating this finite sum of trigonometric functions by a power series in $(\varphi - \varphi')$ over the range $0 \leq |\varphi - \varphi'| \leq (\alpha - \beta)$. For values of the quantity $(\alpha - \beta)$ which are smaller than $0.1\,\pi$ we will use the following expression for

$$K(\varphi - \varphi') = \frac{2\pi}{L}\left(a_0 - a_1(\varphi - \varphi')^2\right). \tag{6.A.13}$$



Graph 6



Graph 7

The validity of the power series approximation depends upon the range $|\varphi - \varphi'| \leq (\alpha - \beta)$. Since we are interested in a space-charge limit, it is important that this approximation be valid for large values of the phase density. We will show later that as the phase density σ becomes large the quantity $(\alpha - \beta)$ becomes small so that the above expression for $K(\varphi - \varphi')$ will be valid for large phase densities. Care must be exercised to make sure that the quantity $(\alpha - \beta)$ does not become so small as to be of the order of π / n_{max} or else the previous approximation will be invalid.

We now return to the equation $P_b(\varphi)$ in which we have substituted for $U(\varphi)$ its equivalent as an expression in terms of $P_b(\varphi)$.

$$H_b = \frac{P_b^2(\varphi)}{2B} + \frac{e f_0 T}{\omega}(\varphi \cos\varphi_s - \sin\varphi)$$
$$+ \frac{8\pi^2 e^2 \sigma}{\omega L^2} \int_\beta^\alpha |P_b(\varphi')| (a_0 - a_1(\varphi - \varphi_1)^2)\, d\varphi' \tag{6.A.14}$$

The term sin φ will next be expanded as a power series about the value of $\varphi = \varphi_s$. Since we will be concerned with large values of σ, the range of the expansion will be small, and we can approximate sin φ by a cubic polynomial in the variable $(\varphi - \varphi_s)$. We now proceed to find a self-consistent solution for $P_b(\varphi)$ by assuming that we can write

$$P_b^2(\varphi) = \delta^2\left[(\alpha-\varphi)(\varphi-\beta)(\varphi-\gamma)\right],$$

in which the quantities δ, α, β and γ are to be determined.

We now make a change of variables from the variable φ to a variable $\bar{\varphi} = (\varphi - \varphi_s)$. The equation for $P_b(\bar{\varphi})$ becomes

$$\bar{H}_b = \frac{P_b^2(\bar{\varphi})}{2B} + \frac{e f_0 T}{\omega}\left(\frac{\bar{\varphi}^2}{2}\sin\varphi_s + \frac{\bar{\varphi}^3}{6}\cos\varphi_s\right) \tag{6.A.15}$$

$$\frac{8\pi^2 e^2 \sigma}{\omega L^2}\int_{\bar{\beta}}^{\bar{\alpha}} (2a\,\bar{\varphi}\bar{\varphi}' + a_1\bar{\varphi}^2)\,|P_b(\bar{\varphi}')|\,d\bar{\varphi}',$$

with

$$\bar{\alpha} = (\alpha - \varphi_s) \qquad \bar{\beta} = (\beta - \varphi_s), \qquad \bar{\gamma} = (\gamma - \gamma_s)$$

$$P_b^2(\bar{\varphi}) = \delta^2\left[(\bar{\alpha}-\bar{\varphi})(\bar{\varphi}-\bar{\beta})(\bar{\varphi}-\bar{\gamma})\right], \tag{6.A.16}$$

and

$$\bar{H}_b = H_b - \frac{e f_0 T}{\omega}(\sin\varphi_s - \varphi_s\cos\varphi_s)$$

$$+ \frac{8\pi^2 e^2 \sigma}{\omega L^2}\int_{\bar{\beta}}^{\bar{\alpha}} (a_0 + a_1\bar{\varphi}'^2)\,|P_b(\varphi')|\,d\varphi'.$$

The function $P_b(\bar{\varphi})$ is dependent upon the value of the constant $\bar{H}_b$. For certain values of $\bar{H}_b$ two of the three roots $\bar{\beta}$ and $\bar{\gamma}$ are complex and the phase curve $P_b(\bar{\varphi})$ is not closed. Since we are interested in calculating a space-charge limit, we will pick the value for $\bar{H}_b$ such that $\beta = \gamma$ or $\bar{\beta} = \bar{\gamma}$, which is the condition for the maximum area inside of a closed phase curve.

