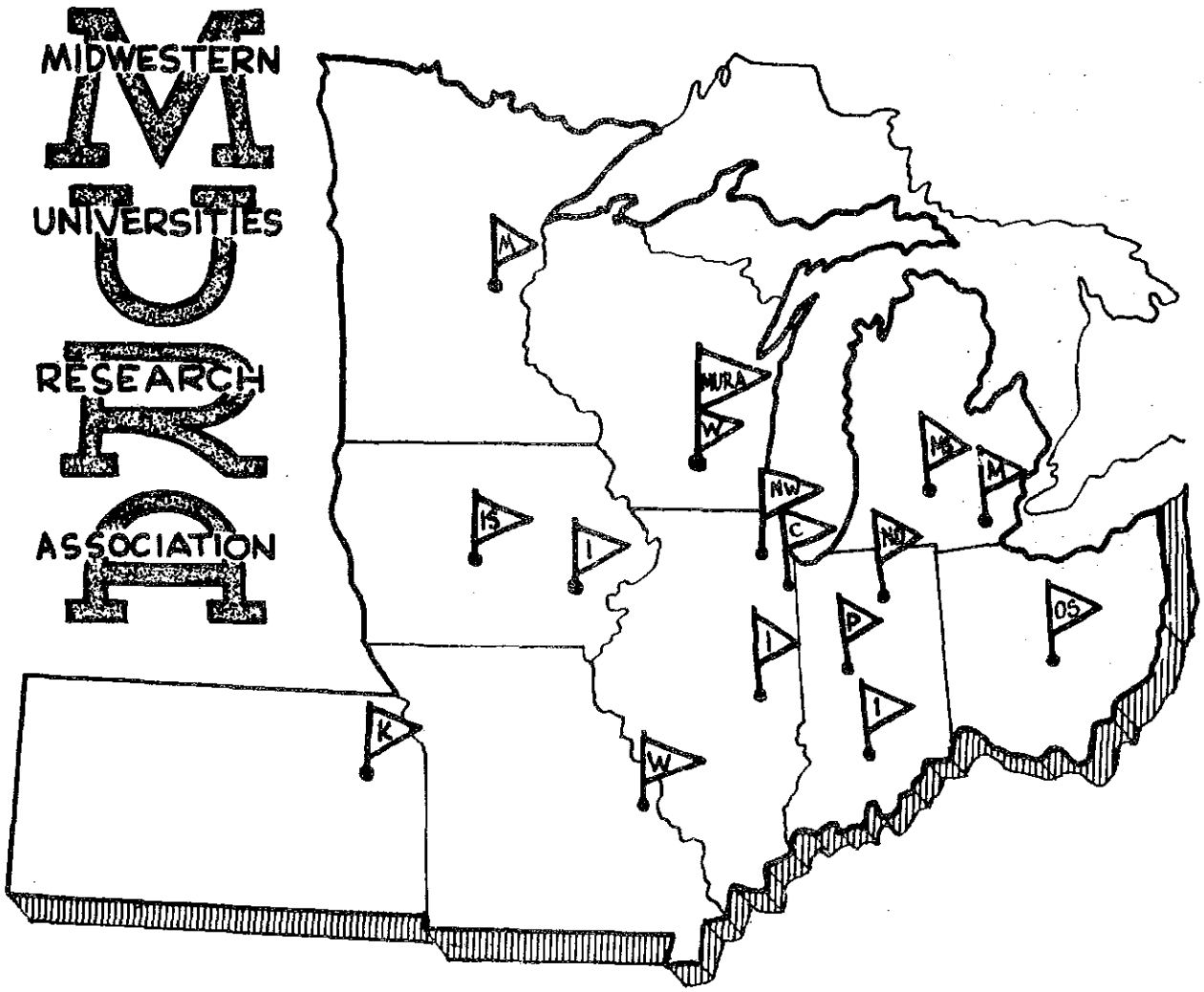

ON THE STABILITY LIMIT OF ORBITS IN A SPIRAL MAGNETIC FIELD

W. N. Wong

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2203 University Avenue, Madison, Wisconsin

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ABSTRACT

The general behavior of plane orbits in a simple spiral magnetic field is studied both analytically and numerically (for a specific sample case). Techniques employed are heavily geometrical in nature and the motivation is the search for some simple conditions in the field parameters which control orbit stability and orbit economics. Pragmatic simplicity dictates a choice of approach which is only justified by a final numerical test. But reasonable rigor is accorded the fundamental equations in order to avoid getting lost from the very start.

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I. Introduction

An orbit problem, which is of more practical interest than it may seem at first look, can be conceived in the following manner: Given a small number, A , representing the maximum oscillation amplitude of the orbits a system can tolerate, given a large number, p , representing the minimum desired time span within which these orbits are to have their oscillation bounded by A ; then let F enumerate the parameters which characterize the magnetic field structure under study. We ask what conditions (the best choice, a compromise, etc.) there are among the F's so that a maximum number of orbits can be admitted by the structure under the desired conditions in the most economical manner. Such a problem is not trivial, neither is it hypothetical. The trouble with it is that it is too idealistic, and, with our present available techniques, barring individual case studies through the aid of digital computation in a purely numerical manner, it is unlikely that we can give it a rigorous analytical survey and arrive at some intelligibly simple conclusion. In such a problem, two aspects of the orbit properties are involved, viz., orbit stability and orbit economics. To achieve good orbit economics (such as the usual notion of a small circumference factor of a structure), one has to deal with such structures that a simple linear approach to their stability properties is in general not realistically adequate. Confining oneself to a simple spiral magnetic field and taking only a one dimensional problem (plane orbits) into consideration, so that

the dynamics involved is Hamiltonian in nature, one would immediately have the notion that to follow Moser's general transformation theory and carry out a resonance neighborhood study for those important resonances considered as in Sturrock's theory, one should be on the right track to break through the stability aspect of the problem. However, such a handling, conceptually simple, is technically greatly difficult, particularly in respect to the transformation of working variables. Nonlinear problems create new transcendental functions which are intimately associated with irrational numbers and the analytical manipulation of them is discouragingly cumbersome.

This study, initiated by George Parzen, is aimed at a much restricted interest regarding the above problem. Although it has been constantly kept in mind how far we can go in order to at least scratch the surface of such a general problem, due preparation is always made for eventual retreat into an oversimplified situation by giving up much of the theoretical delicacies in order to obtain some numerical idea from a purely pragmatic viewpoint. A simple spiral field with Parzen's prearranged parameters is taken as a numerical sample case. The general theory of plane orbits is examined at first in search of some working variables believed to be appropriate for this purpose. Some kind of stability criterion is looked for, based on realistic simplicity rather than sufficiency and necessity. Numerical study on some sample orbits is carried out in order to gain some sophistication in the general

tendency of orbit behavior so that an "orbit model" can be conceived. Such a model eases out much of the transformation procedure. It is rather an awkward approach but is perhaps the only short cut that can lead to some simple result.

Most of the numerical intuition needed in this study is supplied by George Parzen. Background in acceleratior physics is obtained from F. T. Cole's Notes on Accelerator Theory (TN-259), and findings resulting from numerous discussions with H. K. Meier are freely used in this report. Computer numericals are based on FLEXIBLE FIVER (Program 280, MURA-604) with the aid of M. R. Storm.

Relevant plots regarding the behavior of these sample orbits are partially made available in the figures with some brief explanations.

II. General

The theory of plane orbits of a charged particle in a magnetic field can be most conveniently formulated by basing the geometrical aspect of the problem on one of the Frenet formulas which essentially defines a curvature function, and then incorporating half a physics law, viz., the magnetic part of Lorentz force law, into the problem. In Fig. 1 is shown a piece of plane curve, which, if represented in polar coordinate system, can be put as

$$r = f(\theta),$$

i.e., the radius vector is a function of the polar angle. To avoid the occurrence of loops and cusps, one may require that this $f(\theta)$ be a single valued function of θ so that the arc

length measured from any arbitrarily chosen reference point on this curve, $s(\theta)$, is a monotonically increasing function of θ. Then, using s itself as a parameter, we see the three intrinsic unit vectors associated with this curve:

$\hat{e}_s$: defined positive along increasing s,

$\hat{e}_t$: directed upward from the plane of the curve and kept constant (for plane curves),

$\hat{e}_n$: defined by $\hat{e}_s = \hat{e}_n \times \hat{e}_t$ (right-handed convention),

are functions of s. The Frenet formula says

$$\frac{d\hat{e}_s}{ds} = |K|\,\hat{e}_n, \quad |K| \text{ usually defined as a scalar}$$

and the law of physics says

$$-\hat{e}_n\,|K| = \frac{1}{p}\,\hat{e}_s \times \hat{e}_t\,|H(s)|$$

in which p is the particle scalar momentum, $H(s)$ is the magnetic field intensity as a function of the arc length. The electronic charge and light velocity are set equal to unity here. Then with appropriate sign convention one has

$$K(s) = \frac{-1}{p} H(s) .$$

Since $H(s)$ is almost always prescribed as a function of $r = f(\theta)$, i.e., $H(s) = \mathcal{H}(r(\theta), \theta)$, the most natural course to follow is to express $K(s)$ in terms of $r(\theta)$ and its derivatives in the polar system

$$K(s) = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} .$$

The differential equation

$$\frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} = -\frac{1}{p} \mathcal{H}(r(\theta), \theta) \qquad (1)$$

serves as the starting point of plane orbit study. Of course, one can always choose a Lagrangian or a Hamiltonian function in one form or another to start with and arrive at the same result in a more elegant manner and better prepared for systematic approximation in the event (almost a certainty) the equation cannot be rigorously solved. No matter which course is followed, the basic difficulty in approximating this equation is a tremendous one, particularly when $\mathcal{H}$ is of periodic structure, the only case of practical interest. More particularly, one's interest is not just to obtain an approximation for a prescribed field function; rather, it is instead to explore the behavior of the solutions in their dependence upon the parameters which characterize the field function. For this kind of interest, a systematic procedure, such as Moser's theory,[1] becomes technically too arduous to apply. In this note, an approach motivated by this interest is explored, emphasizing the geometrical aspect of the problem, with the hope that while analytical expressions may be lengthy and their meaning obscure, some inference from the behavior of those variables which have a precise geometrical meaning will help to soothe the difficulty.

One can start the problem from the viewpoint of elementary plane geometry, such as the following angle relations among the three angles associated with the curve

in Fig. 1

$$\psi = \frac{\pi}{2} - \alpha + \theta \ ,$$

in which all angles are defined positive in the sense shown and the requirement that $r(\theta)$ be everywhere single-valued implies ψ being bounded within $\pm \frac{\pi}{2}$. Obviously, ψ and α are just the direction angles of the unit vector $\hat{e}_n$ and $\hat{e}_s$ in the polar system. They are functions of the polar angle θ. So analytically

$$\psi(\theta) = \frac{\pi}{2} - \alpha(\theta) + \theta \ . \qquad (2)$$

A closer examination shows that, unlike $\alpha(\theta)$, $\psi(\theta)$ is numerically invariant under coordinate rotation and should be regarded as more suitable for the description of the curve than $\alpha(\theta)$. We, therefore, set out to eliminate α in the following manner.

$$\alpha(\theta) = \int_{\alpha(0)}^{\alpha(\theta)} d\alpha(\theta) + \alpha(0) = \int_0^{\theta} d\theta \, \frac{d\alpha}{ds} \frac{ds}{d\theta} + \alpha(0) \ .$$

Elementary calculus supplies the following information:

$$\frac{d\alpha}{ds} = K(s)$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + r'^2} \qquad \text{where prime refers to } \frac{d}{d\theta}$$

and $$\tan\psi = \frac{r'}{r} \ .$$

Consequently

$$\psi(\theta) = \frac{\pi}{2} + \theta - \alpha(0) - \int_0^{\theta} d\theta \frac{r}{\cos} K(s)$$

$$= \theta + \Omega + \frac{1}{p} \int_0^{\theta} d\theta \, \frac{r}{\cos\psi(\theta)} \mathcal{H}(r(\theta),\theta)$$

in which the constant $\frac{\pi}{2} - \alpha(o)$ is replaced by the symbol Ω representing the value of $\psi(\theta)$ on the initial polar axis $\theta = o$. This expression may be called an "integral relation" which can readily be converted into an integral equation. For example, by integrating $\tan\psi$, we have

$$\int_o^\theta d\theta \tan\psi = \ln r(\theta) - \ln r(o)$$

$$\text{or} \quad r(\theta) = \Lambda\, e^{\int_o^\theta d\theta \tan\psi}$$

in which the symbol Λ is used to designate the value of r on the initial axis. While $\psi(\theta)$ itself is numerically invariant under rotation of the initial axis, its integral

$$\Psi(\theta) \equiv \int_o^\theta d\theta \tan\psi(\theta)$$

is numerically invariant under scale transformation (i.e., independent of the unit used in measuring r) as is evident from its definition.

We now have a transcendental integral equation:

$$\psi(\theta) = \theta + \Omega + \frac{\Lambda}{p}\int_0^\theta \frac{d\theta}{\cos\psi(\theta)}\, e^{\int_o^\theta d\theta \tan\psi(\theta)}\, \mathcal{H}\left(\Lambda \int_o^\theta d\theta \tan\psi(\theta), \theta\right) \tag{3}$$

which, along with the easily proved kinematical relations:

$$\tan\psi(\theta) = \frac{r'(\theta)}{r(\theta)} = \frac{p_r}{p_\theta} \qquad \begin{matrix} p_\theta \equiv p\cos\psi(\theta) \\ p_r \equiv p\sin\psi(\theta) \end{matrix} \quad (p = \text{scalar momentum}), \tag{4}$$

constitutes the dynamics of plane orbits.

