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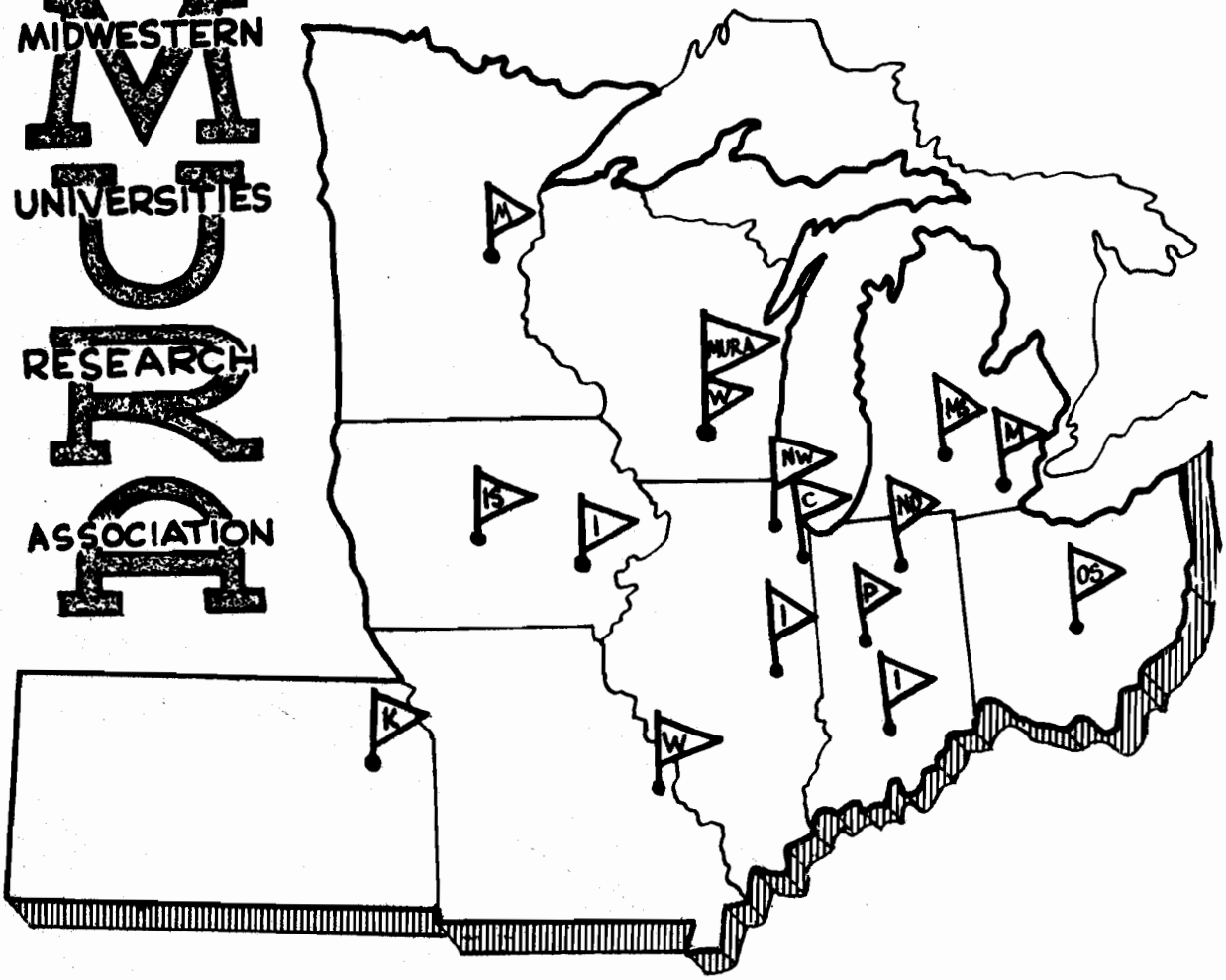

TRANSVERSE SPACE CHARGE EFFECTS IN PARTICLE ACCELERATORS

DORNIS C. MORIN, JR.

REPORT NUMBER 649

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IN PARTICLE ACCELERATORS

BY

DORNIS C. MORIN, JR.

