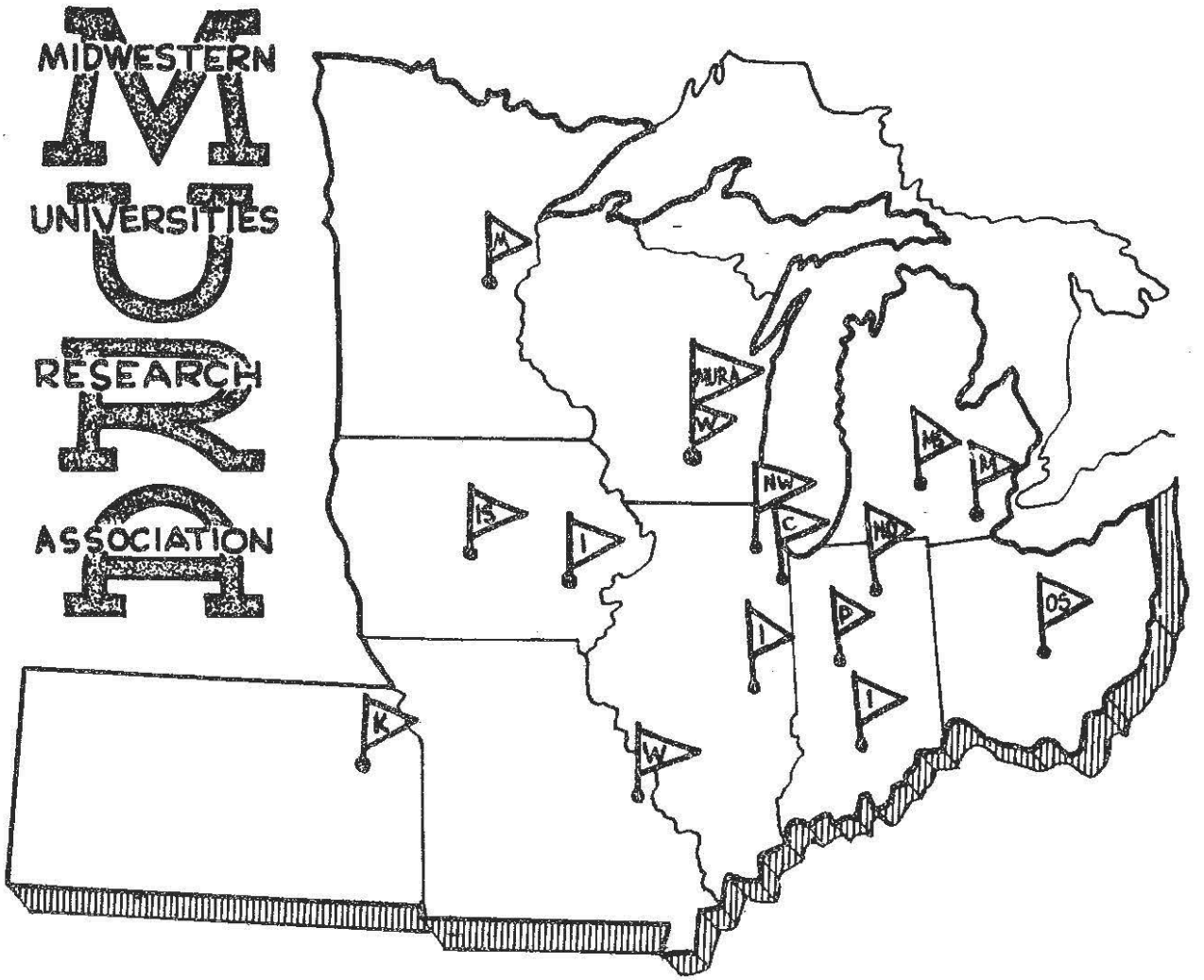

PERTURBATION THEORY FOR ACCELERATORS WITH A GENERAL MAGNETIC FIELD - II

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George Parzen**

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ABSTRACT

The effects of a perturbing field on the linear properties of an accelerator with a general magnetic field are calculated. The perturbing magnetic field in the median plane is assumed to have only a vertical component and to contain no harmonics having the same periodicity as the unperturbed field. Results are found for the change in the equilibrium orbit and in the betatron oscillation frequencies caused by the perturbation. This paper is a continuation of a previous paper in which the perturbing field was assumed to have the same period as the unperturbed field.