We now substitute for $P_b(\bar{\varphi})$ the expression

$$|P_b(\bar{\varphi})| = \delta(\bar{\alpha} - \bar{\varphi})^{\frac{1}{2}}(\bar{\varphi} - \bar{\beta}) \tag{6.A.17}$$

into the integral equation for $P_b(\bar{\varphi})$ and equate coefficients of similar powers of $\bar{\varphi}$ to obtain the following conditions:

$$\delta^2 = \frac{e f_0 T B}{3\omega} \cos\varphi_s$$

$$\frac{\delta^2}{2B}(\bar{\alpha} + 2\bar{\beta}) + \frac{e f_0 T}{2\omega} \sin\varphi_s = \frac{8\pi^2 e^2 \sigma}{\omega L^2} \frac{4}{15} a_1 \delta (\bar{\alpha} - \bar{\beta})^{5/2}$$

$$\frac{\delta^2}{2B}(\bar{\beta}^2 + 2\bar{\alpha}\bar{\beta}) = \frac{-16\pi^2 e^2 \sigma}{\omega L^2} a_1 \left[\frac{4}{15}\delta\bar{\beta}(\bar{\alpha} - \bar{\beta})^{5/2} + \frac{16}{15}\delta(\bar{\alpha} - \bar{\beta})^{7/2}\right], \tag{6.A.18}$$

and

$$\bar{H}_b = \frac{\delta^2}{2B} \alpha \beta^2 .$$

In the limit that $\sigma \longrightarrow \infty$ we have the conditions that

$$\delta^2 = \frac{e f_0 T B}{3\omega} \cos\varphi_s \quad , \quad \bar{\alpha} = -\tfrac{3}{4}\bar{\beta} \tag{6.A.19}$$

and

$$(\bar{\alpha} - \bar{\beta})^{5/2} = \frac{e f_0 T}{2\omega}\left(\frac{15\,\omega L^2}{32\pi^2 e^2 \sigma a_1}\right) \sin\varphi_s .$$

From these expressions we see that $\bar{\alpha}$ and $\bar{\beta}$ go as $\sigma^{-2/5}$ and hence we are justified in using the quadratic approximation for $K(\varphi - \varphi')$ when σ is large.

The following formula gives the total number of particles, N, accelerated per period of the rf field

$$N = 2\sigma \int_{\bar{\beta}}^{\bar{\alpha}} |P_b(\varphi)|\, d\varphi = \frac{8}{15}\, \sigma\, \delta\, (\bar{\alpha} - \bar{\beta})^{\frac{5}{2}}. \qquad (6.A.20)$$

We substitute for δ, $\bar{\alpha}$ and $\bar{\beta}$ the previous expression and obtain

$$N = \frac{e f_0 T L^2}{8 \pi^2 e^2 a_1} \qquad (6.A.21)$$

The maximum current that one can obtain out of a linac is equal to the charge of the proton times the frequency of the rf field times the quantity N. This gives for the maximum beam current, I, the expression

$$I = \left(\frac{e f_0 T L}{E_e\, a_1}\right)\left(\frac{L \sin \varphi_s}{8 \pi^2 R_0}\right)\left(\frac{e \omega}{2\pi}\right), \qquad (6.A.22)$$

in which E_e is the rest mass energy of the electron (0.511 Mev) and R_0 is the classical radius of the electron (2.82×10^{-13} cm).

The value for a_1 is obtained from Graph 6 and is $a \approx 20$; the values for f_0, T, L and sin φ_s are obtained from Table 1. We substitute these numbers into the formula for I to obtain $I \approx$ 50 ma as the longitudinal space-charge limit in the MURA linac. Although beam current expected from the MURA linac is 30, less than the

calculated longitudinal space-charge limit, the accuracy of the calculated limit cannot be trusted to the extent that we can really say that the designed current will be less than this limit. The inaccuracy of the calculation comes from two conflicting assumptions. The first assumption is that the variation in the charge density $\lambda(\bar{\varphi})$ is small over a distance $\Delta\bar{\varphi} \sim \pi/n_{max}$ where n_{max} is the highest harmonic considered in the potential kernel, while the second assumption is that difference $(\alpha - \beta)$ between the two zero points of $\lambda(\bar{\varphi})$ is small enough to permit approximation of the potential kernel by a quadratic in the variable $(\bar{\varphi} - \bar{\varphi}')$.