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I. INTRODUCTION

A. Brief Description of Space Charge Effects

Particle accelerators are constructed to produce beams of charged particles which have sufficient kinetic energy so that they can be used as projectiles on targets in particle scattering experiments. The purpose of such experiments is to induce nuclear reactions, either for their direct interest or for the purpose of generating secondary beams of particles for other scattering experiments, or to probe the interaction forces between particles. The desired reactions or events created in the target occur at a rate measured by the number of events per unit time. In order to perform an experiment in a reasonable amount of time or to obtain a desired degree of statistical accuracy, the number of events per unit time must be sufficiently large. Generally the number of events per unit time is a function of both the energy and the intensity of the accelerator beam. Here the beam intensity is the time-averaged number of particles per unit time supplied by the beam to the target area. For a fixed energy of the beam the number of events per unit time is directly proportional to the beam intensity. Hence, an understanding of the factors which may influence the beam intensity is desirable.

The types of accelerators specifically in mind are the cyclotron, the synchrocyclotron, and the synchrotron. The particles are constrained by a magnetic field to move in orbits which are near a closed reference path lying in a horizontal plane; the reference path may or may not be a circle, it may change size slowly in time, and the orbits themselves need not be in the horizontal plane. The beam then surrounds the closed reference path as shown in Fig. 1. The beam as a whole is contained within a vacuum chamber so that the particles are impeded as little as possible by gas scattering.

For any particle at a given time there is a nearest point on the reference path. Let the vertical plane containing the particle and this nearest point be called the "transverse" plane. The transverse plane is perpendicular to the reference path and it is unique if the particle is not too far from the reference path depending on how much the curvature of the reference path varies from point to point. The direction perpendicular to the transverse plane is called the "longitudinal" direction. Any motion or force in the longitudinal direction is referred to as a "longitudinal" motion or force, and any motion or force perpendicular to the longitudinal direction, and thus lying in the transverse plane, is referred to as a "transverse" motion or force.

The forces acting on the particles may be classified into "external" forces and "internal" forces. The "external" forces are the forces produced by the magnetic field which constrains the particles to