However, this equation, as it stands, promises little hope of soothing our original difficulty. Integral equations of such complicated structure discourage further attempt at

proceeding with the problem in any mathematically rigorous manner. Nevertheless, this equation, from a physicist's point of view, can serve a rich source of information in connection with certain aspects of the orbit problem. So the study here, instead of seeking a solution, will be centered on how to extract as much information as possible to meet with our special interest.

Incidentally, it should be pointed out that the equation, except for minute details, is really just the first integral of the original differential equation. In fact, if we differentiate the angle relation (2) with respect to θ, we can obtain a "differential relation"

$$1 - \psi'(\theta) = K(\delta) \frac{d\delta}{d\theta} \tag{5}$$

which is generically the predecessor of a differential equation. For, by eliminating $\psi'(\theta)$ in favor of $r(\theta)$ through the elementary formula

$$\psi'(\theta) = \frac{d}{d\theta} \arctan \frac{r'}{r} = \frac{rr'' - r'^2}{r^2 + r'^2} ,$$

this relation leads back to equation (1).

The justification for investing time and effort in the study of the orbit problem in this approach is based on the belief that:

(1) Whilst differentiation is basically a coarsening operation, integration tends to smoothen out roughness. Approximations based on integral methods can lead to analytical results of relatively compact structure, if a systematic procedure striving for

better accuracy is not a desired feature.

(2) Whilst causality is "hidden" in a differential equation, it is brought out in some tangible forms in an integral equation. The retrospect nature of integral equation should be more appropriate for the study of problems in which initial conditions are of prime interest. The formal presence of initial parameters from the very start could be visually a guide to searching for a method suitable for a particular aspect of the problem.

We specialize the curvature function $\mathcal{H}(r(\theta),\theta)$ to that of a spiral field having the following structure

$$\mathcal{H}(r(\theta),\theta) = -B\, r^{k}\, h\,[\,N\theta - K \ln r + \gamma\,]$$

in which k is the momentum compaction constant, K the spiral ridge wave number and N the number of periods of the field structure in an angular span of 2π (i.e., N sectors per one complete revolution). γ is an arbitrary phase subject to choice at one's option.[2] The function h is to have its fundamental normalized to unity and is to have only one leading harmonic of amplitude not greater than unity. Essentially, we shall take

$$h\,(\Phi) \equiv 1 + \sin\Phi \qquad \Phi = N\theta - K \ln r + \gamma$$

as the object function for discussion and, for numerical illustration, Parzen's numbers N = 48, k = 63, K = 450 will be used (to be referred to as #4863450).

It is appropriate, at this early stage, to narrow down our interest in the usual manner. That is, we shall not be

concerned with motions vigorously transient in nature. For a relatively steady oscillation of the particle within a finite region in the field, the initial parameters, particularly Λ, the accepting radius, must be prescribed compatibly for a given scalar momentum. Such a compatibility requirement can be stated in the following manner: For a given scalar momentum p, there is a unique choice of R (with the dimension of length) such that

$$p = BR^{k+1} \tag{6}$$

that the exponent should be k+1 is dictated by dimensional requirements. Then, for Λ to be compatible to this p, it must be sufficiently close to R such that if

$$\frac{\Lambda}{R} = e^{\rho} , \tag{7}$$

then $e^{\rho} \simeq 1 + \rho$ shall always prevail, implying ρ must be reasonably smaller than unity. R so defined can be conveniently (but not necessarily) used as the unit of length in the system. As usual, the unit circle defined by R will be referred to as the reference circle. ρ will be referred to as the oscillation parameter and the condition in (6) is to be referred to as the unit gauge.

In terms of the above language, we summarize: an orbit characterized by two initial parameters (ρ, Ω_ρ) is represented by an orbit state function $\psi_\rho(\theta)$ defined by the equation

$$\psi_\rho(\theta) = \theta + \Omega_\rho - \int_0^\theta \frac{d\theta}{\cos\psi(\theta)} e^{(k+1)[\Phi(\theta)+\rho]} h\left[N\theta - K(\Phi(\theta)+\rho)\right] ,$$

where $\Phi(\theta) \equiv \int_0^\theta d\theta \tan\psi(\theta)$

subject to the gauge condition $p = BR^{k+1}$ and the scale condition $\frac{\Lambda}{R} = e^{\rho}$. The corresponding differential equation is

$$\psi'_{\rho}(\theta) = 1 - \frac{1}{\cos \psi_{\rho}(\theta)} e^{(k+1)[\Psi(\theta)+\rho]} h[N\theta - K(\Psi(\theta)+\rho)].$$

It is apparent that the equation will respond only to the numerical values of ρ fed into it and is unaware of the extra scale-gauge conditions adopted by the observer. So ψ is twofold degenerate with respect to the parameters Λ and R (which defines p). This degeneracy precisely implies the well-known scaling symmetry of the orbits in a spiral field.

The orbit-state function ψ and its integral Ψ mutually define each other through the medium of the initial parameters and the field parameters in a very intricate manner. In addition to their geometrical meaning, they have a very direct connection with the conventional canonical variables $p_x(\theta)$ and $x(\theta)$ as used in computer programs from which numerical results are derived for comparison with theoretical prediction. Essentially

$$e^{\Psi(\theta)+\rho} - 1 = (\Psi(\theta)+\rho) + \frac{(\Psi(\theta)+\rho)^2}{2!} + \ldots$$

should be identified with the total oscillation $x(\theta)$ while

$$p \sin \psi(\theta) = p\,\psi(\theta) - p\frac{\psi^3(\theta)}{3!} + \ldots$$

is obviously equivalent to the oscillation momentum $p_x(\theta)$. These apparently rather complicated relations are actually only formal, for insofar as the practical ranges of numerical

values are concerned, the simple approximation

$$x(\theta) \simeq \Psi(\theta)+\rho \qquad x(o) = \rho \quad (p=1 \text{ understood})$$

$$p_x(\theta) \simeq \psi(\theta) \qquad p_x(o) = \Omega_\rho$$

always prevails. The conditions for validity of such simplification lie in the fact that in no case shall we be interested in orbits with oscillation momentum (p_x) exceeding 10% of the total scalar momentum (e.g., in a particle accelerator of 1 Bev. energy, the betatron oscillation momentum certainly is well below $\frac{1}{10}$ Bev./c). Confining interest to this range of p_x, one can always take the advantage of the extremely accurate numerical approximation:

$$\tan\psi \simeq \psi \qquad \text{if} \qquad |\psi| \lesssim 0.1. \tag{10}$$

We shall refer a problem in which this approximation is valid as a problem of 0.1-boundedness interest and in problems of this nature, (9) and (10) are understood to be numerically good.

Before concluding the general discussion in this section, let us describe in a very preliminary manner the topic of interest in this study. The word "stability" needs a precise definition in order to attain an unambiguous and concrete meaning. In this respect, we shall define an ideal stable orbit as one which will remain oscillating about the reference circle for an infinitely long time with its maximum amplitude bounded by some prescribed finite quantity. Such an idealization implies that only two cases are possible:

(1) The orbit is a "closed" orbit, by which is meant that if the orbit is represented by the state (ρ , Ω) on sector axis no. I, it will come back to this same state after a finite number of sectors and in the transit, its maximum oscillation amplitude stays bounded within some finite prescribed value.

(2) The other alternative is that it must be "quasi-closed", i.e., it will come back arbitrarily close to its initial phase of the motion but never precisely so. This is merely a consequence of the ergodic hypothesis which we shall take for granted.

This idealization is now relaxed. We shall say an orbit is stable with respect to a spatial limit A_L (finite), if its oscillation will stay within this bound for a very long time p_L (e.g., time measured in number of sectors and p_L stands for some large integer). In this sense, the meaning of the word "stability" is brought out by two definitive numbers and their dependence upon the field parameters is what we want to study.

III. Formulation of the Problem

Equation (8) defines the orbit-state function $\psi_\rho(\theta)$ labeled by two parameters (ρ , Ω_ρ), in the same sense as a quantum mechanical state labeled by two quantum numbers. Its dependence on field parameters is not notationally brought out and it shall be understood that these field parameters are not firmly prescribed entities. Two numbers

(ρ^{I}, Ω^{I}) given on the initial axis ($\theta = 0$) define an orbit state $\psi^{I}(\theta)$. As the motion is propagated to the axis II ($\theta = \frac{2\pi}{N}$), the function $\psi^{I}(\frac{2\pi}{N})$ assumes a numerical value Ω^{II}, while the function $\Psi^{I}(\frac{2\pi}{N}) + \rho^{I}$ assumes a numerical value ρ^{II}. These two numbers (ρ^{II}, Ω^{II}) certainly can also be used on axis^{I} to define an orbit state $\psi^{II}(\theta)$ representing the same orbit but at a different inital phase. And, in the sense of stability defined earlier, this difference reflects one important fact: if state (ρ^{I}, Ω^{I}) is expected to have a stable life of p_L sectors, the state (ρ^{II}, Ω^{II}) can only be expected to have a stable life one sector shorter. In other words, states (ρ^{I}, Ω^{I}) and (ρ^{II}, Ω^{II}) represent the same orbit at different "ages", (ρ^{I}, Ω^{I}) being younger than (ρ^{II}, Ω^{II}) by one sector. This situation can be generalized to any number of sectors and we conclude that different pairs of (ρ, Ω) specified on an axis do not necessarily represent different orbit identities; they might represent the same orbit at its different age states (different substates of the same orbit). We now wish to set up a chronological standard through which the age of an orbit can be referenced in order to pick out only those states which, by some arbitrary convenient standard, represent the "youngest" states of each individual orbit for study. Such a procedure is obviously very arbitrary; nevertheless, it is a useful notion when one wishes to compare the relative properties of different orbits. To achieve this aim, one must ask the question: what intrinsic

property does a state function ψ have that characterizes the identity of the orbit irrespective of its state of age. An answer to this question is that the absolute maximum numerical value of ψ may be used to characterize the orbit identity. (The statement is equivalent to: the maximum value of the oscillation momentum p_x attainable by a particle along its orbit characterizes the identity of this orbit). The geometrical meaning of ψ hints that wherever the orbit has a point of inflection, ψ has a local maximum (or minimum) so the absolute maximum of ψ occurs at one of these local maxima. We now demonstrate that, in a 0.1 bounded problem, this absolute maximum can be made to occur in the immediate vicinity of the origin (but never precisely at the origin), if the field function has the form $1 + \sin\left[N\theta - K(\bar{\psi} + \rho)\right]$ which is the function we have chosen for study.

If ψ is to have a maximum at $\theta = 0$, $\psi'(\theta = 0)$ must vanish, and if this maximum is in the immediate neighborhood of the origin, ρ must be very small. Then the differential relation in (8) gives:

$$\psi'(\theta = 0) = 0 = 1 - \frac{e^{(k+1)\left[\bar{\psi}(0)+\rho\right]}}{\cos \psi(0)}\left\{1 + \sin\left[-K(\bar{\psi}(0)+\rho)\right]\right\}$$

$$= 1 - \frac{e^{(k+1)\rho}}{\cos W}\left\{1 - \sin K\rho\right\} \quad \text{where} \quad \psi(0) \equiv W.$$

Since W is 0.1-bounded such that $\cos W \cong 1 - 1/2W^2$ always prevails, and ρ is to be made small, we have

$$\rho = \frac{1}{2}\,\frac{W^2}{K - (k+1)} \quad .$$

Consequently, given a number W representing the maximum numerical value of ψ , defines an orbit state on axis I:

$$\left(\rho^{I} = \frac{1}{2} \frac{W^2}{K-(k+1)} \simeq 0 , \quad \Omega^{I} = W \right).$$

Since the numerical significance of ρ^{I} is almost nil, we may often regard the simpler state ($\circ$, W) as the "birth state" of an orbit whose identity is recognized through a single label W. In a sense, W on axis I serves as a total quantum number while a pair of (ρ , Ω) on successive axes may be regarded as subquantum numbers that remove the many-fold degeneracy in the problem. By referring to Fig. (6), in which a conventional phase plot (constructed with the aid of the computer) for several sample orbits is illustrated, it can be easily seen that W is nothing but the pinnacle point in each of these idealized phase "curves" while (ρ , Ω)'s represent other points in the plot. As George Parzen puts it, the computer numericals are a theoretical worker's experimental data; he is saying the pairs (ρ , Ω) are dynamical 'observables." Such observables supply us an abundance of information regarding orbit behavior, but they need not be the appropriate variable for use in theoretical analysis. It is because of this very difficulty that we cannot straightforwardly follow Moser's rigorous theory in this study. For, transformation back and forth, with different working variables carrying along in the transformation coefficients the complicated dependence on field parameters, is algebraically very difficult even in a lowest order estimate. In the

interest of simplicity, we have to approach the problem in a less rigorous and systematic manner by recourse to some kind of simplified physical picture of the orbits, for example, a model, which eases the transformation difficulty at the expense of theoretical soundness.