While there appears to be a range in the variable $(\bar{\alpha} - \bar{\beta})$ such that both assumptions are justified, for the case where the phase density becomes infinite, the quantity $(\bar{\alpha} - \bar{\beta})$ goes to zero so that the number of harmonics needed to approximate the potential kernel becomes infinite. The result for the space-charge limit is also rather pessimistic since it assumes that longitudinal defocusing force experienced by an off-axis particle is equal to the longitudinal defocusing force acting on an axis particle, but the shielding effect of the drift-tube walls is such as to weaken the defocusing force for off-axis particles. This calculation shows that the approximate space-charge limit is of the order of the design beam current in the MURA case.

It appears that the longitudinal space-charge effect will be very important in limiting beam intensities from linacs. More exact calculations of longitudinal space-charge forces should be done in the future design of linacs.

APPENDIX I

First we perform the integral

$$I = \int_{\beta}^{\alpha} \sqrt{\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\, a^3 \ell + a^2 x - x^3}\; dx ,$$

in which α, β, and γ are roots of the cubic equation

$$\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\, a^3 \ell + a^2 x - x^3 = 0 ,$$

and $\alpha \geq \beta \geq \gamma$. The roots are functions of ℓ and a. However, the roots are linear in a so that we can write

$k^2(\ell) = \left(\frac{\alpha-\beta}{\alpha-\gamma}\right)$ and $\left(\frac{\alpha-\gamma}{a}\right)$ as a function of ℓ only,

and also we write

$$I = \int_{\beta}^{\alpha} \sqrt{(\alpha-x)(x-\beta)(x-\gamma)}\; dx.$$

We make the substitution

$$x = \frac{\beta - \gamma k^2 \sin^2\varphi}{1 - k^2 \sin^2\varphi} ,$$

and obtain

$$I = 2k^3(\alpha-\beta)^{\frac{1}{2}}(\beta-\gamma)\int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi \cos^2\varphi \; d\varphi}{(1-k^2\sin^2\varphi)^{\frac{2}{3}}}$$

We integrate by parts and obtain

$$I = \tfrac{2}{5} k^2 (\alpha - \gamma)^{5/2} (1-k^2)^2 \int_0^{\pi/2} \frac{2\sin^2\varphi - 1}{(1-k^2\sin^2\varphi)^{5/2}}\, d\varphi \, .$$

We let

$$K(k^2) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \, ,$$

and

$$E(k^2) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi \, .$$

in which K and E are called complete elliptic integrals of the first and second kind. We use the relations

$$\frac{dE}{dk} = \frac{E-K}{k} \, , \qquad \frac{dK}{dk} = \frac{E}{k(1-k^2)} - \frac{K}{k} \, ,$$

and the expressions for E and K and we obtain

$$\int_0^{\pi/2} \frac{2\sin^2\varphi - 1}{(1-k^2\sin^2\varphi)^{5/2}} = \frac{2(1-k^2+k^4)\,E(k^2) - (2-3k^2+k^4)\,K(k^2)}{3k^2(1-k^2)^2} \, .$$

This gives

$$I = \frac{2}{15}(\alpha-\gamma)^{5/2}\left\{2(1-k^2+k^4)E - (2-3k^2+k^4)K\right\}.$$

Since $(\alpha-\gamma/a)$ and k^2 are independent of a, we can write

$$I = \frac{2}{15}\, a^{5/2}\, f_2(\ell)$$

Next we want to determine

$$\frac{d\ell}{df_2} = \left(\frac{df_2}{d\ell}\right)^{-1}$$

Since

$$f_2 = \frac{15}{2}\, a^{-5/2} \int_\beta^\alpha \sqrt{\tfrac{2}{3}\sqrt{\tfrac{1}{3}}\, a^3\ell + a^2x - x^3}\; dx,$$

then

$$\frac{df_2}{d\ell} = \frac{5}{2}\sqrt{\frac{1}{3}}\; a^{1/2} \int_\beta^\alpha \frac{dx}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)}}\;.$$