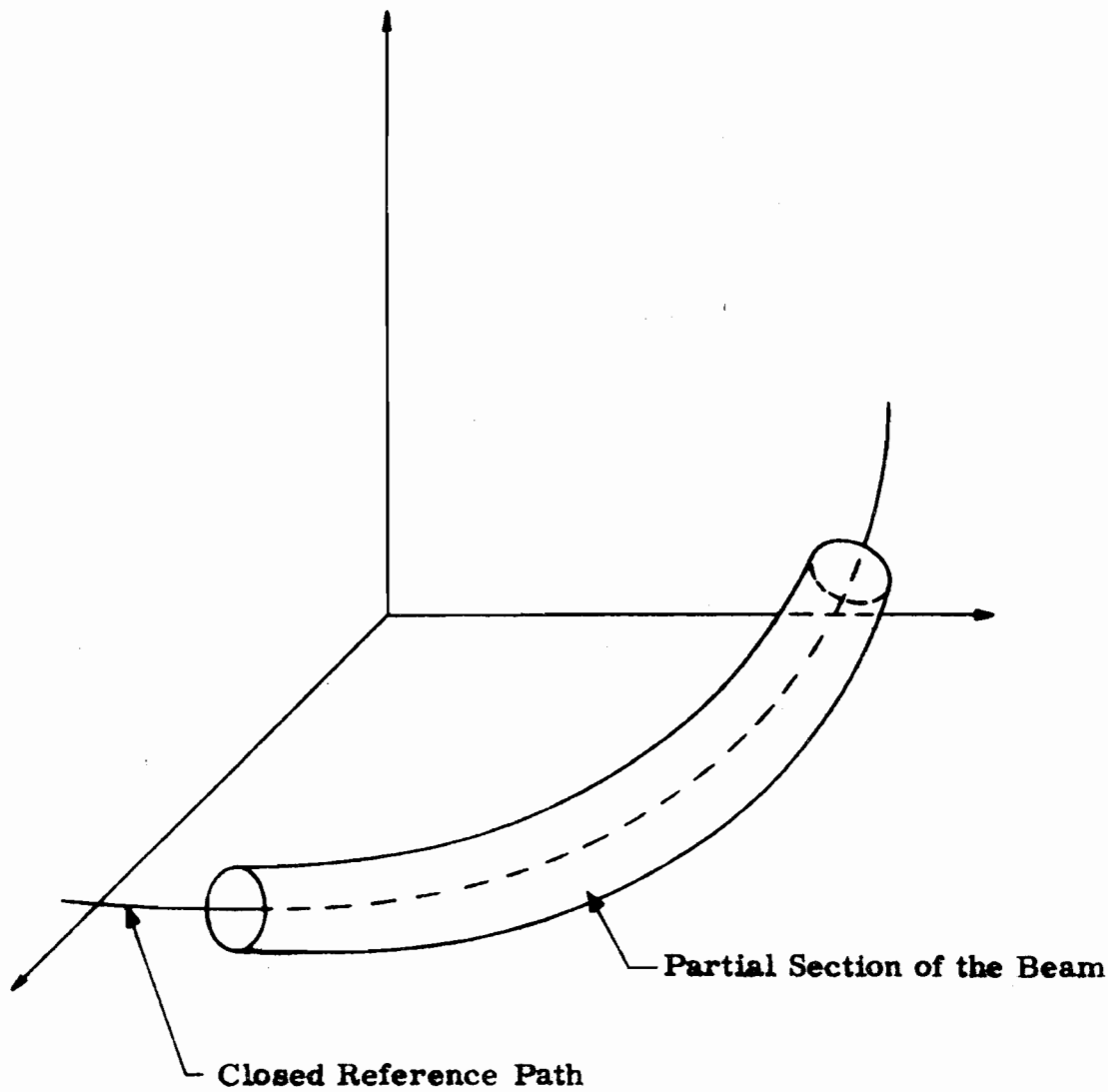


Figure 1. PICTURE OF THE BEAM

the vacuum chamber and the forces produced by the electric fields which are used to accelerate the particles. The "internal" forces are the interaction forces between the particles of the beam, the interaction forces between the particles of the beam and residual gas molecules in the vacuum chamber, and the interaction forces between particles of the beam and charged particles created by ionization of the residual gas by the beam. The total force acting on a particle of the beam caused collectively by all the other charged particles within the vacuum chamber (i.e., the other particles of the beam and charged particles created by ionization) is referred to as the "space charge" force. It may be resolved into two components: the longitudinal space charge force and the transverse space charge force.

For a first approximation the internal forces may be neglected to predict the behavior of the beam. Deviations from this behavior, when they are caused by space charge forces, are called "space charge" effects. In general, if these effects are serious, they result in beam instabilities which cause a loss of particles and therefore a reduction of the beam intensity. Usually, for simplicity, space charge effects are analyzed as two separate effects. Under appropriate assumptions the influence of the transverse space charge force can be decoupled from the influence of the longitudinal space charge force and conversely. Then the space charge effects separate into two independent effects: the "transverse space charge" effect and the "longitudinal space charge"

effect. Of course, such a separation is an approximation; the two effects are interdependent and the true situation is more complex.

To obtain a qualitative picture of the transverse space charge effect, consider a beam of charged particles which is a torus with a circular cross section and let the particles be uniformly distributed within the torus. The particles have the same charge so that the electrostatic force between them is repulsive. This tends to make the beam expand transversally and produces a transverse defocusing force on a particle. The longitudinal motion of the particles gives an electric current which creates a magnetic field surrounding the center of the beam. This produces a transverse force tending to contract the beam and gives a focusing force on a particle. The electrostatic force is always larger than the magnetic force but the two nearly cancel one another at relativistic energies. Thus the net transverse space charge force is defocusing. (This result is not necessarily true when account is made of additional charged particles due to gas ionization.) Increasing the number of particles in the beam increases the space charge defocusing force. The space charge force thus modifies the focusing force produced by the external magnetic force resulting in a net focusing force which depends on the number of particles in the beam. Since the focusing force determines the frequency of betatron oscillations, increasing the number of particles changes the frequency. This can put the oscillation on a resonance which causes the betatron oscillation to increase

in amplitude driving a large number of the particles into the walls of the vacuum chamber.