No matter how the particle is accepted into the field, the mere fact that it is to stay oscillating about the reference circle implies that the state of the motion must inherit from the structure of the field certain modes of vibration characterized by the field parameters. We conclude that the basic frequency N of the field must constitute one of the modes of vibration in the orbit. The fact that the harmonic part of the field, viz., the sine term in h, should on the average fluctuate out implies that in the long range behavior of the orbit there must be exhibited a mode of vibration whose frequency is predominantly dependent upon k, which dictates the average intensity of the field. This frequency, referred to as the propagation frequency ν ,[3] depends on N and K only weakly and its dependence on the state of orbit W will be discussed in section IV and V. Therefore as a first survey we may conceive a model based on a doubly periodic system with both the oscillating periods confined on the real axis (N is given real, while ν must be arranged real). The commensurability of these two numbers, reflected in the orbit behavior as a result of interaction of these two modes of vibration, is the center of all complications that affects orbit stability. These complications cannot be

directly examined if the orbit states are represented, as so far it has been done, in the ρ-representation alone. The usual notion of phase shift must be introduced, amounting to a transformation from the ρ-representation to some other representation more intimately associated with the properties of the two numbers N and ν.

In this respect, we must first examine in a purely formal manner, how motion is propagated, once the particle is accepted into the field, from the viewpoint of the definition of the state function $\psi_\rho(\theta)$.

If initially at $\theta = 0$ the state of motion is (ρ, Ω_ρ), then at $\theta - \eta$ the state function becomes:

$$\psi_\rho(\theta-\eta) = \theta - \eta + \Omega_\rho - e^{(k+1)\rho} \int_0^{\theta-\eta} \frac{d\theta}{\cos\psi_\rho(\theta)} e^{(k+1)\bar{\Psi}(\theta)} h\,[N\theta - k(\bar{\Psi}(\theta)+\rho)] \qquad (11)$$

A dummy variable transformation brings this equation to the form

$$\psi_\rho(\theta-\eta) = \theta - \eta + \Omega_\rho + \left[\int_0^{\eta} - \int_0^{\theta} \right] \frac{e^{(k+1)\rho}}{\cos\psi(\theta-\eta)} d\theta\, e^{(k+1)\left[\int_0^{\theta} - \int_0^{\eta}\right] d\theta \tan\psi(\theta-\eta)} \cdots$$

$$\cdots\ h\left[N(\theta-\eta) - k\rho - k\left(\int_0^{\theta} - \int_0^{\eta}\right) d\theta \tan\psi(\theta-\eta)\right]$$

At certain values of η such that

$$\rho = \int_0^{\eta} d\theta \tan\psi_\rho(\theta-\eta) \left(= -\int_0^{-\eta} d\theta \tan\psi_\rho(\theta)\right) \qquad (12)$$

is satisfied, a new constant Ω_η can be defined

$$\Omega_\eta = \Omega_\rho - \eta + \int_0^\eta d\theta \, \frac{e^{(k+1)\,\Psi(\theta-\eta)}}{\cos\psi_\rho(\theta-\eta)} \; h\left[N(\theta-\eta)-K\Psi(\theta-\eta)\right] \tag{13}$$

Then equation (11) adjusted in notation can be put in the form

$$\psi_\eta(\theta) = \theta + \Omega_\eta - \int_0^\theta \frac{d\theta}{\cos\psi_\eta(\theta)} \, e^{(k+1)\Psi_\eta(\theta)} \; h\left[N(\theta-\eta)-K\Psi_\eta(\theta)\right] \tag{14}$$

which defines a function $\psi_\eta(\theta)$ in the same manner as equation (8) defines $\psi_\rho(\theta)$, except that now a phase η in the field function has taken the place of ρ in the exponential. The meaning of this change of representation is really trivial if one takes a look at Fig. (4), in which the geometrical situation is quite lucid. The transformation equations (12) and (13) are quite involved; however, we shall use them only in a very simplified case. Essentially, the two functions $\psi_\rho(\theta)$ and $\psi_\eta(\theta)$ are orbit-state functions in two different representations. In the ρ-representation, two numbers (ρ, Ω_ρ) are specified on the sector axis representing the state of the orbit: ρ stands for the oscillation while Ω_ρ stands for the oscillation momentum, both being computer observables. In the η-representation, two numbers (η, Ω_η) are specified on the reference circle representing the state of the orbit: η stands for the location where the orbit has a zero, while Ω_η stands for the oscillation momentum at this zero. In the limiting situation $\rho \simeq 0$ and $\eta \simeq 0$, $\Omega_\rho \simeq \Omega_\eta \simeq W$, so for the birth state, the two representations almost coincide.

We are going to compare all orbits in their birth states on the initial $\theta = 0$ axis. As motion is propagated to

successive axes, each orbit becomes capable of two very different representations. By studying the transformation equations, we expect to find some kind of criterion to judge the stability of these orbits as a function of the state number W and the field parameters.

To establish this connection, we have to introduce the notion of the conventional equilibrium orbit which defines the central fixed point in the phase plot. This orbit, for all intents and purposes, can be defined (i.e., to avoid bothering about its existence) to be periodic of a period equal to that of the field structure. The state of this orbit is therefore $(\frac{1}{2}\frac{W_o^2}{K-(k+1)} \simeq o\ ,\ W_o)$ and it has no substate. The numerical value of W_o as a function of field parameters will be discussed in section IV. The unique properties this orbit possesses can be summarized:

(1) It is immortal: once a birth state is defined for it (o, W_o), it will never age.

(2) It is simply periodic and only one mode of vibration of frequency number N characterizes the motion.

(3) Its long range average behavior is that of a circle, i.e., the amplitude associated with the propagation part of its motion is zero.

This orbit owes its immortality to its capability of rejuvenating its state after each cycle of the motion. If such a rejuvenating process can also be incorporated into other orbits, we can expect long stable life of these orbits too. We certainly cannot achieve this for a wide range of

W values; however, we may attempt to arrange the situation such that each orbit within a narrow W-cone at the origin after a certain period (e.g., after every propagation period) will come back to a state resembling its birth state. Such a rejuvenation process is not complete, but we at least can expect a slower rate of aging of these orbits.

The birth state, as defined, is characterized by the simultaneous smallness of ρ and η in the two representations discussed. With ρ and η numerically small, the transformation equations (12) and (13) render the very simple relations:

$$\rho \simeq \eta \Omega_\eta$$

$$\frac{\Omega_\rho - \Omega_\eta}{\eta} = \left[K - (k+1)\right] \eta \Omega_\eta - \frac{\Omega_\rho^2}{2} \tag{15}$$

They are solved for η and Ω_η :

$$\Omega_\eta = \frac{\Omega_\rho}{2}\left[1 + \sqrt{1 + 2\rho - 4\left[K-(k+1)\frac{\rho^2}{\Omega_\rho^2}\right]}\right] \tag{15a}$$

$$\eta = \frac{\rho}{\Omega_\eta}$$

Since Ω_η must be real, the following condition must be observed:

$$\left|\frac{\rho}{\Omega_\rho}\right| \leq \frac{1}{2}\sqrt{\frac{1+2\rho}{K-(k+1)}} \simeq \frac{1+\rho}{2\left(K-(k+1)\right)^{\frac{1}{2}}} \simeq \frac{1}{2\left(K(k+1)\right)^{\frac{1}{2}}} \tag{16}$$

which says that for any substate (ρ, Ω_ρ) to "resemble" the birth state, this inequality must be satisfied in addition to ρ and η being both very small. The requirement that the orbits after approximately each propagation period will

assume a state satisfying this condition implies that we must arrange the field parameters so that the orbits will have a spectrum of zeroes, rather regularly spaced, to fall within a "linear" neighborhood of a sector axis which will be referred to as the "propagation axis" (on which the condition (16) is to be observed). The word "linear' is used instead of "very small" so as to make the condition more concrete in the following sense: If the zero occurs at a distance η from the foot of the propagation axis, then η must be sufficiently small as to satisfy $\sin\eta \simeq \eta$. Figure 3 illustrates the situation.

We must now define what a propagation axis is. We shall see that the propagation frequency ν in general is an irrational number smaller than N. One propagation period is therefore an angular span of $2\pi/\nu$ corresponding to

$$\frac{\frac{2\pi}{\nu}}{\frac{2\pi}{N}} = \frac{N}{\nu} \equiv m - \delta \text{ sectors, with m an integer and } |\delta| < \frac{1}{2}$$

Let a state W start its trip from axis I. After a little more or less than one propagation period, it arrives at axis M (m sectors away from axis I $\theta = 0$) to assume a substate (ρ^M, Ω_ρ^M) which shall be arranged to satisfy (16). Starting from axis M, it continues its trip another propagation period and arrives at an axis on which it is to assume a substate to satisfy (16) again. This axis would be 2 m sectors away from axis I if $2|\delta| < \frac{1}{2}$ and would be $2m \pm 1$ sectors away from axis I

if $2|\delta| > \frac{1}{2}$. (Equality never occurs since ν is irrational). In general, after p (integer) propagation periods, the propagation axis on which the propagation condition (16) is to be observed may be represented by the symbol

$$\left\langle \frac{pN}{\nu} \right\rangle \equiv q \text{ sector away from axis I.} \qquad (17)$$

The integer q is defined by: Let $p\delta$ = an integer i + a fraction f,

then $q = mp - i$ if $f < \frac{1}{2}$

$q = mp - i - 1$ if $f > \frac{1}{2}$.

For lack of a good general orbit solution, to formulate a stability condition in which initial parameters, field parameters and "desired" parameters (such as the parameters in connection with the minimum stable life time and maximum tolerable oscillation amplitude the user may desire) all appear in a fully analytical manner, in terms of some necessary and sufficient language, is believed to be impossible. The criterion established in (16) is neither necessary nor sufficient; it is just a humble criterion for the particular case of a simple spiral field with parameters in the range of practical interest. Only actual numerical sophistication can lure one into the confidence that this condition could serve as a reliable guide to the problem we wish to explore. Although considerable time and effort have been invested in the search of a "better" condition, nothing more intelligible and simple can be materialized than (16). The rest of the study is devoted to interpreting the meaning of this condition.

IV. The Central Fixed Point and Harmonic Gauge Curves

It was defined that the central fixed point (corresponding to the equilibrium orbit) shall have the state $(\frac{1}{2}\frac{W_o^2}{K-k-1} \simeq o,\ W_o)$ with W_o dependent upon the field parameters. This dependence is now to be studied with the aid of the integral equation in (8). We shall approach the problem by a method which can only be modestly called "probing for information." Nevertheless, this method will give us a very accurate numerical result in connection with the f.p., (central fixed point) state and a good physical picture in connection with other orbits.

From the analytical point of view, the equilibrium orbit is perhaps the most singular orbit, but looking at it just as a curve, a geometrical entity, it has the simplest outward form; for whatever it is, it by definition should be representable by

$$\sum_n E_n \sin (nN\theta - \epsilon_n) \quad .$$

And, corresponding to a simple one-harmonic field function as the one under study, an approximate representation $E \sin(N\theta - \epsilon)$ should be adequate for the purpose. We shall refer the following one-harmonic form

$$E \sin \mu (\theta - \epsilon) \qquad (18)$$

as a one-harmonic prober, to be fed into equation (8) in order to look for conditions that bind the three parameters E, μ and ϵ. In the event μ is made to approach N, we obtain information regarding the fixed point state. If μ is made to

approximate ν, we obtain some first-order information regarding other orbits, such as the general propagation properties of these orbits in their one-harmonic average. We are here taking advantage of the integral structure of our fundamental equation (8), which overlooks local roughness of a function and is willing to respond to an oversimplified probing function by yielding information which is not critically dependent upon such local roughness as possessed by the true solution. Just like any approximation method in mathematical physics, be it perturbational or variational in nature, one needs a zeroth-order function to start with. Form (18) serves a similar purpose. We replace $\bar{\Psi}(\theta) + \beta$ in equation (8) by this form:

$$\psi(\theta) = \theta + \Omega - \int_0^{\theta} \frac{d\theta}{\cos\psi} e^{(k+1)E\sin\mu(\theta-\epsilon)} \left\{1 + \sin\left(N\theta - KE\sin\mu(\theta-\epsilon)\right)\right\} \tag{19}$$

This function $\psi(\theta)$, thus defined, represents only the first harmonic average of the orbit (and is not the true orbit-state function). So at $\theta = \frac{\pi}{2\mu} + \epsilon$, $\psi(\theta = \frac{\pi}{2\mu} + \epsilon)$ should have a zero. Equation (18) then gives an informative relation:

$$0 = \frac{\pi}{2\mu} + \epsilon + \Omega - \int_0^{\frac{\pi}{2\mu}+\epsilon} \frac{d\theta}{\cos\left[\mu E\cos\mu(\theta-\epsilon)\right]} e^{(k+1)\left[E\sin\mu(\theta+\epsilon)\right]} \left\{1 + \sin\left[N\theta - KE\sin\mu(\theta-\epsilon)\right]\right\} \tag{20}$$

The evaluation of integrals of the type in this expression is quite a difficult problem; however, approximation methods for

two extreme cases are rather well explored. In the case (k+1)E is a large positive number, then an asymptotic treatment in one form or another (such as the saddle point-method, or a very simple Laplace's method of critical points) is in line.[4] If E is sufficiently small, such that expansion of the integrand in terms of Bessel's functions converge very rapidly that only the leading term is important, then an accurate and simple result can be made available although at times the calculation may be lengthy. The details of this second method are presented in the appendix. The result of the evaluation brings equation (20) into the form

$$\Omega - \frac{k+1}{\mu^2}\Omega \simeq \frac{1-\cos\left(\frac{N}{\mu}\frac{\pi}{2}+N\epsilon\right)}{N} - \frac{E}{\left(\frac{N}{\mu}\right)^2-1}\frac{1}{\mu}\Bigg[\tag{21}$$

$$\frac{N}{\mu}(k+1)\left\{\cos\left(\frac{N}{\mu}\frac{\pi}{2}+N\epsilon\right)+\sin\mu\epsilon\right\}+K\left\{\frac{N}{\mu}\sin\left(\frac{N}{\mu}\frac{\pi}{2}+N\epsilon\right)-\cos\mu\epsilon\right\}\Bigg]$$

This result is accurate to a maximum error of 1% if (1) the numerical values of E and K ($k \ll K$ assumed) are such that $E \lesssim \frac{1}{5K}$ and (2) if the ratio $\frac{N}{\mu}$ is kept away from an integer > 1 by at least two orders of KE. Both these conditions come about because of truncation of the series involving Bessel's functions in the approximation and are in general very easily satisfied in a 0.1-bounded problem.