Again we substitute

$$x = \frac{\beta - \gamma k^2 \sin^2\varphi}{1 - k^2\sin^2\varphi}$$

which gives

$$\frac{df_2}{dl} = \frac{5}{\sqrt{3}} \left(\frac{a}{\alpha - \gamma} \right)^{1/2} K(k^2) ,$$

or the inverse

$$\frac{dl}{df_2} = \frac{\sqrt{3}}{5} \left(\frac{\alpha - \gamma}{a} \right)^{1/2} \frac{1}{K(k^2)} .$$

APPENDIX II

The Hamiltonian considered here is of the form

$$\mathcal{H}_a = \frac{P_x^2}{2} + g(\theta)\frac{x^2}{2} .$$

The equations of motion are

$$x' = P_x \qquad \text{and} \qquad P_x' = -g(\theta)\, x ,$$

or rewriting as one second-order differential equation, we obtain

$$x'' + g(\theta)\, x = 0 .$$

This equation has two independent solutions μ_1 and μ_2, where we write $\mu_1 = e^{i\nu_x\theta}\,\psi(\theta)$ and $\mu_2 = e^{-i\nu_x\theta}\,\bar{\psi}(\theta)$. If $g(\theta)$ is nearly a periodic function, i.e., $g(\theta) = \sum g_n e^{in\theta}$ where g_n is a slowly varying function of θ, then ψ and $\bar{\psi}$ are nearly periodic functions of θ. We can then write

$$\psi = \sum \psi_n e^{in\theta} \qquad \text{and} \qquad \bar{\psi} = \sum \bar{\psi}_n e^{-in\theta},$$

where ψ_n and $\bar{\psi}_n$ are slowly varying functions of θ. We substitute this expression for ψ into the differential equation, and neglecting the slow variance of g_n and ψ_n, we obtain

$$(n+\nu_x)^2 \psi_n = \sum_m g_m \psi_{n-m} .$$

We set $n = 0$, and we obtain the equation of the tune

$\nu_x^2 = \frac{1}{\psi_0} \sum g_m \psi_m$. In order to solve for ψ_n, we define

ψ_0 as our normalizing constant and proceed to iterate the equation for ψ_n. Since $g_n \ll 1$, we assume for our first approximation that $\psi_n / \psi_0 = 0$ for $n \neq 0$ and derive a second approximation

$$\psi_n = \psi_0 \frac{g_n}{(n+\nu_x)^2} \quad , \quad n \neq 0$$

We use this approximation to obtain a third approximation

$$\psi_n \approx \psi_0 \frac{1}{(n+\nu_x)^2} \left\{ g_n + \sum_m \frac{g_{n-m}\, g_m}{(n-m+\nu_x)^2} \right\}, \quad n \neq 0$$

We can repeat this procedure any number of times and for $g_n < 1$ it will converge if ν_x is not close to integer. We stop with the second approximation for ψ_n, and we obtain for ν_x the following "smooth" approximation:

$$\nu_x^2 = g_0 + \sum_{n \neq 0} \frac{g_{-n}\, g_n}{(n+\nu_x)^2}$$

Since $\nu_x^2 \ll 1$, we can write

$$\nu_x^2 = g_0 + \sum_{n \neq 0} \frac{g_{-n}\, g_n}{n^2}$$

Notice that we cannot write $\nu_x^2 \approx g_0$ since even for $|g_n| \ll 1$, it is possible that $|g_n| \gg g_0$ for $n \neq 0$. However, it is possible to write $\psi_n / \psi_0 \ll 1$, $n \neq 0$ for

many other cases, and in particular, in the expressions for $\Omega^{(4)}_{1,k,l,k',l',m}$ we will be justified to use only the ψ_0 term.