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This paper is a continuation of the report MURA-634 on the effects of a perturbing field H_z on the linear orbit properties of an accelerator with a general magnetic field. In MURA-634, the perturbation H_z was assumed to have only those harmonics G_n, for $n = 0, \pm N, \pm 2N$; that is, the perturbation has the same period, $2\pi/N$, as the unperturbed field. We now treat the case where the perturbing field has all the harmonics G_n for which $n \neq 0, \pm N, \cdots$.

It will be found that the perturbing harmonics G_n, $n \neq 0, \pm N, \cdots \pm 2N \cdots$, will produce a second-order shift in the tune in contrast to the first-order shift produced by the G_n, $n=0, \pm N, \pm 2N \cdots$. The higher order of the shift in tune is counterbalanced to some extent by the resonance effects that enter here, particularly for harmonics whose n is near the tune of the unperturbed machine.

I. Equilibrium Orbit

The first step is to find the effect on the equilibrium orbit. Expanding around the equilibrium orbit of the unperturbed machine, $r_s(\theta)$, we write $r = r_s(\theta) + \rho$, where satisfies

$$\rho'' - \rho = \frac{er^2}{pc} H_z - \left(\frac{er^2}{pc} H_z\right)_{r_s} . \tag{2.1}$$

We expand in powers of ρ keeping up to linear terms in ρ, and writing $\rho = \bar{x}R$, where R is the average radius of the unperturbed orbit with the same momentum, we find

$$\bar{x}'' + \bar{n}(\theta)\, \bar{x} = \Delta f\ (\theta), \tag{2.2a}$$

$$\bar{n}(\theta) = -\left[\frac{\partial}{\partial r}\left(\frac{er^2}{pc}\bar{H}_z\right)\right]_{r_s} - 1 \tag{2.2b}$$

$$= -\frac{er_s}{pc}\left[r\,\bar{H}_{z,r} + 2\bar{H}_z\right]_{r_s} - 1$$

$$= -\frac{eR}{pc}\left[\left(r\,\bar{H}_{z,r} + 2\bar{H}_z\right)_R + x\left(R^2\,\bar{H}_{z,rr} + 4\,r\,\bar{H}_{z,r} + 2\bar{H}_z\right)_R\right] - 1,$$

$$f(\theta) = \frac{1}{R}\frac{er^2}{pc}\left[\Delta H\right]_{r_s} \tag{2.2c}$$

$$\doteq \frac{er}{pc}\left[H_z + x\left(r\Delta H_{z,r} + 2\Delta H_z\right)\right]_R.$$

We solve Eq. (2.2) is a similar way as was done in MURA-634. We write $\bar{n}(\theta)$ as

$$\bar{n}(\theta) = \bar{E}_s - \bar{g}(\theta). \tag{2.3}$$

We note that $E_s = E_o$, since $G_n = 0$, for $n = 0$, $\pm N$, $\pm 2N\cdots$, and $x_n = 0$ for $n \neq 0$, $\pm N$, $\pm 2N\cdots$. Also

$$\bar{g}_m = g_m + \Delta g_m \tag{2.4a}$$

$$g_m = -\frac{eR}{pc}\left(R\,\Delta G_m' + 2\Delta G_m\right), \tag{2.4b}$$

$$g_m = -\frac{eR}{pc}\left(R\,G_m' + 2\,G_m\right). \tag{2.4c}$$

The harmonics of $\Delta f(\theta)$ are given by

$$f_s = 0; \tag{2.5a}$$

$$f_m = \frac{eR}{pc}\left\{-\Delta G_m - \sum_s x_s\left(R\,\Delta G_{m-s}' + 2\,\Delta G_{m-s}\right)\right\}. \tag{2.5b}$$

In Eq. (5) we use the notation that the subscript s takes on the values 0, $\pm N$, $\pm 2N\cdots$, while the subscript m takes on all values except $m = 0$, $\pm N$, $\pm 2N\cdots$.