If we let $\mu \to N$ and approximate $\epsilon \simeq 0$ and further observe that if the initial parameter Ω in (21) is that of the equilibrium orbit, W_o, so that $E \simeq \frac{W_o}{N}$, we obtain the result:

$$W_o \simeq \frac{1}{N - (1+\frac{\pi}{4})\frac{k+1}{N} + \frac{K}{2N}} \quad . \tag{22}$$

Simple as it is, this formula is quite accurate. For the field #4863450, it yields $W_0 \simeq 1.985 \times 10^{-2}$, while the computer result is 1.98783×10^{-2}.[5]

We can see from the left-hand side of equation (21) that if μ is made close to $\sqrt{k+1}$, then the dependence on Ω (the initial condition, and numerically equal to W for a birth state) in this relation is minimized. This implies that the propagation frequency ν discussed in III must be something like $\sqrt{k+1}\,(1+\Delta_1)$ with a small Δ_1. This serves as a guide to the study of the propagation properties of the orbits in V.

Let us prescribe a family of harmonic curves of arbitrary amplitude A and constant frequency ν : $A \sin(\nu\theta - \gamma)$ with $\gamma \simeq 0$, in the field structure, in the same manner as we prescribed a reference circle before. Just as the reference circle was used to define a constant unit gauge (Eq. 6), these harmonic curves define a system of oscillating gauges. Further, just as the reference circle, which is itself not a possible orbit, represents the average motion of the equilibrium orbit, these harmonic gauge curves, which are not orbits themselves, represent the average motion of the other orbits in a W-cone for which a constant ν is a meaningful concept. A mode of vibration with frequency equal to the frequency of the structure riding on the reference circle yields the equilibrium orbit; a mode of vibration with frequency almost equal to the frequency of the structure riding on a harmonic gauge curve A should yield the orbit corresponding to an average oscillation amplitude A (which should be a function of W). This mode of

vibration will be termed the "modulator" as against the gauge curve itself which will be called the "propagator." The concept of propagator vs. modulator is of course relative in nature. The observer may at his option use the mode N as a gauge curve and treat the mode ν as the riding component such as in the conventional approach in the linear betatron oscillation theory. For orbits of oscillation so large that the equilibrium orbit as seen by these orbits is just a small wiggling component, the approximation procedure in which the ν-mode is used to define the gauge curve should be in favor. To complete the description of the orbit model here, we need the answer to a very important question. An orbit with state number W geometrically means it is accepted into the field on the initial axis at an angle W with the circle. To this orbit W corresponds a gauge curve A which makes an angle νA with the circle at the origin. We ask the question at what angle relative to the propagation is the modulator accepted into the field, i.e., what is the difference $W - \nu A \equiv W_s$ expressed as a function of A and field parameters? The nature of this question is shown in Fig. 2. A rigorous answer to this question calls for a good orbit solution which we do not have. Nevertheless, since the question concerns only the limiting situation $\theta \to 0$, an answer approximate to within a first-order infinitesimal can be supplied using a scaling trick in the result already obtained for the equilibrium orbit. For very small $\nu\theta$ (which eventually is to go to zero), the gauge $A \sin \nu\theta$ behaves like $A\nu\theta$ and may be replaced by a tangent

line $A\nu\theta$ at the origin. The situation then is similar to that of the equilibrium orbit problem with the coordinate system rotated counterclockwise an angle νA. This rotation incurs two consequences: the apparent periodicity of the structure is now $N-KA\nu$ instead of N, and the unit gauge condition is now θ-dependent, being scaled by a factor $e^{(k+1)\nu\theta A}$. These are seen simply by replacing the form $A\nu\theta + E\sin\mu\theta$ for (18) so that the integrand in (19) now has the form

$$e^{(k+1)A\nu\theta} + (k+1)E\sin\mu\theta\left\{1+\sin\left[(N-KA\nu)\theta - KE\sin\mu\theta\right]\right\}$$

For arbitrarily small $A\nu\theta$, $e^{(k+1)A\nu\theta} \to 1+(k+1)A\nu\theta$, so that integrand after some manipulation can be made to equal

$$e^{(k+1)E\sin\mu\theta}\left\{1+\sin\left[(N-(K-k-1)\nu A)\theta - KE\sin\mu\theta\right]\right\} + O^2,$$

where O^2 is a term which approaches zero faster than $\nu\theta$ and is one order smaller than the first term, being of the form

$a(1+\sin b - \cos b)$ with $a \sim O(A)$

and $\lim_{\nu\theta\to 0} a \to 0$, $\lim_{\nu\theta\to 0}(1+\sin b - \cos b) \to 0$.

O^2 is therefore negligible. This approximation amounts to transferring the effect due to a small difference ΔR in the gauge condition (8) to the equivalent effect due to a phase $\Delta\gamma$ in the field function. For R sufficiently small, it is numerically very accurate.[5]

Following the same procedure as in the fixed point problem, we should come to a result similar to (22) with all N replaced by $N-(K-k-1)\nu A$, i.e.,

$$W_s \cong \frac{1}{N-(K-k-1)\nu A+\frac{K}{2}-(1+\frac{\pi}{4}(k+1)}$$

$$= \frac{1}{N} \frac{1 - s\nu A}{(1- s\nu A)^2+\left(\frac{1}{NW_o}-1\right)} \qquad \text{with } s=\frac{K-k-1}{N}$$

This result will be used in the modulation theory in Section VI.

V. Theory of Propagation

In this section we will analyze in some detail the important number ν defined to represent the mode of vibration in the average long range behavior of those orbits within a particular W-cone. If, as we expect, this ν should depend on W very weakly, it is more appropriate for us to start the investigation from the differential relation in (8) in which the initial parameters are not present:

$$\psi'(\theta) = 1- \frac{1}{\cos\psi(\theta)} e^{(k+1)(\Phi(\theta)+\rho)}\left\{1+ \sin\left[N\theta -K(\Phi(\theta)+\rho)\right]\right\}$$

For notational simplicity, let us denote $\Phi(\theta)+\rho \equiv \chi(\theta)$ so that

$$\chi'' = 1 - \frac{1}{\cos\chi'} e^{(k+1)\chi} \left\{ 1 + \sin (N\theta - Kx)\right\} \qquad (24)$$

If this equation can be solved for χ as a function of θ, we may expand the solution about some neighborhood center a_o so that

$$\chi(\theta) = \frac{1}{a_1} (\theta - a_o)- \frac{a_2}{a_1^3} (\theta - a_o)^2+ \frac{1}{a_1^5} (2a_2^2-a_1a_3)(\theta -a)^3+\ldots \qquad (25)$$

in which the expansion coefficients are so arranged that if we invert the series, we may have[6]

$$\theta = a_o + a_1 x + a_2 x^2 + \ldots . \qquad (25a)$$

The procedure is similar to inverting a sine series to obtain its cyclometric counterpart. We note that the coefficients a_n's are functions of the neighborhood center a_o. They depend on how we choose a_o. The following series expansions are then readily obtained:

$$x' = \frac{1}{a_1} - \frac{2a_2}{a_1^2} + \left(\frac{4a_2^2}{a_1^3} - \frac{3a_3}{a_1^2}\right) + \ldots \left(\because \psi \sim \frac{1}{a_1}\right)$$

$$\frac{1}{\cos x'} = 1 + \frac{1}{2a_1^2} - \frac{2a_2}{a_1^3} x + \left(\frac{6a_2^2}{a_1^4} - \frac{3a_3}{a_1^3}\right) x^2 + \ldots \qquad (26)$$

$$\sin(N\theta - Kx) \doteq \sin Na_o + x(Na_1 - K)\cos Na_o - x^2 \frac{1}{2}(Na_1 - K)^2 \sin Na_o + \ldots$$

$$e^{(k+1)} = 1 + (k+1)x + \frac{(k+1)^2}{2}x^2 + \ldots$$

and equation (24) assumes the form

$$x'' = 1 - \left(1 + \frac{1}{2a_1^2}\right)(1 + \sin Na_o)$$

$$- x\left[\left(1 + \frac{1}{2a_1^2}\right)(Na_1 - K)\cos Na_o + (1 + \sin Na_o)\left(\left(1 + \frac{1}{2a_1^2}\right)(k+1) - \frac{2a_2}{a_1^3}\right)\right]$$

$$- x^2\left[(Na_1 - K)\cos Na_o \left(\left(1 + \frac{1}{2a_1^2}\right)(k+1) - \frac{2a_2}{a_1^3}\right)\right.$$

$$\left. - \frac{1}{2}\left(1 + \frac{1}{2a_1^2}\right)\sin Na_o\,(Na_1 - K)^2\right.$$

$$+(1+\sin Na_o)\left((1+\frac{1}{2a_1^2})\frac{(k+1)^2}{2} - (k+1)\frac{2a_2}{a_1^3}\frac{6a_2^2}{a_1^4}\frac{3a_3}{a_1^3}\right)\Big] \quad (27)$$

$$+ O(\varkappa^3) + \ldots$$

$$\cong G - P\varkappa - D\varkappa^2 - M\varkappa^3 + \ldots .$$

We wish to extract from this equation of motion the first harmonic average part of $\varkappa(\theta)$ and find out what is the natural frequency associated with this average harmonic motion. The conception of such a natural frequency is useful and will depend on the state of the orbit W very weakly only if the maximum value of $\varkappa$, say $\varkappa_m$ is sufficiently small that $O(\varkappa_m^3) \ll \varkappa_m$. If this is not the case, introduction of such a natural frequency, although it would be still meaningful and could be unambiguously done, would be of little use in the problem. In either case, we may justify the truncation of the series (27) at $O(\varkappa^3)$ term by defining a "practical zero" in the problem. Whatever $\varkappa_m$ is (it is always much smaller than unity by definition), we agree to treat $\varkappa_m^3 \cong 0$ so that $\varkappa_m^{-3} \cong \infty$. Such a notion permits us to set all expansion coefficients a_n with $n \geq 3$ equal to zero, and, in exchange, we have placed ourselves under the obligation that any results deduced from the theory should be interpreted as valid at most for a time span of $O(\varkappa_m^{-3})$ beyond which the deductions are meaningless.

We now choose the first neighborhood center at $\theta = \gamma \simeq 0$ (for numerically very small γ) in the expansion (25), the second neighborhood center at $Na_o + \gamma$ and the third at $2Na_o + \gamma$, etc. This procedure implies that we mean to extract from $x(\theta)$ its odd harmonic component (i.e., the sinusoidal part which is $\simeq 0$ at $\theta = \gamma \simeq 0$) which has a period of Na_o, with Na_o yet to be sought. Since in the odd function there should be no even power terms, the contribution from the coefficient a_2 should be rejected. Consequently, we have

(1). The coefficient of x^2 term in (27)

$$D: \quad (1+\frac{1}{2a_1^2})\Big[(Na_1-K)\cos Na_o\,(k+1)-\frac{1}{2}(Na_1-K)^2\sin Na_o + (1+\sin Na_o)\frac{(k+1)^2}{2}\Big] \tag{28}$$

(2). The coefficient of x term in (27)

$$P: \quad (1+\frac{1}{2a_1^2})\Big[(Na_1-K)\cos Na_o+(1+\sin Na_o)(k+1)\Big] \tag{28a}$$

We are going to minimize D (setting it equal to 0) by an appropriate choice of a_o and a_1. The condition is fed into P, which then plays the role of the natural frequency ν^2 in a typical simple harmonic motion. Then by definition of Na_o being the period associated with a frequency $\nu \equiv \sqrt{P}$, we have

$$Na_o = \frac{2\pi}{\nu}$$

from which the number ν is obtained in terms of field parameters through a_o. The procedure so far outlined involves rather complicated algebraic operations. We simplify the work by

introducing a small quantity Δ_1 as has been commented in Section IV, such that P can be put as

$$\left(1+\frac{1}{2a_1^2}\right)(k+1)\left[\frac{Na_1-K}{k+1}\cos Na_o + 1 + \sin Na_o\right] \cong (k+1)(1+\Delta_1)^2 \tag{29}$$

and the expression D that is to be minimized is

$$2\frac{Na_1-K}{k+1}\cos Na_o + 1 + \sin Na_o - \left(\frac{Na_1-K}{k+1}\right)^2 \sin Na_o \longrightarrow 0 \tag{29a}$$

which together with $Na_o = \frac{2\pi}{\nu} \cong \frac{2\pi}{\sqrt{k+1}(1+\Delta_1)}$ constitute a set of equations for us to investigate Δ_1.