One type of longitudinal space charge effect called the "negative-mass" instability can be described as follows. Consider an accelerator operating above the transition energy so that $df/dE < 0$ where f is the particle's frequency of revolution around the accelerator and E is the particle's energy. Let the beam be monoenergetic and longitudinally bunched. A particle approaching a bunch from behind is repelled by the electric field of the bunch and the particle loses kinetic energy. Consequently the particle's frequency of revolution increases so that the particle catches up with the bunch, just as if it had a negative mass. This situation is unstable and will cause the beam to collapse into small bunches.

B. Methods of Accelerator Theory

Various methods of developing the theory of accelerators may be used depending on the approximations made and the degree of accuracy desired. Their usefulness rests on whether or not the equations which are set up can be solved, at least, to some approximation. A complete theory, after solving the equations of motion, would describe the motion of each particle of the beam under the influence of all forces. This is not possible because of the complexity created by the interactions of such a large number of particles (the number of particles in the beam is of the order 10^{10} to 10^{15}). Even if it were possible to give

such a description, by itself it would not be useful since one cannot read and interpret 10^{10} items of data. Some statistical measures of the data would be needed.

One method of approaching accelerator theory is to assume that the internal forces are negligible. In this approximation each particle then moves independently of the others and its equation of motion is that for a single particle under the influence of the external forces. This method is called the "single-particle" theory. With various additional simplifying assumptions the single-particle theory has been used most extensively; it is now well developed.[1,2] The fault of the single-particle theory is that it ignores the internal forces which can have a large influence on the beam.

An approach which attempts to account for the internal forces is to assume the particles are distributed in the beam in some manner and calculate the internal force on a single particle which results from the assumed distribution. The calculated internal force is then inserted as a correction into the equation of motion in the single-particle theory. Investigations of the transverse space charge effects have been made primarily in this manner.[3,4,5] In this method the effect of the internal forces is only partially taken into account since no allowance is made for the change in the distribution of particles brought about by the internal forces.

Another method, more general in that an attempt is made to describe the collective motion of all the particles, is based on a statistical approach. A distribution function is defined which specifies how all the particles are distributed in position and momentum; i.e., the distribution function gives the density of particles in the phase space. This approach seeks to determine how the distribution function changes in time starting from some initial distribution. The distribution function satisfies a partial differential equation known as the Boltzmann equation which depends on the forces acting on the particles.[6]

Besides the Boltzmann equation, equations governing the forces are required. In particular the equation for the internal force is functionally dependent on the distribution function. The Boltzmann equation and the force equations together form a system of equations which completely determines the distribution function at any time. Solving the system of equations is generally impossible; simplifying assumptions are usually necessary to obtain exact solutions or to obtain approximate information concerning the distribution function.

Investigations of space charge effects by means of the Boltzmann equation have heretofore been almost exclusively on the longitudinal effects.[7,8,9] The present investigation of transverse space charge effects is made through the Boltzmann equation for the purposes of putting previous results on a firmer theoretical basis and obtaining a better understanding of beam dynamics.

C. Previous Investigations of Space Charge Effects

The transverse space charge effect has been studied by a number of people. Kerst made calculations for the maximum beam current which could be contained within the dimensions of the vacuum chamber of a betatron by the external focusing force against the electrostatic repulsion force of space charge.[3] The calculations were based on assuming a uniformly distributed beam of circular cross section and the magnetic force of the space charge was ignored because he was interested in conditions at the injection energy at which it was negligible. However, he mentioned that the magnetic and electrostatic space charge forces nearly cancel at relativistic energies. This calculation was refined by Blewett who showed that a uniformly distributed beam in a betatron would have an elliptical cross section for the particular form of the external magnetic field.[10]

In a study of injection processes Judd stated that the transverse space charge force would change the frequency of the betatron oscillations.[4] He suggested that this was the reason why some betatrons which had been designed to operate on a resonance worked better than had been expected; namely, the space charge effect took the frequency off the resonance. Barden has discussed the limitations placed on the beam intensity by the space charge force changing the frequency to a resonance in alternating-gradient synchrotrons.[5] Jones investigated the influence on the transverse space charge effect due to the variation

of the beam's cross section produced by the alternating gradients in an alternating-gradient accelerator.[11] This was found to be small. Modifications of the space charge force created by the charge and current induced in the walls of the vacuum chamber by the beam have been considered by Mills and by Cole et al.[12,13] These are small unless the beam nearly fills the aperture. The change of betatron oscillation frequency for a uniform beam of elliptical cross section has been studied by Teng.[14]

None of the above-mentioned studies of transverse space charge effects made use of the Boltzmann equation. An investigation which did was made by Courant.[9] He investigated the stability of a beam when it is uniformly distributed longitudinally but distributed transversally in such a manner that the density of particles oscillates coherently with the betatron oscillations. He concluded that the beam would be stable.