We expand $\bar{x}$ in the set of function $\bar{u}_n(\theta)$ which are the solutions of the equation

$$\frac{d^2}{d\theta^2}\,\bar{u}_n + (\bar{E}_n - \bar{g}(\theta))\,\bar{u}_n = 0. \tag{2.6}$$

The E_n are chosen so that the $\bar{u}_n(\theta)$ have the tune $\nu = 0, \pm 1, \pm 2, \pm 3 \cdots$. We write

$$x = \sum_n a_n\, u_n(\theta),$$

when the a_n are given by

$$a_n = \frac{1}{\bar{E}_s - \bar{E}_n}\,\frac{1}{2\pi}\int_0^{2\pi} d\theta\; \bar{u}_n{}^*(\theta)\,\Delta f(\theta) \tag{2.7}$$

If a_n is computed to first order in the perturbation, then it is sufficient to compute $\bar{E}_s$, $\bar{E}_n$ and $\bar{u}_n$ without including the effect of the perturbation. Thus

$$u_n(\theta) = e^{in\theta}\left\{1 - \sum_{s=\pm N} \frac{g_s}{(s+n)^2 - E_n}\, e^{is\theta}\right. \tag{2.9a}$$

$$\bar{E}_n = E_n = n^2 - \sum_{s=\pm N} \frac{|g_s|^2}{s^2 - 4n^2} \tag{2.9b}$$

Eqs. (29) are clearly not valid when $n = N/2$. We will assume that a_n for $n = N/2$ is not important and can be neglected in the following, which is usually the case. If this is not the case, the procedure can be modified to handle this case without a great deal of trouble.

Now since E_o is related to the unperturbed tune by

$$E_o = \nu_r^{\,2} + \sum_{s=\pm N} \frac{|g_s|^2}{s^2 - 4r^2}\ , \tag{2.10}$$

we find that

$$\bar{E}_s - \bar{E}_n = \nu_r^{\,2} - n^2\ . \tag{2.11}$$

Thus the solution given by Eq. (2.7) exhibits a resonant behavior for $n \simeq \nu_r$. We find for the a_n

$$a_n = \frac{1}{\nu_r^2 - n^2}\left\{\Delta f_n - \sum_{s=\pm N} \frac{g_s}{(s+n)^2 - E_n} f_{n-s}\right\}. \tag{2.12}$$

Thus

$$a_s = 0, \tag{2.13a}$$

$$a_m = \frac{1}{m^2 - \nu_r^2}\left\{\frac{eR}{pc}\Delta G_m + \frac{eR}{pc}\sum_{s=\pm N} x_s \left(R\Delta G'_{m-s} + 2\Delta G'_{m-s}\right)\right.$$

$$\left. + \sum_{s=\pm N} \frac{g_s}{(s+m)^2 - E_m} \frac{eR}{pc}\Delta G_{m-s} \cdots .\right\}. \tag{2.13b}$$

We write Eq. (2.13) as

$$a_s = 0 \tag{2.14a}$$

$$a_m = \frac{1}{m^2 - \nu_r^2}\left\{\frac{eR}{pc}\Delta G_m + \left(\frac{eR}{pc}\right)^2 \sum_{s=\pm N}\left[\frac{G_s}{s^2}\left(R\Delta G'_{m-s} + 2\Delta G_{m-s}\right)\right.\right.$$

$$\left.\left. + \frac{(RG_s + 2G_s)}{s^2}\Delta G_{m-s}\right]\right\}. \tag{2.14b}$$

We have further assumed in Eq. (2.14), that $m < N$ gives the interesting a_m.