The problem is approached by the method of perturbation starting from the assumption that Δ_1 defined is small and by invoking the negligibility of the geometrical factor $\frac{1}{2a_1^2}$ which by definition of a_1 in (26) in a 0.1-bounded problem has at most an effect of 0.005 compared with 1, amounting to a numerically very insignificant correction which can be incorporated afterwards, if such a connection is desired.

Let $\Delta_1^{(0)}$ denote the first approximation of Δ_1 by setting $\frac{1}{2a_1^2} \sim 0$ in (29), so that

$$2\,\Delta_1^{(0)} \cong \frac{Na_1-K}{k+1}\cos Na_o + \sin Na_o \,. \tag{29b}$$

Neglecting the $\Delta_1^{(0)2}$ term in (29a), we can solve for

$$\sin Na \cong \frac{-1+\sqrt{1+4(1+4\,\Delta_1^{(0)})}}{2} \tag{29c}$$

which, when equated with $\sin \frac{2\pi}{\sqrt{k+1}(1+\Delta_1)} \simeq \sin \frac{2\pi}{\sqrt{k+1}} -$

$\frac{2\pi}{\sqrt{k+1}} \Delta_1^{(0)} \cos \frac{2}{k+1}$ yields the approximate solution

$$\Delta_1^{(0)} = \frac{\frac{1}{2} - \frac{\sqrt{5}}{2} + \sin \frac{2\pi}{\sqrt{k+1}}}{\frac{2\pi}{\sqrt{k+1}} \cos \frac{2\pi}{\sqrt{k+1}} + \frac{4}{\sqrt{5}}} \tag{30}$$

So in the first approximation, Δ_1 depends only on k+1. Its dependence on N and K is brought in through the geometrical connection, which can be effected in the following manner: From 29b and 29c we approximate

$$\frac{N\alpha_1 - K}{k+1} \simeq \frac{2\Delta_1^{(0)} - \sin N\alpha_o}{\cos N\alpha_o} \simeq \sqrt{\frac{\sqrt{5}-1}{2}} \left[1 + \frac{9-\sqrt{5}}{8} \Delta_1^{(0)}\right]$$

so that

$$\frac{\nu^2}{k+1} \equiv (1+\Delta_1)^2 \simeq (1+2\Delta_1^{(0)}) \left\{1 + \frac{1}{2} \frac{N^2}{\left[K+(k+1)\sqrt{\frac{\sqrt{5}-1}{2}}\left(1+\frac{9-\sqrt{5}}{8}\Delta_1^{(0)}\right)\right]^2}\right\}$$

and $$\Delta_1 \simeq \Delta_1^{(0)} + \frac{1}{4} \frac{N^2}{\left[K+(k+1)\sqrt{\frac{\sqrt{5}-1}{2}}\right]^2} \tag{30a}$$

For the sample field #4863450, $\Delta_1^{(0)}$ is 0.0379 and the correction is about 0.0023, giving $\nu \simeq 8.32$. The computer result is $\nu = 8.3140004$ which is obtained under the guise of "linear tune number" which has a slight difference in meaning from the propagation frequency ν being referred to here.

VI. Theory of Modulation

We are now in a position to introduce an explicit representation of our orbit model with our interest retreated into the following hideout:

(1) Spatially we stay within a narrow W-cone which enumerates all those orbit states with average oscillation amplitude A satisfying

$$A^3 \ll A \text{ and } A^3 \cong 0$$

(2) Temporally we can only afford to look forward to a finite future defined by

$$\theta < A^{-3}$$

Under these circumstances, two aspects of the field properties, which may be deemed as intrinsic in the structure and as completely independent of the orbit state, can be summarized by two numbers, viz., the fixed point state number W_o and the propagation frequency ν .

To every orbit state W, we associate a number A, thus defining a propagator for this state $A \sin (\nu\theta - \gamma)$ with γ dependent on W (or A) analytically but numerically $\gamma \simeq 0$ shall always be understood. This propagator is now subject to a process of modulation as a refinement toward the true orbit picture. The process of modulation is certainly not unique. Different approaches can lead to equivalent approximate results and choice of the type of modulation in the representation calls for parameters of different nature which supply different information one desires to learn. The

purpose here is to study the condition (16) for which we desire to learn only the orbit behavior in a small neighborhood about the foot of every propagation axis and for such neighborhoods, both $\nu\theta$ (mod. 2π) and $N\theta$ (mod. 2π) are very small. Further, we are concerned only with standard initial states ($\frac{1}{2}\frac{W^2}{K-k-1} \simeq 0$, W) on the $\theta \simeq 0$ axis, so a modulator approximated by one odd harmonic function of the form B sin $(N\theta - \gamma_0)$ with $\gamma_0 \simeq 0$ should be adequate. However, the modulation amplitude B must itself be further modulated by the propagator. Whether such a modulation process should be effected through amplitude modulation or phase modulation is immaterial; we may generally assume B to be a function of both θ and A, i.e., $B(A, \theta)$ so that whatever $B(A, \theta)$ is, the orbit picture should look like

$$A \sin(\nu\theta - \gamma) + B(A, \theta) \sin(N\theta - \gamma_0) \qquad (31)$$

with γ and γ_0 very small. The structure of $B(A, \theta)$ can be as complicated as we wish to imagine. Nevertheless, the requirement that as $A \to 0$, the whole thing should approach the representation of the equilibrium orbit, which in its one-harmonic approximation has the form $\frac{W_0}{N} \sin(N\theta - \epsilon)$ with $\epsilon \simeq 0$, defines the limiting behavior of $B(A, \theta)$:

$$\lim_{A \to 0} B(A, \theta) \to \frac{W_0}{N} \text{ for all } \theta \ .$$

And as has been discussed in Section III, as $\theta \to 0$ (or both $\nu\theta$ (mod. 2π) and $N\theta$ (mod. 2π) becomes very small), we should have

$$\lim_{\theta \to 0} B(A, \theta) \to \frac{W_S}{N} \text{ with } W_S \text{ dependent on A.}$$

This much information regarding $B(A, \theta)$ is all we need. It should be pointed out here that the propagator amplitude A is defined to depend sheerly on the state W, implying that it is temporally a constant. Consequently, if there is any monotonic growth in the oscillation amplitude of the orbit, such growth will have to be taken care of by the function $B(A, \theta)$. If the monotonic dependence on θ in $B(A, \theta)$ is strong, condition (16) cannot be satisfied for any time duration long enough to be of interest. We can therefore afford to stay ignorant of what the detailed structure of $B(A, \theta)$ is and get along with this representation (31) which actually is more general than its simple form would suggest.

The two lowest derivatives of this representation are:

$$\psi(\theta) = \nu A \cos(\nu\theta - \gamma) + B'(A,\theta)\sin(N\theta - \gamma_0) + N B(A,\theta)\cos(N\theta - \gamma_0)$$

$$\psi'(\ ) = -\nu^2 A \sin(N\theta - \gamma) + 2NB'(A,\theta)\cos(N\theta - \gamma_0) + B''(A,\theta)\sin(N\theta - \gamma_0) - N^2 B(A,\theta)\sin(N\theta - \gamma_0).$$

To satisfy the standard initial conditions, the following relations are observed: ($\gamma_0 \simeq 0$ used)

$$-A \sin\gamma \simeq \frac{W^2}{2[K-k-1]} \simeq 0 \text{ so that } \gamma \simeq 0$$

$$\nu A \cos\gamma + NB(A,0) = W \text{ so that } W \simeq \nu A + NB(A,0) \qquad (32)$$

$$\nu^2 A \sin\gamma + 2NB'(A,0) = 0 \text{ so that } B'(A,0) = \frac{\nu^2}{2N}\frac{W^2}{2[K-k-1]} \simeq 0.$$

On any sector axis, say on the qth axis, the state of the orbit is

$$\begin{aligned} \xi &\sim A \sin\left(2\pi q \frac{\nu}{N} - \gamma\right) - B\left(A, \frac{2\pi q}{N}\right) \sin\gamma_0 \\ \Omega &\sim \nu A \cos\left(2\pi q \frac{\nu}{N} - \gamma\right) + NB\left(A, \frac{2\pi q}{N}\right) \cos\gamma_0 \\ &\quad - B'\left(A, \frac{2\pi q}{N}\right) \sin\gamma_0 . \end{aligned} \tag{33}$$

If this qth axis is a propagation axis as defined in Section III, $q \equiv \left\langle \frac{pN}{\nu} \right\rangle$, we can approximate $B\left(A, \frac{2\pi q}{N}\right)$ by a linear extrapolation from $B\left(A, \frac{2\pi p}{\nu}\right) \simeq B(A,0)$ so that

$$B\left(A, \frac{2\pi q}{N}\right) = B(A,0) + B'(A,0)\, \frac{2\pi}{N} \left\{ \left\langle \frac{Np}{\nu} \right\rangle - \frac{Np}{\nu} \right\} .$$

This correction term is of the same order as γ and γ_0 and if we mean to neglect γ and γ_0, this small correction can also be consistently disregarded. In the interest of analytical simplicity with little loss of numerical accuracy, this will be done.

Condition (16) can now be put in its simplest possible form:

$$\left| \frac{\sin\left\{ \frac{2\pi\nu}{N} \left\langle \frac{Np}{\nu} \right\rangle \right\}}{\cos\left\{ \frac{2\pi\nu}{N} \left\langle \frac{Np}{\nu} \right\rangle \right\} + \Gamma} \right| \lesssim \frac{\frac{\nu}{N}}{2\sqrt{\frac{S}{N}}} \tag{34}$$

where $\Gamma \equiv \frac{NB(A,0)}{\nu A} = \frac{\omega_s}{\nu A} = \frac{1}{\nu N A} \frac{1 - S\nu A}{\left[(1 - S\nu A)^2 + \left(\frac{1}{NW_0} - 1 \right) \right]}$ (35)

This inequality, along with (32), which is now expressed as

$$W = \nu A + \frac{1}{N} \frac{1 - S\nu A}{(1-S\nu A)^2 + (\frac{1}{NW_0} - 1)} \tag{36}$$

and its derivative with respect to A

$$\bar{\Phi} \equiv \frac{\delta W}{\delta A} = \nu + \frac{S\nu}{N} \frac{1}{(1 - S\nu A)^2 + (\frac{1}{NW_0} - 1)} \tag{37}$$

which shall be called the "admissibility" of the field structure, constitute the set of equations we wish to study. In looking for information from these equations, one must clearly observe their limitations. In addition to the conditions that must be fulfilled in deriving the numbers ν and W_0, the neglect of small numerically insignificant quantities like γ and the rather arbitrary choice of an explicit representation in (31), all should have profound influence on the kind of information they are capable of supplying. From the manner these equations are derived, we cannot expect them to yield such delicate predictions as concerning a particular resonance phenomenon. They are only ready to give numerical ideas pertinent to the over-all properties of the field structure, the orbit states, and their mutual dependence. They do give some guidance in such matters as the choice and the compromise to be made among the field parameters, but whenever a quantity whose numerological structure enters into the problem in some critical manner, to look for information from these equations

without refined considerations extended to the analytically non-vanishing small phases like γ , γ_0 and other connections so far disregarded, will do us no justification.

VII. Conclusion

To apply and discuss the equations (34-37), we start from the admissibility function, the definition of which implies that a structure with large Φ is preferred. For if a slight increase in A would bring about a large increase of W this implies more orbits can be admitted for a prescribed fixed average oscillation amplitude. Since $\nu \approx \sqrt{k+1}$, large k is of first choice. This deduction is consistent with the fact that large momentum compaction gives small circumference factor of a structure. Equation (37) also tells that if the f.p., state number W_0 is made such that $\frac{1}{NW_0} \sim 1$, Φ can be enhanced. W_0 as a function of the field parameters (equation (22)) is subject to the validity condition in the evaluation: $\frac{W_0}{N} \lesssim \frac{1}{5K}$ (Section IV). So the best we can do is to arrange

$$NW_0 \sim \frac{N^2}{5K} \sim 1 .$$

For the field #4863450, this is $\frac{48^2}{5 \times 450} = \frac{2304}{2250} \simeq 1.03$.

If these preliminaries are observed, the quantities Γ and W assume much simpler dependence on other parameters:

$$\Gamma \simeq \frac{1}{\nu NA} \quad \frac{1}{1-S\nu A} \quad \left(\text{or } \nu A = \frac{1}{2S} - \sqrt{\frac{1}{4S^2} - \frac{1}{NS\Gamma}}\right) \tag{38}$$

$$W \simeq \nu A + \frac{1}{N} \frac{1}{1-S\nu A} \tag{39}$$

and we may proceed to ask what conditions there are in the ratio $\frac{\nu}{N}$ (or $\frac{N}{\nu}$), so that the satisfaction of the inequality (34) can most favorably be effected with as small a value of Γ as possible for all possible integers p. We note here, small Γ imples large A (or W) and therefore large stability limit.