Nielsen and Sessler have studied longitudinal space charge effects with radio-frequency acceleration, and, for this case, have obtained a stationary solution of the Boltzmann equation.[7] Longitudinal instability (i.e., the negative-mass instability) has been investigated by Nielsen, Sessler, and Symon by means of a linearized Boltzmann equation.[8] It was found that this instability will not occur if the energy spread of the particles is sufficient.

D. Experimental Observations

Experimental observations of transverse space charge effects

have been reported by Haxby et al., and Kerst et al.[15,16] The measurements were made on an electron spiral sector fixed-field alternating-gradient (FFAG) accelerator at the Midwestern Universities Research Association. An injection current greater than the calculated space charge limit was injected over a relatively long period of time and the output current was observed. Over the time of injection the output current behaved roughly as shown in Fig. 2. The initial output current I_o was found to be about 30 to 50% of the calculated space charge limit. Then ions formed in the residual gas begin to be collected in the beam and neutralize the electrostatic space charge force but not the magnetic space charge force of the beam. The current then builds up to I_{max} which perhaps represents the maximum current available from the injector. The measured frequencies of the betatron oscillations were found at the time of I_o to be lower than for small injection currents and to increase as ions were collected in the beam. This is in agreement with the qualitative picture of space charge effects. Eventually enough ions are collected that the frequency of betatron oscillations increases to a resonant value and the beam is lost. In a short time enough ions migrate to the walls that the beam is again established at I_{max}. This process keeps repeating as shown in Fig. 2.

Measurements of longitudinal instabilities in the Cosmotron have been reported by Barton and Nielsen.[17] The results are in agreement with the linearized theory of negative-mass instability in the

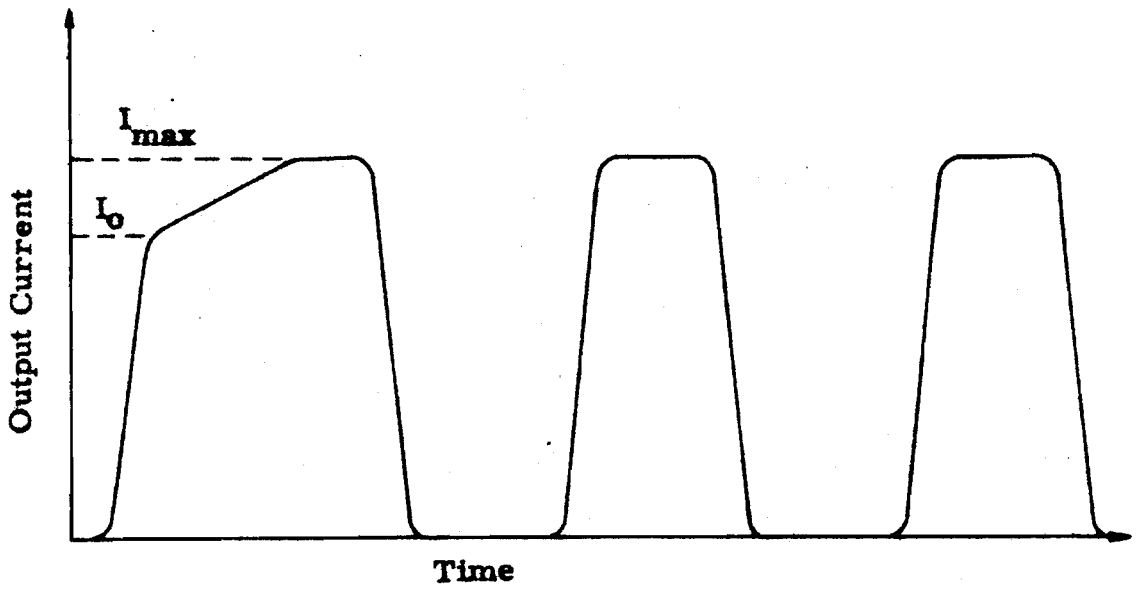


Figure 2. OUTPUT CURRENT FOR HIGH INJECTION CURRENT

range where it is applicable.