The harmonics in $x(\theta)$ may now be found,

$$\bar{x}(\theta) = \sum_n a_n e^{in\theta}\left\{1 - \sum_{s=\pm N} \frac{g_s}{(s+n)^2 - E_n} e^{is\theta}\right\}, \tag{2.15}$$

and thus

$$\bar{x}_n = a_n - \sum_{n',\, s=\pm N} a_{n'} \frac{g_s}{s(s+n')} \delta(s+n', n), \tag{2.16}$$

where $\delta(s+n',n) = 1$ if $s+n' = n$ and vanishes otherwise. We still assume that the largest $\bar{x}_n$ occurs for the value of n closest to ν_r which will be called $n = \nu$. We then find that

$$\bar{x}_\nu = \frac{1}{\nu^2 - \nu_r^2} \frac{eR}{pc} \Delta G_\nu \tag{2.17a}$$

$$\bar{x}_m = \frac{1}{m^2 - \nu_r^2} \left(\frac{eR}{pc}\right) \Delta G_m \tag{2.17b}$$

$$- \sum_{s=\pm N} \left\{ a_\nu \frac{g_s}{s(s+2\nu)} \delta(s+\nu,m) + a_{-\nu} \frac{g_s}{s(s-2\nu)} \delta(s-\nu,m) \right\},$$

for $m \neq \pm\nu$. One may note that the second term in Eq. (2.17b) only contributes when $m = N \pm \nu$.

Equations (2.17) give the shift in the equilibrium order correct to first order in the perturbation. It will be seen later that the shift in tune is a second-order effect being proportional to the square of the perturbing field. In order to be able to correctly calculate the shift in tune, it is found necessary to compute the $\bar{x}_n$ for $n = 0, \pm N$ correct to second-order terms of the perturbation. We will now retrace our steps in order to get the second-order terms in $\bar{x}_0$ and $\bar{x}_n$. Keeping up to second-order terms we find for $f(\theta)$

$$f(\theta) = \frac{1}{R}\left[\frac{er^2}{pc} \Delta H_z + \frac{1}{2} \frac{\partial^2}{\partial r^2} \left(\frac{er^2}{pc} H_z\right) R^2 \bar{x}^2 ..\right]_{r_s}$$

$$f(\theta) = \frac{1}{R}\left[\frac{er^2}{pc} \Delta H_z + \frac{eR^2}{2pc} (r^2 H_{z,rr} + 4rH_{z,r} + 2H_z)\, \bar{x}^2\right]_{r_s} \tag{2.18}$$

$$f(\theta) = \frac{eR}{pc}\left[H_z + x\,(R\ H_{z,r} + 2\ H_z) + (R^2 H_{z,rr} + 4RH_{z,r} + 2H_z)\, \bar{x}^2 \right]_R .$$

The presence of the $\bar{x}^2$ term in $\Delta f(\theta)$ makes Eq. (2.2) a nonlinear equation in $\bar{x}$ which we solve by successive approximations by substituting for $\bar{x}$ in the $\bar{x}^2$ term the first-order solution found for $\bar{x}$ above.

We then find for the harmonics of $\Delta f(\theta)$

$$\Delta f_s = \frac{-eR}{pc} \sum_{s',m,m'} (R^2 G''_{s'} + 4RG'_{s'} + 2G_{s'}) \Delta x_m \Delta x_{m'} \delta(m+m'+s;s) \tag{2.19}$$

$$\Delta f_s = -\left(\frac{eR}{pc}\right) \sum_{s;m,m'} (R^2 G''_{s'} + 4RG'_{s'} + 2G_{s'}) \frac{\Delta G_m}{m^2 - \nu_r^2} \frac{\Delta G_{m'}}{m'^2 - \nu_r^2} \delta(m+m'+s;s). \tag{2.20}$$

In Eq. (2.9) for $\bar{u}_n$ we must keep up to first-order terms, and we find for the a_n to replace Eq. (2.12)

$$a_n = \frac{1}{\nu_r^2 - n^2} \left\{ \Delta f_n - \sum_{n'n''} \frac{\bar{g}_{n''} \Delta f_{n'}}{(n''+n')^2 - E_{n'}} \delta(n'+n'';n) \right\} \tag{2.21}$$

and the $\bar{x}_n$ are given by

$$\bar{x}_n = a_n - \sum_{n;n''} a_{n'} \frac{\bar{g}_{n''}}{(n''+n')^2 - E_{n'}} \delta(n'+n'';n) \quad . \tag{2.22}$$