Certainly, if the argument

$$Q \equiv \frac{2\pi\nu}{N} < \frac{Np}{\nu} >$$

can be arranged to be small for all integers p, this inequality would accommodate small values of Γ for fixed field parameters. To explore this condition, let

$$\frac{N}{\nu} \equiv \frac{m}{1+\Delta} = m - \frac{m\Delta}{1+\Delta} , \qquad (40)$$

in which m is an integer and Δ a small number, being very involvedly dependent on k, N, Δ_1, (c.f. $\frac{N}{\nu} = \frac{N}{\sqrt{k+1}(1+\Delta_1)}$ in Section V). Then

$$< \frac{pN}{\nu} > = pm - < \frac{pm\Delta}{1+\Delta} >$$

and

$$Q(\text{mod. } 2\pi) = 2\pi \frac{1+\Delta}{m} \left\{ \frac{pm\Delta}{1+\Delta} - < \frac{pm\Delta}{1+\Delta} > \right\}, \qquad (40a)$$

in which the brace, by definition, is a number always smaller than $\frac{1}{2}$ in absolute value and the ratio $\frac{1+\Delta}{m}$ is just $\frac{\nu}{N}$. Apparently a large m will insure a small Q whatever p is. However, we cannot make m very large since on the RHS of this inequality there is also a factor $\frac{\nu}{N}$ (in addition to other conflicts which may arise for too large an m). The compromise

is to arrange m sufficiently large such that if the value of the brace is stretched to its maximum possible $\frac{1}{2}$, Q will satisfy

$$\sin Q \simeq Q ;$$

i.e., Q assumes a value about or smaller than $\frac{\pi}{6}$ or $m \gtrsim 6$ so that the factor $\frac{\nu}{N}$ in Q and same in the bound on RHS may in effect cancel out. This cancellation implies that, when m is sufficiently large, the dependence on m in the satisfaction of (34) becomes very weak, being only in the brace whose value is firmly bounded by $\frac{1}{2}$. If one notices that the ratio $\frac{\nu}{N}$ is equivalent to $\frac{\sigma}{2\pi}$, where σ is the linear phase shift in Floquet's theory one immediately sees that choice of m large enough implies keeping $\frac{\sigma}{2\pi}$ away from such dangerous fractions as $\frac{1}{3}$ or $\frac{1}{4}$. For $m \gtrsim 6$, the lowest resonance nearby would be $\frac{1}{5}$ which is not very harmful (even exactly at resonance). This latter statement is concluded from Moser-Sturrock's theories, a coordinated study of which is available in Cole's notes.

With m chosen sufficiently large being agreed upon, the propagation condition may now be put in the simplest possible form

$$\frac{\left| \frac{p_m \Delta}{1+\Delta} - \left\langle \frac{p_m \Delta}{1+\Delta} \right\rangle \right|}{\cos\left\{ 2\pi \frac{1+\Delta}{m} \left| \frac{p_m \Delta}{1+\Delta} - \left\langle \frac{p_m \Delta}{1+\Delta} \right\rangle \right| + \Gamma \right\}} \lesssim \frac{1}{4\pi} \sqrt{\frac{N}{S}} \qquad \begin{matrix} m \gtrsim 6 \\ \left| \frac{m\Delta}{1+\Delta} \right| < \frac{1}{2} \end{matrix} \tag{41}$$

The argument Q in the cosine function is formally very complicated, but its numerical value is quite constant, being usually 0.92 $\pm$ small fluctuations depending on choice of $m \gtrsim 6$.

Without going into details regarding the small number Δ, we can now derive an estimate of the stability limit as a function of field parameters.

Take the extreme situation $\left|\frac{p\,m\Delta}{1+\Delta} - \left\langle\frac{p\,m\Delta}{1+\Delta}\right\rangle\right| \simeq \frac{1}{2}$ whatever p and Δ may be, and define the marginal value of Γ by the equality

$$\frac{\frac{1}{2}}{\cos Q + \Gamma_L} \simeq \frac{1}{4\pi}\sqrt{\frac{N}{S}}\,, \tag{42}$$

so that

$$\nu A_L \simeq \frac{1}{2S} - \frac{1}{2S}\sqrt{1 - \frac{4S}{N}\,\frac{1}{2\pi\sqrt{\frac{S}{N}} - \cos Q}} \tag{43}$$

and

$$W_L \simeq \nu A_L + \frac{1}{N}\,\frac{1}{1 - S\nu A_L}$$

W_L gives the size of the stable cone and A_L gives the propagation amplitude at the stability limit. If to A_L is added the maximum modulation amplitude, which differs negligibly from that of the equilibrium orbit amplitude, one obtains the numerical value of this stability limit usually referred to. For the field #4863450, m is 6 so that cos Q $\simeq$ 0.88, S is 8.04 with $\nu \simeq 8.314$ so that $A_L = 0.0017$ while $W_L \simeq 0.038$. The amplitude of the equilibrium orbit is approximately 0.0004. The estimate thus gives a stability limit of about 0.002 corresponding to a phase plot curve with a pinnacle value at 0.038. Such a number is meaningful for a time span estimated to be of order $(2 \times 10^{-3})^{-2} \sim 10^5$ sectors or better. George Parzen's computer numerical analysis also concludes a number about 0.002 with a stable life expectation adequately long for application.

We conclude this study with a discussion of the small number Δ defined in (40) by raising the question: What kind of small member this Δ we would like to choose and why? The question is meaningful only if more refined considerations are incorporated into the propagation condition. The neglect of small phases and the notion of standardization of all orbit initial states are no longer justifiable. In fact, the assumed existence of a continous W-cone at the origin which defines the limit of stability in such a clear-cut manner as so far has been conceived is really a fiction. Condition (41) can only supply us some idea in the limiting case $\beta_L \to \infty$ (∞ defined in the sense of Section VI) and $A \to A_L$. To associate some A beyond this A_L with some β smaller than this β_L in order to answer the question mentioned in the Introduction will call for an approach in which every substate of the same orbit be considered on an equal footing and only under this general situation will some scrutiny into the structure of Δ be meaningful and essential.

Appendix

We are concerned with the evaluation of the following two integrals:

$$\mathcal{J}_0 \equiv \int_0^{\theta_1} \frac{d\theta}{\cos\left[\mu E \cos\mu(\theta-\epsilon)\right]}\, e^{(k+1)\left[E \sin\mu(\theta-\epsilon)\right.}$$

$$\mathcal{J}_+ \equiv \int_0^{\theta_1} \frac{d\theta}{\cos\left[\mu E \cos\mu(\theta-\epsilon)\right.}\, e^{iN\theta}\, e^{k_1 e^{i\phi} E \sin\mu(\theta-\epsilon)}$$

where $k_1^2 \equiv K^2 + (k+1)^2$ and $\phi \equiv \tan^{-1}\frac{-K}{k+1}$.

The transformation $\Theta \equiv \mu(\theta-\epsilon)$ and the approximation

$$\cos\left[\mu E \cos\Theta\right] \simeq 1 - \frac{\mu^2 E^2 \cos^2\Theta}{2}$$

are first effected. Then up to order $O(E)^2$, $\mathcal{J}_0$ can be very simply handled by quadratizing the integrand. Straightforward integration leads to the result:

$$\mu\,\mathcal{J}_0 = \mu\theta_1 + \frac{1}{4}E^2(\mu^2+(k+1)^2)\mu\theta_1 - (k+1)E\left[\cos\mu(\theta_1-\epsilon) - \cos\mu\epsilon\right] + \frac{1}{8}E^2(\mu^2-(k+1)^2)\left[\sin 2\mu(\theta_1-\epsilon) + \sin 2\mu\epsilon\right]$$

up to $O(E)$ with $\theta_1 = \frac{\pi}{2\mu} + \epsilon$, this yields

$$\mathcal{J}_0 \simeq \frac{1}{\mu}\left\{\left(\frac{\pi}{2} + \mu\epsilon\right) + \left[(k+1)E\cos\mu\epsilon\right]\right\}.$$

For $\mathcal{J}_+$, the expansions

$$e^{k_1 e^{i\phi} E \sin\Theta} = \sum_{m=-\infty}^{\infty} e^{im\Theta} J_m (i k_1 e^{i\phi} E)$$

and

$$J_m(z) = \frac{(\frac{1}{2} z)^m}{m!} - \frac{(\frac{1}{2} z)^{m+2}}{1!(m+1)!} + \ldots$$

are used. Only the leading terms in $J_m(z)$ need be considered if the argument Z is bounded as $|z| \leq \frac{1}{5}$, or approximately

$$|E| < \frac{1}{5k_1} \sim \frac{1}{5K}$$

for an accuracy of about 1% in error. Then up to $O(k_1^2 E^2)$, integration leads to (in terms of variable Θ)

$$\mu \mathcal{J}_+ e^{-iN\epsilon} \simeq (1 + \tfrac{1}{4}\mu^2 E^2) \frac{e^{i\frac{N}{\mu}\theta_1}}{i\frac{N}{\mu}} + \frac{k_1 e^{i\phi} E}{2} \left[\frac{-\cos(1-\frac{N}{\mu})\theta_1 + i \sin(1-\frac{N}{\mu})\theta_1}{1-\frac{N}{\mu}} \right.$$

$$\left. + \frac{-\cos(1+\frac{N}{\mu}) - i\sin(1+\frac{N}{\mu})\theta_1}{1+\frac{N}{\mu}} + \frac{1}{8}(\mu^2 E^2 - k_1^2 E^2 e^{2i\phi}) \left[\frac{\sin(2-\frac{N}{\mu})\theta_1 + i\cos(2-\frac{N}{\mu})\theta_1}{2-\frac{N}{\mu}} + \frac{\sin(2+\frac{N}{\mu})\theta_1 - i\cos(2+\frac{N}{\mu})\theta_1}{2+\frac{N}{\mu}} \right] \right]$$

Up to $O(k_1 E)$ and transformed back to variable θ , one has

$$\mu \mathcal{J}_+ \simeq \frac{e^{iN\theta_1} - 1}{i\frac{N}{\mu}} + \frac{k_1 e^{i\phi} E}{2} \left[\frac{e^{i(N-\mu)\theta_1 + i\mu\epsilon} - e^{i\mu\epsilon}}{\frac{N}{\mu} - 1} - \frac{e^{i(N+\mu)\theta_1 - i\mu\epsilon} - e^{-i\mu\epsilon}}{\frac{N}{\mu} + 1} \right]$$

With $\theta_1 = \frac{\pi}{2\mu} + \epsilon$ as the integration limit, one has

$$\mu \mathcal{J}_+ \simeq \frac{\sin(\frac{N}{\mu}\frac{\pi}{2} + N\epsilon)}{\frac{N}{\mu}} + \frac{E}{(\frac{N}{\mu})^2 - 1} \left\{ \frac{N}{\mu} \left[(k+1)\sin(\tfrac{N}{\mu}\tfrac{\pi}{2} + N\epsilon) - K\cos(\tfrac{N}{\mu}\tfrac{\pi}{2} + N\epsilon) - K\sin\mu\epsilon \right] \right.$$

$$\left. - (k+1)\cos\mu\epsilon \right\} + i \left\{ \frac{1 - \cos(\frac{N}{\mu}\frac{\pi}{2} + N\epsilon)}{\frac{N}{\mu}} - \frac{E}{(\frac{N}{\mu})^2 - 1} \left\{ \frac{N}{\mu} \left[(k+1)\cos(\tfrac{N}{\mu}\tfrac{\pi}{2} + N\epsilon) \right. \right. \right.$$

$$\left. \left. \left. + K\sin(\tfrac{N}{\mu}\tfrac{\pi}{2} + N\epsilon) + (k+1)\sin\mu\epsilon \right] - K\cos\mu\epsilon \right\} \right\}$$

The complex conjugate of $\mathcal{J}_+$ defines $\mathcal{J}_-$. The three integrals $\mathcal{J}_0$, $\mathcal{J}_+$, $\mathcal{J}_-$ are used in Section IV.

References

1. The relevant parts of Moser's work and Sturrock's work mentioned here are available in Cole's Notes, TN-259. Original references are listed there.

2. It is understood these parameters K, N, k are already loosely bound by the requirements of vertical motion stability and avoidance of those most harmful coupling resonances. Both these latter requirements are not given any further considerations in this study. The structure is a perfect one.

3. ν customarily is used to denote the linear tune number which is connected with linear phase shift σ as $\nu = \frac{N\sigma}{2\pi}$. This same ν is used here to denote the propagation frequency, although there is some difference which is numerically insignificant because of the small equilibrium orbit amplitude.

4. Both these methods have been studied for application here

   1. A. Erdelyi: Asymptotic Expansions (Dover) §2.3

   2. Morse and Feshbach: Method of Theoretical Physics §4.6

The results are not intelligibly simple for the purpose.