E. Outline of Investigation

The following chapters present an analysis of transverse space charge effects by a method using the Boltzmann equation. Various simplifying assumptions are made so that the general Boltzmann equation can be reduced to a one-dimensional, collisionless Boltzmann equation. A technique for solving the reduced Boltzmann equation by using moments is developed which gives an infinite set of moment equations to be solved. In general a finite number of the moment equations do not form a closed set but instead they are coupled to the remaining set of equations. By means of two essential approximations a closed set of moment equations is obtained. The first of these approximations is that the forces acting on a particle are linear with respect to its position. The second approximation is that the space charge force depends on the beam distribution only by being dependent on the position of the center of the beam and on the width of the beam. The behavior of the beam is then described by the resulting closed set of moment equations.

It is found that the motion of the center of the beam is independent of the space charge forces and that space charge effects cannot put the beam onto an integral resonance. Conditions are obtained for an approximate stationary solution for which the center and width of the beam remain constant. In the absence of field errors and if the conditions for a stationary solution are not satisfied, the beam width per-

forms oscillations which are always stable. A linear theory for the effect on the beam width due to gradient errors in the external magnetic field indicates that unstable resonances may exist.

Chapter II gives an elementary treatment of transverse space charge effects. In Chapter III some considerations of the Boltzmann equation and the initial assumptions are presented which allow the general Boltzmann equation to be reduced to a one-dimensional Boltzmann equation. The technique for solving the Boltzmann equation by using moments is developed in Chapter IV and a linear approximation for the space charge force is obtained which enables one to get a closed set of moment equations. The behavior of the beam described by the moment equations is discussed in Chapter V. Finally, in Chapter VI, some of the limitations of the results arising from the initial assumptions are discussed and possible ways of extending the analysis are presented.

Gaussian units are used throughout the analysis with the unit of electric current being statcoulomb/sec. All integrals without specified limits are over the complete range of the integration variables.

II. ELEMENTARY THEORY OF TRANSVERSE SPACE CHARGE EFFECTS

A. Review of Single-Particle Theory

An elementary treatment of transverse space charge effects is presented in this chapter. Before doing so, a brief account of that part of the single-particle theory needed for understanding in this and the remaining chapters is given without derivation. The literature can be consulted for details concerning derivations and justification for the approximations.[1,2,18,19]

The single-particle theory is based on ignoring the internal forces and considers the motion of a single particle. For a particle of charge q, velocity $\vec{v}$, and momentum $\vec{p}$, the equation of motion is taken to be

$$\frac{d\vec{p}}{dt} = q\,\vec{E} + \frac{q}{c}\,\vec{v} \times \vec{B} \tag{1}$$

where t is the time, $\vec{E}$ is the accelerating electric field, $\vec{B}$ is the magnetic field for constraining the particle within the vacuum chamber, and c is the speed of light. An approximate analysis of the transverse motion is made by assuming that the effect of $\vec{E}$ on the transverse motion is negligible and that $\vec{B}$ is constant in time. Then

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\,\vec{v} \times \vec{B};\ \vec{B} \text{ constant in time.} \tag{2}$$

Consequently, the magnitude of the momentum and the energy of the particle are constants of the motion.

Consider a cylindrical coordinate system r, θ, z with the $r - \theta$ plane horizontal. As an idealization it is assumed that the magnetic field $\vec{B}$ is such that $B_r = B_\theta = 0$ on some horizontal plane which is called the "median" plane. Let $z = 0$ on the median plane. For a particle of a given energy, or momentum p, there exists an orbit which lies in the median plane and is closed; i.e., the orbit is specified by $r = R(\theta)$, $z = 0$, where $R(\theta)$ is periodic in θ. This orbit is called the "equilibrium" orbit of the particle corresponding to its momentum p.