Thus $\bar{x}_n$ is given by

$$\bar{x}_n = \frac{\Delta f_n}{\nu_r^2 - n^2} - \frac{1}{\nu_r^2 - n^2} \sum_{n;n''} \frac{\bar{g}_{n''} \Delta f_{n'}}{n^2 - E_{n'}} \delta(n'+n'';n) \tag{2.23a}$$

$$- \sum_{n'n''} \frac{\Delta f_{n'}}{\nu_r^2 - n'^2} \frac{\bar{g}_{n''}}{n^2 - E_{n'}} \delta(n'+n'';n),$$

$$\bar{x}_n = \frac{\Delta f_n}{\nu_r^2 - n^2} - \sum_{n;n''} \frac{\bar{g}_{n''} \Delta f_{n'}}{m^2 - n'^2} \delta(n'+n'';n) \tag{2.23b}$$

$$\times \left[\frac{1}{\nu_r^2 - n^2} + \frac{1}{\nu_r^2 - n'^2} \right] \quad .$$

We find then for x_s,

$$\bar{x}_s = -\frac{1}{\nu_r^2-s^2}\frac{eR}{pc}\sum_{s',m,m'}(R\,G''_{s'}+4RG'_{s'}+2G_{s'})\tag{2.24}$$

$$\frac{\Delta G_m}{m^2-\nu_r^2}\frac{\Delta G_{m'}}{m'^2-\nu_r^2}\,\delta(m+m'+s;s)$$

$$+\left(\frac{eR}{pc}\right)^2\sum_{m,m'}\frac{1}{m^2}\,\Delta G_m\,(R\,\Delta G'_{m'}+2\,\Delta G_{m'})\,\delta(m+m;s)$$

$$\times\left[\frac{1}{\nu_r^2-s^2}+\frac{1}{\nu_r^2-m'^2}\right]\quad.$$

II. Tune Shift

We now find the shift in radial tune, ν_r. We proceed as in MURA-634 to obtain the linear equation for the oscillations around the equilibrium orbit

$$\rho'' + \bar{n}_u(\theta) = 0,\tag{3.1a}$$

$$\bar{n}_u(\theta) = -\frac{e}{pc}\left[r^2\bar{H}_{z,r}+2r\bar{H}_z\right]_{\bar{r}_s} -1,\tag{3.1b}$$

where $\bar{r}_s(\theta)$ is the displaced orbit found in section II. We expand $\bar{n}_u$ in powers of $\bar{x}(\theta)$, and keep up to second terms in the perturbation,

$$\bar{n}_u(\theta) = -\frac{e}{pc}\Big\{(r^2\bar{H}_{z,r}+2r\bar{H}_z)_{r_s}\tag{3.2}$$

$$+\frac{\partial}{\partial r}(r^2\bar{H}_{z,r}+2r\bar{H}_z)_{r_s}R\,\bar{x}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial r^2}(r^2\bar{H}_{z,r}+2r\bar{H}_z)_{r_s}R^2\bar{x}^2+\ldots\quad.$$

In Eq. (3.2), the coefficients of the powers of $\bar{x}$ are to be evaluated on the unperturbed equilibrium orbit $r_s(\theta) = R(1+x(\theta))$. We expand these coefficients in powers of x and obtain

$$\bar{n}_u(\theta) = -\frac{eR}{pc}\Big\{ R^2\bar{H}_{z,r}+2\bar{H}_z+x(R^2\bar{H}_{z,rr}+4R\bar{H}_{z,r}+2\bar{H}_z) \tag{3.3}$$
$$+\bar{x}\left[R^2\bar{H}_{z,rr}+4R\bar{H}_{z,r}+2\bar{H}_z+x(R^3\bar{H}_{z,rrr}+6R^2\bar{H}_{z,rr}+6R\bar{H}_{z,r})\right]$$
$$+\frac{1}{2}\bar{x}^2\Big[R^3\bar{H}_{z,rrr}+6R^2\bar{H}_{z,rr}+6R\bar{H}_{z,r}+x(R^4\bar{H}_{z,rrrr}+8R^3\bar{H}_{z,rrr}$$
$$+12R^2\bar{H}_{z,rr})\Big]\Big\}$$
$$+1.$$