5. FLEXIBLE FIVER (MURA-604) Program 280. OVERWRITE 2 is used for fixed point search. The size of unit circle in this program (programmed R=1) is not the same as the size required in the work here. The comparison is made in the following manner. A field function $1 + \sin \Phi$ is used to integrate the equilibrium orbit which is then plotted. The point at which the orbit has its maximum β_x (which is 1.987×10^{-2} for #4863450) can be very accurately located in the plot, and the oscillation there $x \sim 2.2 \times 10^{-4}$ estimated. This β_x by

definition is W_0 in the work here and $\varkappa$ gives the approximate difference in the sizes of the unit circles. To make sure that such is indeed the situation, as well as to have a numerical verification as to the validity of the phase-gauge equivalance approximation which leads to the result in (23) for W_s (only a heuristic argument was used there; it must be numerically justified), a double check is carried out in the following manner. The difference in gauge $\Delta R \sim 2.2 \times 10^{-4}$ is converted into an equivalent phase $\Delta\gamma \sim (K-k-1)\,\Delta R \sim 0.085$ (it could be 0.084 or 0.086, but no such critical consideration is needed). Then a field function $1 + \sin(\Phi - 0.085)$ (actually -0.08425 was used, which covers an unjustified and unnecessary correction, later discarded) is used in the same program to search for the fixed point. The computer yields $p_\varkappa \sim 1.9878307 \times 10^{-2}$, $\varkappa \sim -2.1537204 \times 10^{-4}$ giving an accuracy up to third decimal place. This phase-gauge equivalence approximation should be extremely good in the limit $\Delta R \to 0$.