Now consider the motion of a particle of momentum p which is not necessarily on its equilibrium orbit. It is convenient to describe the motion in terms of a new coordinate system defined with respect to the particle's equilibrium orbit in the following way. For any point P, let Q be its vertical projection onto the median plane (see Fig. 3). Let x be the perpendicular distance from Q to its nearest point O on the equilibrium orbit. x will be taken positive when Q is outside the area enclosed by the equilibrium orbit. Let z be, as before, the distance P is from the median plane. Let s be the arc length along the equilibrium orbit from some arbitrary point to O. Then P is uniquely specified by the new coordinates x, s, and z, provided that $|x|$ is less than the curvature of the equilibrium orbit at O. Let $\hat{e}_x$, $\hat{e}_s$, and $\hat{e}_z$

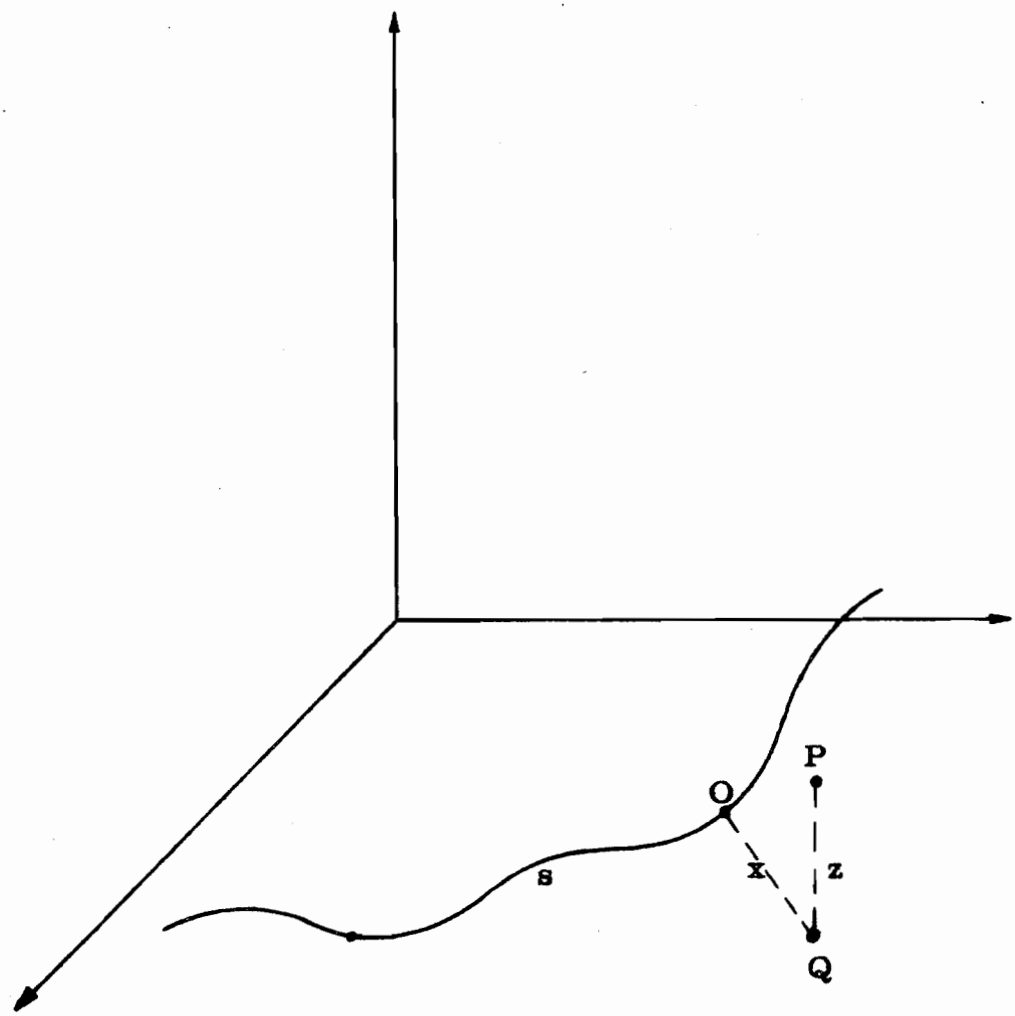


Figure 3. COORDINATE SYSTEM x, s, z

be a right-handed orthogonal set of unit vectors at O. Equilibrium orbits will be used for the closed reference path discussed in Chapter I and the x - z plane then becomes the transverse plane.