We return now to the Hamiltonian

$$\mathcal{H}_a = \frac{P_x^2}{2} + g(\theta)\frac{x^2}{2}$$

We again use μ_1 and μ_2 as the two independent solutions and write

$$\mu_1 = e^{i\nu_x\theta}\psi(\theta) \quad , \quad \mu_2 = e^{-i\nu_x\theta}\bar{\psi}(\theta),$$

$$\mu_1' = i\,e^{i\nu_x\theta}\eta(\theta) \qquad \mu_2' = -i\,e^{-i\nu_x\theta}\bar{\eta}(\theta),$$

where

$$\eta = \nu_x\psi - i\psi' \quad , \quad \bar{\eta} = \nu_x\bar{\psi} + i\bar{\psi}'.$$

From the differential equation, we obtain

$$\eta' = i(g\psi - \nu_x\eta) \quad , \quad \bar{\eta}' = -i(g\bar{\psi} - \nu_x\bar{\eta}).$$

The Wronskian, $(\mu_1\mu_2' - \mu_1'\mu_2)$, is a constant which we will pick for normalization purposes equal to - 4 i. That is

$$(\bar{\eta}\psi + \eta\bar{\psi}) = 4 \ .$$

In transforming from (x, P_x) variables to $(\xi, -i\bar{\xi})$, we use the generating function

$$G = \frac{1}{\bar{\eta}}\left(\frac{i}{2}\eta\xi^2 - 2\xi P_x - \frac{i}{2}\bar{\psi}P_x^2\right),$$

which gives

$$-i\bar{\xi} = -\frac{1}{\bar{\eta}}\{i\eta\xi - 2P_x\} \quad \text{or} \quad P_x = \frac{i}{2}(\eta\xi - \bar{\eta}\bar{\xi}),$$

$$\chi = \frac{1}{\eta}(2\xi + i\Psi P_x) = \frac{1}{2}(\Psi\xi + \bar{\Psi}\bar{\xi}),$$

and $\mathcal{H}_b = \mathcal{H}_a + \frac{\partial G}{\partial \theta} = \nu_x \xi\bar{\xi}$,

where we have used the properties of Ψ, $\bar{\Psi}$, η and $\bar{\eta}$ which were discussed previously.

From the normalization condition we see that if

$$\Psi = \sum_n \Psi_n e^{in\theta} \simeq \Psi_0 \quad \text{and} \quad \bar{\Psi} = \sum \bar{\Psi}_n e^{-in\theta} \simeq \Psi_0,$$

then

$$2\nu_x \Psi_0^2 \simeq 4 \quad \text{or} \quad \Psi_0 = \sqrt{2/\nu_x}.$$

APPENDIX III

In this appendix we will solve for the potential of an on-axis particle as a function of the charge distribution. We will assume that the charge distribution is periodic in the variable $kz - \omega t$ with a period of 2π and can be represented by a step function in the variable r. Thus we assume that the charge density ρ can be written as follows:

$$\rho(r, z) = \sum_{n=0}^{\infty} Q_n(r)\, e^{in(kz - \omega t)},$$

where $Q_n(r) = \bar{Q}_n =$ constant for $0 < r < b$ and $Q_n(r) = 0$ for $r > b$.

We will also assume that this charge distribution is enclosed in an infinitely long circular pipe with infinitely conducting walls. This gives the boundary condition that the potential equals zero at the walls of the pipe. We now make the transformation to a moving coordinate system such that $\bar{z} = z - \frac{\omega}{k} t$ which gives for the charge density

$$\rho(r, \bar{z}) = \sum_{n=0}^{\infty} Q_n(r)\, e^{ink\bar{z}}$$

We let $U(r, \bar{z})$ be the potential function which we assume can be written as

$$U(r,\bar{z}) = \sum_{n=0}^{\infty} U_n(r)\, e^{ink\bar{z}}$$

This potential function must solve the following Poisson's equation as well as the previous boundary condition

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}U(r,z)\right) + \frac{\partial^2}{\partial z^2}U(r,z) = -4\pi\rho(r,z)$$

Upon substituting a Fourier series for $U(r,z)$ and $\rho(r,z)$ Poisson's equation becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}U_n(r)\right) - n^2k^2U_n(r) = -4\pi Q_n(r)$$