In Eq. (3.3), $\bar{H}_z$ and its derivatives are to be evaluated at $r = R$. We write $\bar{n}_u$ as

$$n_u(\theta) = E_o+\Delta E_o-(g(\theta)+\ \ g(\theta)),$$

where $E_o+\Delta E_o$ is the average value of $\bar{n}_u$ over the period of the magnetic field. We find for ΔE_o,

$$\Delta E_o = \frac{eR}{pc}\sum_s \bar{x}_{-s}\ (R\ G_s''+4RG_s'+2G_s) \tag{3.4}$$
$$+\frac{eR}{pc}\sum_{m,m',s}\frac{1}{2}\bar{x}_m\bar{x}_{m'}(R^3G_s'''+6R^2G_s''+6RG_s')\ \delta(m+m'+s,o)$$

We find for the Δg_n,

$$\Delta g_s = -\frac{eR}{pc}\sum_{s'}\bar{x}_{s-s'}(R\ \bar{G}_{s'}''+4RG_{s'}'+2G_{s'}) \tag{3.5a}$$
$$-\frac{eR}{pc}\sum_{mm's'}\frac{1}{2}\bar{x}_m\bar{x}_{m'}(R^3G_{s'}'''+6R^2\bar{G}_{s'}''+6RG_{s'}')\ \delta(m+m'+s',s)$$

$$\Delta g_m = -\frac{eR}{pc}\ (R\Delta G_m'+2\Delta G_m) \tag{3.5b}$$
$$-\frac{eR}{pc}\sum_{m'}\bar{x}_{m'}(R^2G_s''+4RG_s'+2G_s')\ \delta(m'+s,m)\quad .$$

One might notice that to get an answer for the tune shift which is correct to second-order in the perturbing field, one has to calculate ΔE_0 and Δg_s correct to second-order while Δg_m is required only to first-order. For simplicity, terms proportional to x have been omitted in the expression for ΔE_0, Δg_s.

The perturbed tune ν_r may be found from ΔE_0 and Δg_n according to (see MURA-634)

$$\bar{\nu}_r^2 - \nu_r^2 = \Delta E_0 + \sum_{s \neq 0} \frac{g_s \Delta g_{-s} + g_{-s} \Delta g_s}{s(s - 2\nu_r)} + 2 \sum_{m \geq 0} \frac{|\Delta g_m|^2}{m^2 - 4\nu_r^2} , \tag{3.6}$$

where ν_r is the unperturbed tune.

We now find the shift in the vertical tune, ν_z. We proceed as in MURA-634 to obtain the linear equation for the vertical oscillations around the equilibrium orbit

$$z'' + \bar{n}_z(\theta)\, z = 0, \tag{3.7a}$$

$$n_z(\theta) = - \frac{e}{pc} \left[r' \bar{H}_{z,\theta} - r^2 \bar{H}_{z,r} \right]_{\bar{r}_s} , \tag{3.7b}$$

where $\bar{r}_s(\theta)$ is the perturbed equilibrium orbit found in Section II. We expand $\bar{n}_z(\theta)$ in powers of $\bar{x}(\theta)$ and keep up to second-order terms,

$$\begin{aligned} \bar{n}_z(\theta) = - \frac{e}{pc} \Big\{ & (r'\bar{H}_{z,\theta})_{r_s} + (\bar{H}_{z,\theta})_{r_s} R\bar{x}' + (r'\bar{H}_{z,\theta r})_{r_s} R\bar{x} \\ & + \frac{1}{2} (r'\bar{H}_{z,\theta rr})_{r_s} R^2\bar{x}^2 + (\bar{H}_{z,\theta r})_{r_s} R^2\bar{x}\,\bar{x}' \\ & - (r^2\bar{H}_{z,r})_{r_s} - (r^2\bar{H}_{z,rr} + 2r\bar{H}_{z,r})_{r_s} R\bar{x} \\ & - \frac{1}{2} (r^2\bar{H}_{z,rrr} + 4r\bar{H}_{z,rr} + 2\bar{H}_{z,r})\, R^2\bar{x}^2 \Big\} . \end{aligned} \tag{3.8}$$