6. K. Knopp: Infinite sequences and series §4.4

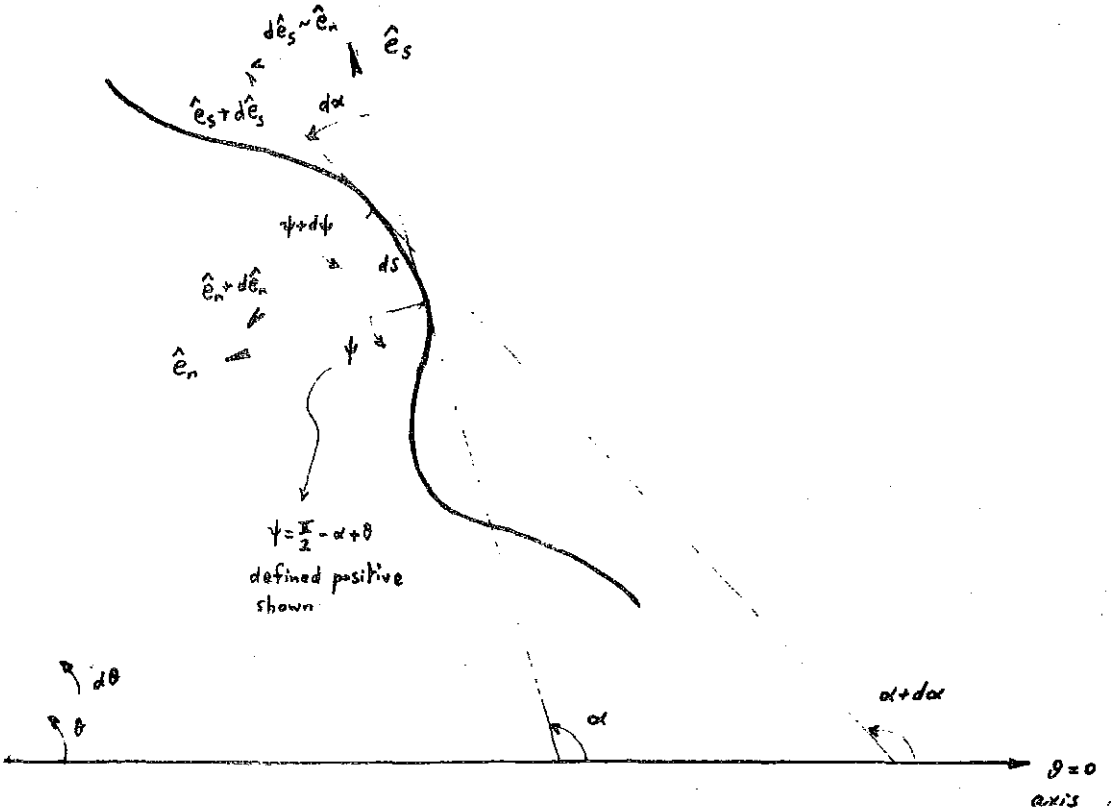


Fig 1 Orbit Geometry

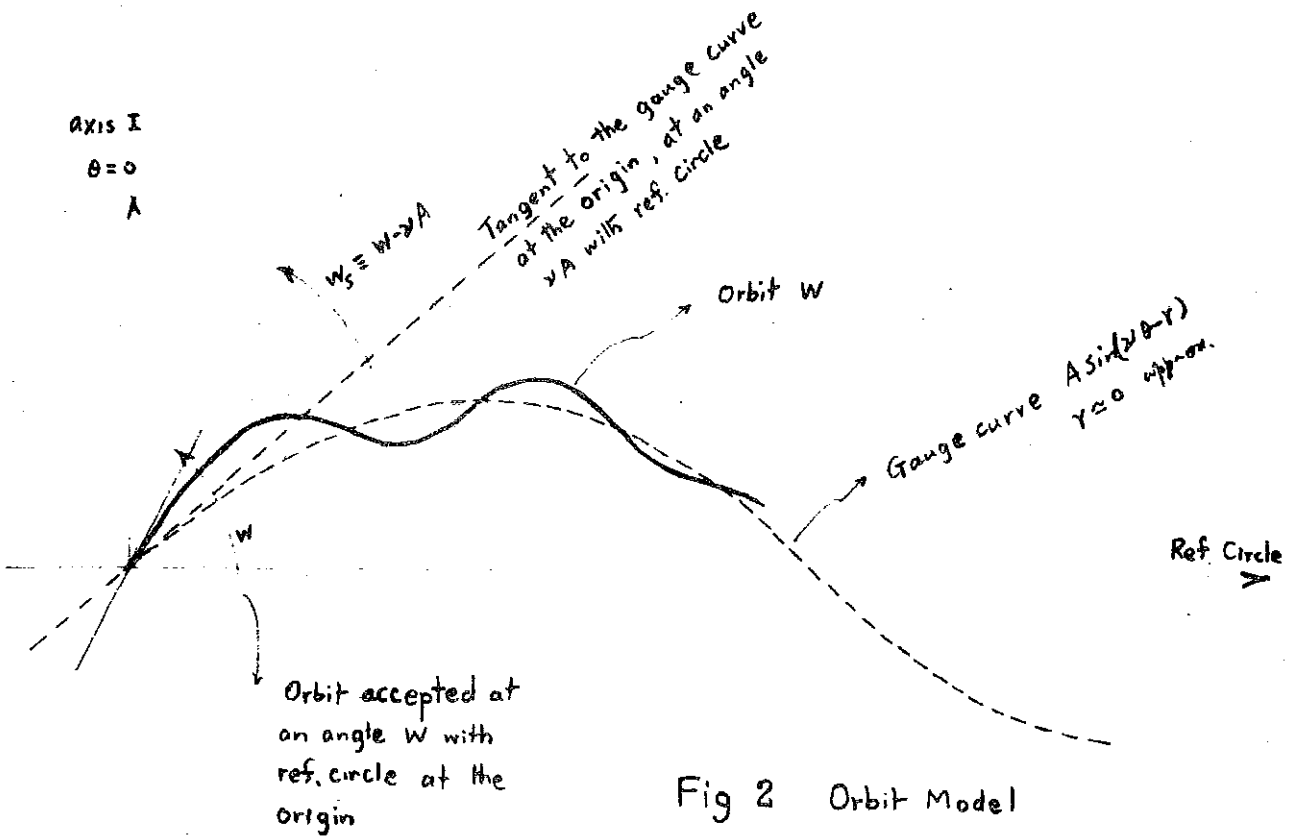


Fig 2 Orbit Model

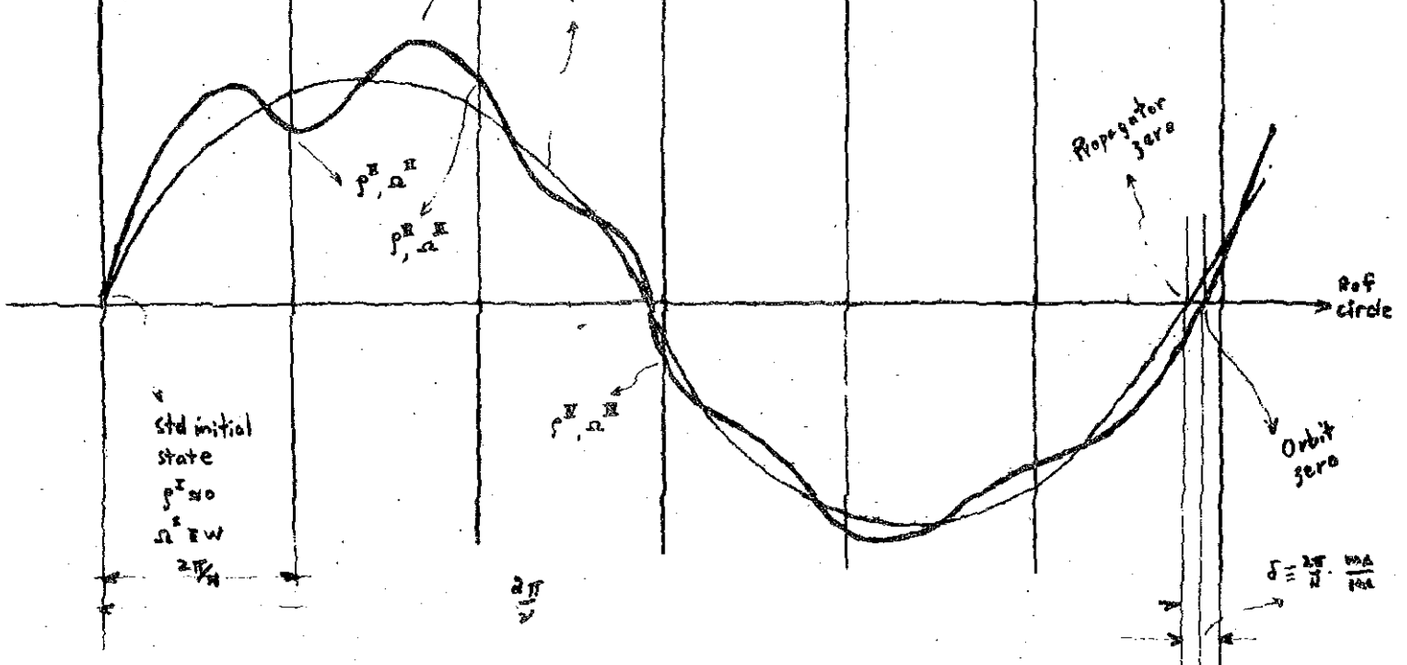


Fig 3 A sketch showing the first propagation period

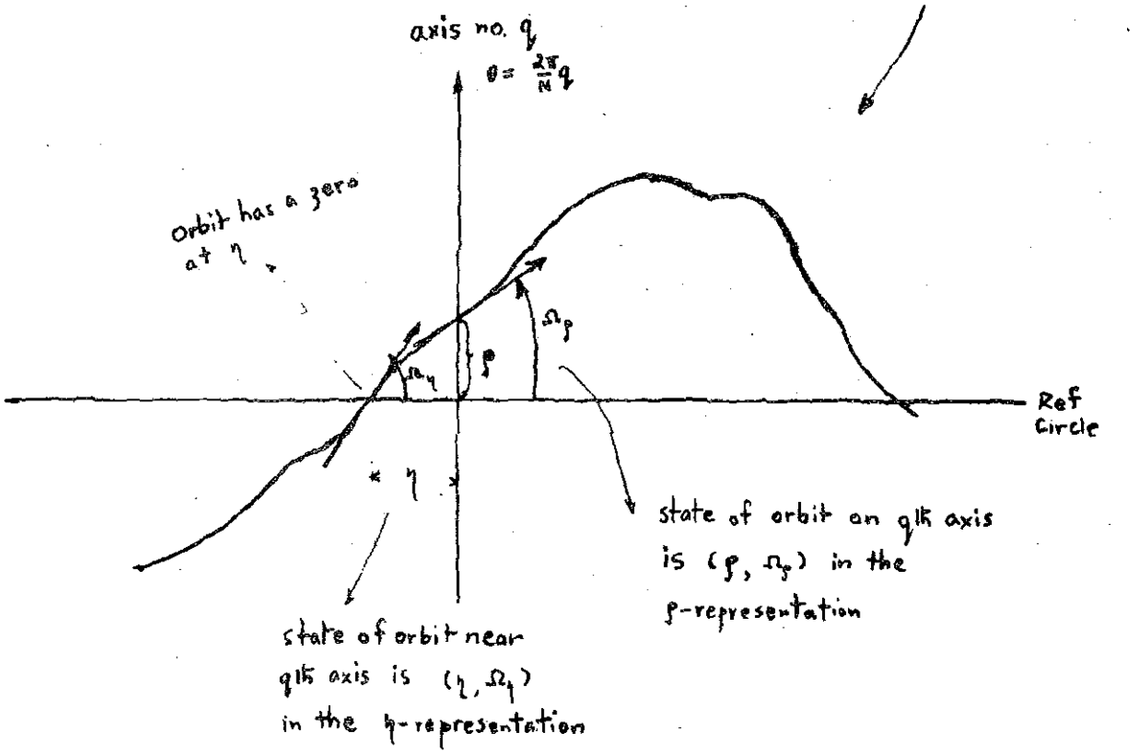


Fig 4 Orbit representations near a propagation axis

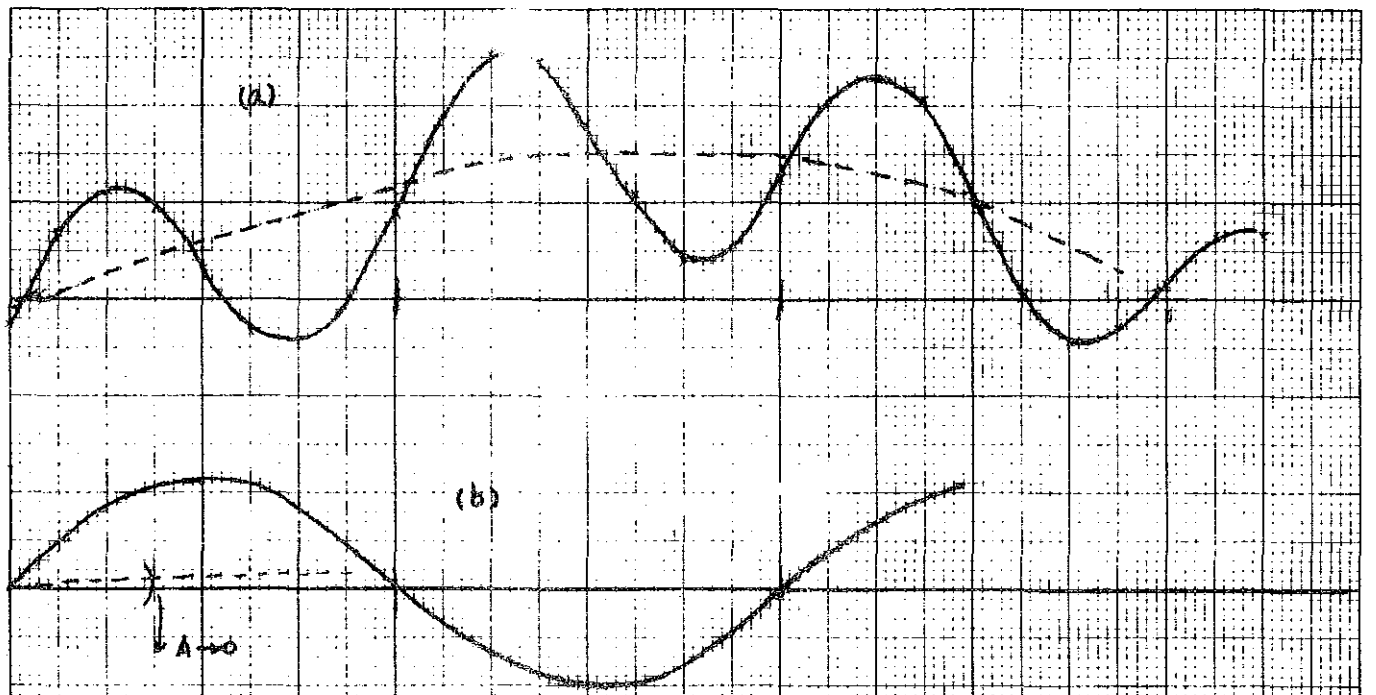
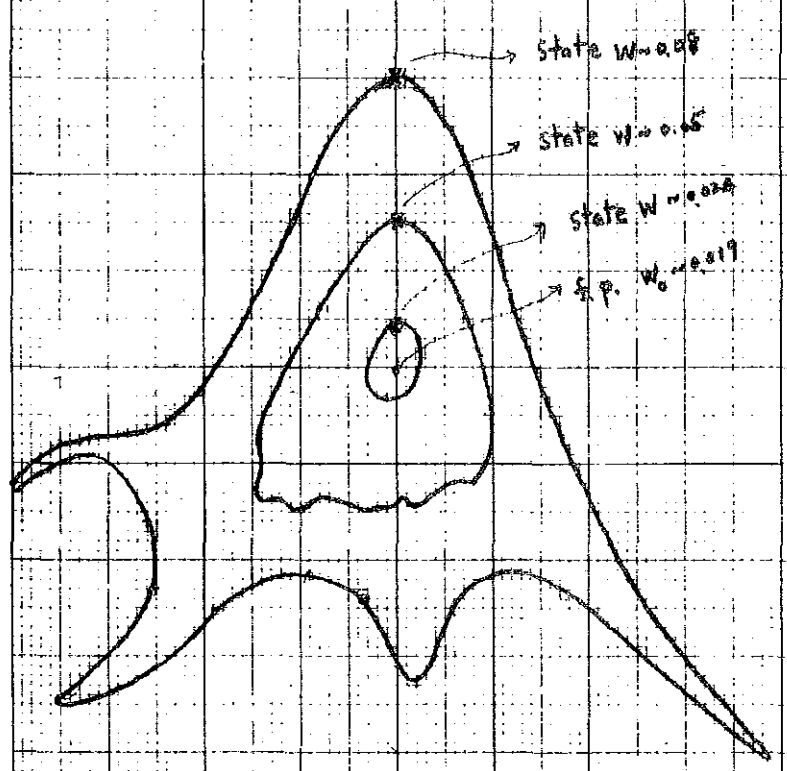


Fig 5 (a) Actual sample orbit $w \simeq 0.024$, propagator amplitude smaller than modulator amplitude

(b) The equilibrium orbit $w_0 \simeq 0.0198$, as the limiting situation $A \to 0$



The phase plot curves of sample orbits representative of general orbit behaviour at stages of increasing w

Fig 6

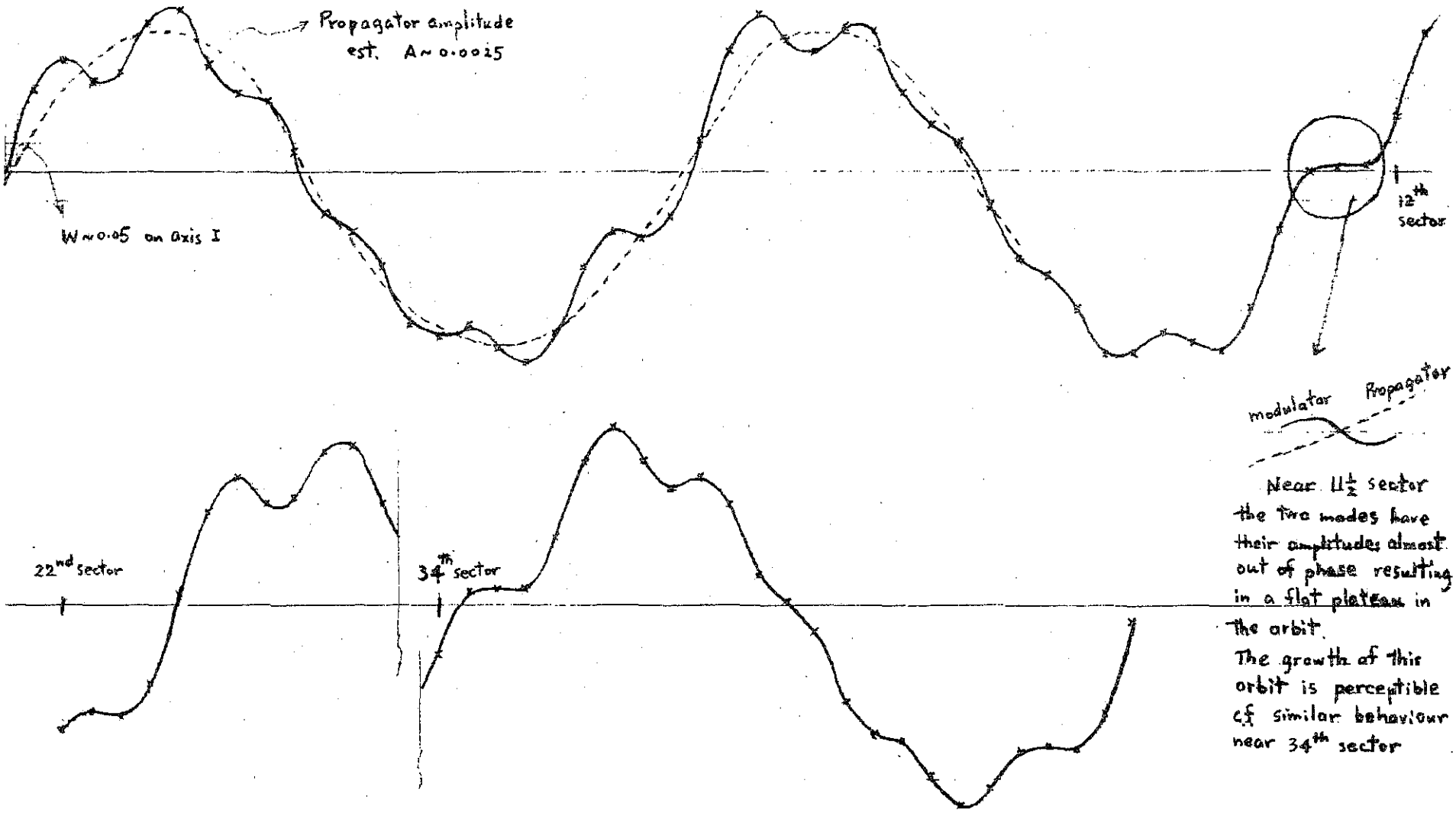
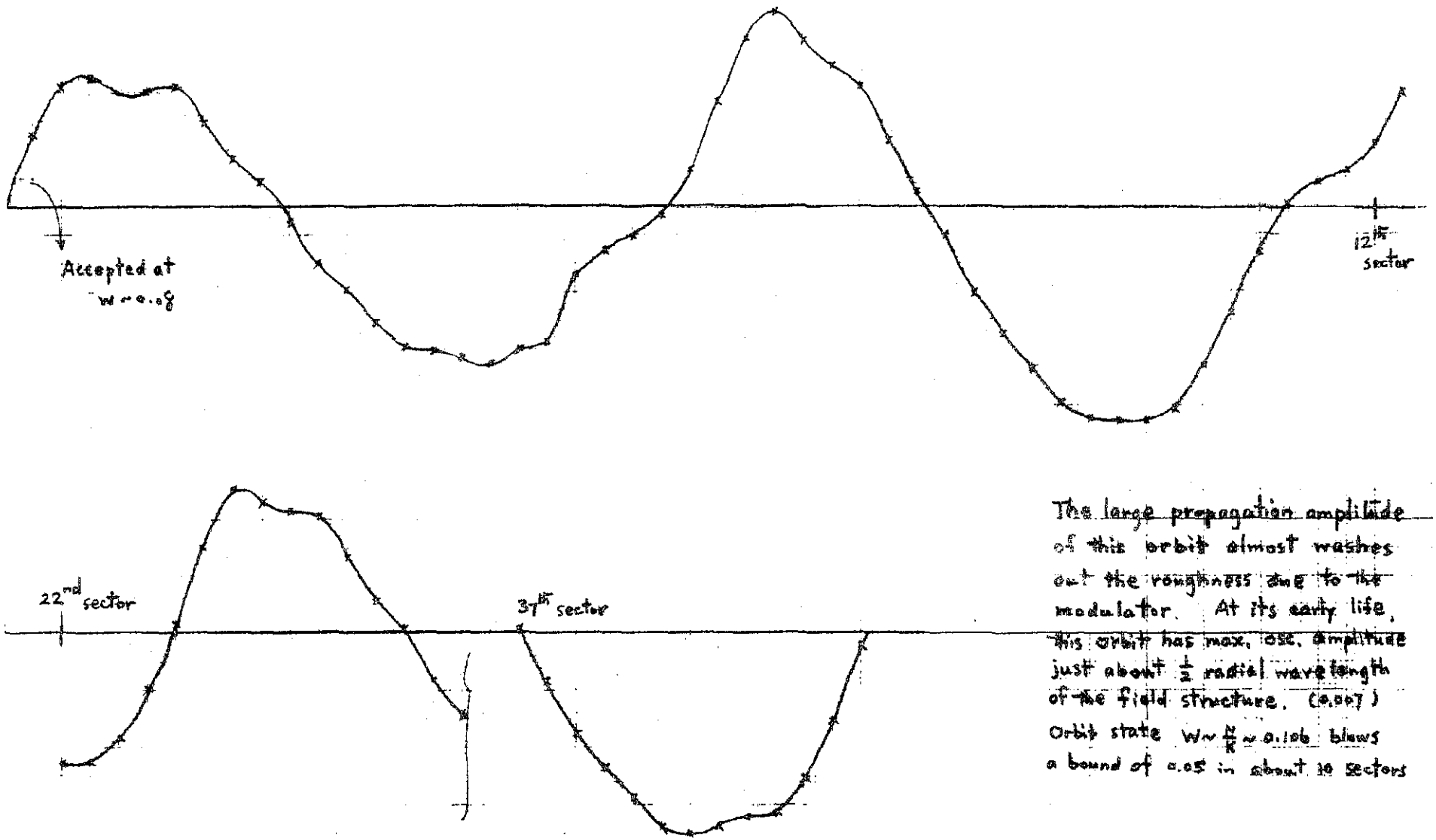


Fig 7 Orbit State W~0.05, just beyond the stable cone

4863450



The large propagation amplitude of this orbit almost washes out the roughness due to the modulator. At its early life, this orbit has max. osc. amplitude just about $\frac{1}{2}$ radial wavelength of the field structure. (0.007)
Orbit state $w \sim \frac{N}{K} \sim 0.106$ blows a bound of 0.05 in about 10 sectors

Fig 8 Orbit State $w \sim 0.08$ (Spiral ridge angle is $\frac{N}{K} \sim 0.106$)

4863450