The solutions to the homogeneous part of this differential equation are zeroth order modified Bessel functions. We represent the two solutions of the homogeneous equation by $y_1(r)$ and $y_2(r)$ where

$$y_1(r) = I_0(nkr) \quad , \quad y_2 = K_0(nkr).$$

The Wronskian of y_1 and y_2 will be denoted by $W(y_1,y_2,r)$ and is given below

$$W(y_1,y_2,r) = y_1y_2' - y_1'y_2 = -\frac{1}{r} \quad .$$

The general solution to the above differential equation is equal to a specific solution plus $C_1 Y_1$ plus $C_2 Y_2$ where C_1 and C_2 are constants which are chosen to match the boundary conditions. The specific solution to the differential equation is given below as

$$V(r,\bar{z}) = \sum_n V_n(r)\, e^{i n k \bar{z}},$$

$$V_n(r) = \int_0^r \frac{-4\pi Q_n(t)}{W(y, y_0, t)} \left(Y_1(t)\, y_2(r) - Y_1(r)\, y_2(t) \right) dt,$$

which becomes for $r > b$

$$V_n(r) = 4\pi \bar{Q}_n \left\{ \left(\int_0^b t\, I_0(nkt)\, dt \right) K_0(nkr) - \left(\int_0^b t\, K_0(nkt)\, dt \right) I_0(nkr) \right\},$$

and for $r < b$

$$V_n(r) = 4\pi \bar{Q}_n \left\{ \left(\int_0^r t\, I_0(nkt)\, dt \right) K_0(nkr) - \left(\int_0^r t\, K_0(nkt)\, dt \right) I_0(nkr) \right\}.$$

We now use the boundary conditions that $U(a, \bar{z}) = 0$ where a is the radius of the cylindrical pipe, and that $U(0, \bar{z})$ be finite to determine the constants C_1 and C_2 in the equation

$$U_n(r) = V_n(r) + C_1 y_1(r) + C_2 y_2(r)$$

It follows from the second boundary condition that C_2 must equal 0, and from the first boundary condition it follows that

$$C_1 = 4\pi \bar{Q}_n \left\{ \left(\int_0^b t K_0(nkt)\,dt \right) - \left(\int_0^b t I_0(nkt)\,dt \right) \frac{K_0(nka)}{I_0(nka)} \right\}.$$

By defining $\varphi = k\bar{z}$ and $\lambda(\varphi)$ as the linear charge density per unit angle φ, we can write

$$\lambda(\varphi) = \sum_{n=0}^{\infty} \lambda_n e^{in\varphi},$$

where

$$\lambda_n = \frac{\pi}{k} b^2 \bar{Q}_n .$$

We substitute this expression for λ_n into the equation for $U_n(r)$ to obtain for the case $r = 0$

$$U_n(0) = k\lambda_n \left\{ \frac{4}{b^2} \left(\int_0^b t K_0(nkt)\,dt \right) - \frac{4}{b^2} \left(\int_0^b t I_0(nkt)\,dt \right) \frac{K_0(nka)}{I_0(nka)} \right\},$$

or

$$U_n(0) = -k\lambda_n \left\{ d_2(nkb) + d_1(nkb)\frac{K_0(nka)}{I_0(nka)} \right\},$$

where

$$d_2(nkb) = \frac{-4}{(nkb)^2}\int_0^{nkb} t\,K_0(t)\,dt,$$

and

$$d_1(nkb) = \frac{4}{(nkb)^2}\int_0^{nkb} t\,I_0(t)\,dt.$$

The coefficient λ_n is equal to $\frac{1}{2\pi}\int_0^{2\pi}\lambda(\varphi)\,e^{-in\varphi}\,d\varphi$

which gives the following expression for the potential function of an on-axis particle

$$U(\varphi) = k\int_0^{2\pi}\left(\sum_n g_n e^{in(\varphi-\varphi')}\right)\lambda(\varphi')\,d\varphi',$$

where

$$g_n = g_{-n} = -\frac{1}{2\pi}\left\{ d_2(nkb) + d_1(nkb)\,\frac{K_0(nka)}{I_0(nka)} \right\},$$

and in the limit that $n \to 0$

$$g_n \to g_0 = \frac{1}{2\pi}\left\{ 1 + 2\ln a/b \right\}.$$

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