Again we write the unperturbed orbit $r_s(\theta) = R(1+x(\theta))$ and expand in powers of x,

$$\bar{n}_z(\theta) = -\frac{eR}{pc}\Big\{-R\bar{H}_{z,r}+x'\bar{H}_{z,\theta}-x(R^2\bar{H}_{z,rr}+2R\bar{H}_{z,r}) \tag{3.9}$$
$$+\bar{x}\Big[-R^2\bar{H}_{z,rr}-2R\bar{H}_{z,r}-x(R^3\bar{H}_{z,rrr}+4R^2\bar{H}_{z,rr}$$
$$+2R\bar{H}_{z,r})+x'(R\bar{H}_{z,\theta r})\Big]$$
$$+\bar{x}'\bar{H}_{z,\theta}$$
$$+\frac{1}{2}\bar{x}^2\Big[-R^3\bar{H}_{z,rrr}-4R^2\bar{H}_{z,rr}-2R\bar{H}_{z,r}-x(R^4\bar{H}_{z,rrr}$$
$$+6R^3\bar{H}_{z,rrr}+6R^2\bar{H}_{z,rr})+x'(R^2\bar{H}_{z,\theta rr})$$
$$+\bar{x}\,\bar{x}'\Big[R\,\bar{H}_{z,\theta r}-x\,R^2\bar{H}_{z,\theta rr}\Big]\Big\}.$$

We write $\bar{n}_z$ as

$$\bar{n}_z(\theta) = E_o' + \Delta E_o' - (f(\theta)+\Delta f(\theta)), \tag{3.10}$$

where $E_o' + \Delta E_o'$ is the average value of $\bar{n}_z$ over the period of the magnetic field. We find for $\Delta E_o'$,

$$E_o' = \frac{eR}{pc}\sum_s \bar{x}_{-s}\,(-R^2G_s''-2RG_s') \tag{3.11}$$
$$+\frac{eR}{pc}\sum_s s^2\,\bar{x}_{-s}\,G_s$$
$$+\frac{eR}{pc}\sum_{m,m',s}\frac{1}{2}\bar{x}_m\bar{x}_{m'}\,(-R\,G_s'',-4R\,G_s''-2RG_s').$$
$$\delta(m+m's,o)$$
$$+\frac{eR}{pc}\sum_{m,m',s}(-ms)\,\bar{x}_m\bar{x}_{m'}R\,G_s'\quad.$$

We find for the Δf_n

$$\Delta f_s = \frac{eR}{pc} \sum_{s'} \bar{x}_{s-s'} (+R^2 G''_{s'} + 2RG'_{s'}) + \frac{eR}{pc} \sum_{mm's'} \frac{1}{2} \bar{x}_m \bar{x}_{m'}$$

$$(+R^3 G'''_{s'} + 4R^2 G''_{s'} - 2RG'_{s'}) \, \delta(m+m'+s; s). \qquad (3.12a)$$

$$+ \frac{eR}{pc} \sum_{mm's'} ms' \, \bar{x}_m \bar{x}_{m'} RG'_{s'} \, \delta(m+m'+s; s)$$

$$\Delta f_m = \frac{eR}{pc} (R \, \Delta G'_m) \qquad (3.12b)$$

$$+ \frac{eR}{pc} \sum_{m's} \bar{x}_{m'} (R \, G''_s + 2RG'_s) \, \delta(m'+s, m)$$

$$+ \frac{eR}{pc} \sum_{m's} m's \, \bar{x}_{m'} G_s \, \delta(m'+s, m).$$

The perturbed tune $\bar{\nu}_z$ may be found from ΔE_o, Δf_n according to

$$\bar{\nu}_z^{\,2} - \nu_z^{\,2} = \Delta E'_o + \sum_{s \neq o} \frac{f_s \, \Delta f_{-s} + f_{-s} \, \Delta f_s}{s(s - 2\nu_z)} \qquad (3.13)$$

$$+ 2 \sum_{m > o} \frac{|\Delta f_m|^2}{m^2 - 4\nu_z^{\,2}} \quad .$$