The magnetic field is specified by giving B_z on the median plane. For constant gradient accelerators B_z on the median plane is usually taken to be

$$B_z = B_o \left(\frac{r}{r_o}\right)^{-n}$$

where B_o is a constant, r_o is a reference radius, and $0 < n < 1$. For most alternating gradient (AG) accelerators the median plane field is taken to be

$$B_z = B_o + B_1 (s)\, x$$

with respect to a particular equilibrium orbit. Here B_o is constant and B_1 (s) is periodic of some square-wave form. For a fixed-field alternating-gradient (FFAG) accelerator the median plane field can be written

$$B_z = B_o \left(\frac{r}{r_o}\right)^{k} f\left(K \ln \frac{r}{r_o} - N\theta\right)$$

where B_o, K, and N are constants, r_o is a reference radius, and f is a periodic function of period 2π. In most machines n, k, and K are independent of r. Along an equilibrium orbit in an AG or FFAG machine B_z or $\frac{\partial B_z}{\partial x}$ as functions of s are periodic with a smallest

period $T < L$ where L is the length of the equilibrium orbit once around the machine. The basic periodic structure is called a "sector" and the number of sectors N is L/T.

Within the vacuum chamber, $\vec{\nabla} \times \vec{B} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$. These relations and the knowledge of $\vec{B}$ on the median plane make it possible to expand B_r, B_θ, and B_z, or instead, B_x, B_s, and B_z about the equilibrium orbit in a double power series in x and z with coefficients which are functions of s. Substituting these expressions into equation (2) and making a change of variable from t to s for the independent variable, the equations of motion in x and z, including relativistic effects, take the form:

$$\frac{d^2x}{ds^2} + K_1(s)\,x = M_1$$
$$\frac{d^2z}{ds^2} + K_2(s)\,z = M_2. \qquad (3)$$

K_1 and K_2 are periodic functions of s with period T. M_1 and M_2 are functions which make the above equations nonlinear. These equations give the motion of a particle relative to its equilibrium orbit. The x- and z-motions according to equations (3) when the acceleration process is ignored are called "betatron" oscillations. Although the x-direction is not the same as the r-direction, x-motion is referred to as "radial" motion, and naturally z-motion is referred to as "vertical" motion.

Let θ be an "angle" variable defined by

$$\theta = \frac{2\pi s}{L} . \tag{4}$$

Here θ is not the azimuthal angle used initially for the cylindrical coordinate system. However, if the equilibrium orbit does not deviate too much from a circle, θ is very nearly equal to the azimuthal angle. Transforming from s to θ, equations (3) can be written:

$$\begin{aligned} \frac{d^2x}{d\theta^2} + K_x(\theta)\, x &= M_x \\ \frac{d^2z}{d\theta^2} + K_z(\theta)\, z &= M_z. \end{aligned} \tag{5}$$

K_x and K_z are periodic functions of θ with period $2\pi/N$. M_x and M_z are the nonlinear terms.

The linearized approximation is obtained by ignoring the nonlinear terms and solving the equations:

$$\frac{d^2x}{d\theta^2} + K_x(\theta)\, x = 0 \tag{6}$$

$$\frac{d^2z}{d\theta^2} + K_z(\theta)\, z = 0. \tag{7}$$

Consider equation (6) for the radial motion. Its solution can be written in the form

$$x = e^{i\nu_x\theta}\, h_1(\theta) + e^{-i\nu_x\theta}\, h_2(\theta)$$

where h_1 and h_2 are periodic functions of θ with period $2\pi/N$ and ν_x is a constant. By "stable motion" it is meant that the motion is bounded. The motion is stable only if ν_x is real and $\nu_x \neq \frac{1}{2} n N$ where n is an integer or zero. The value of ν_x is determined by $K_x(\theta)$ which depends on the parameters of the accelerator. A typical stable solution has the appearance shown in Fig. 4. It looks like a sinusoidal oscillation of ν_x oscillations per revolution around the machine with a superimposed ripple which is periodic with period $2\pi/N$. ν_x is the number of radial betatron oscillations per revolution; it is also called the radial "tune."

Similar results hold for the vertical motion given by equation (7). Namely,

$$z = e^{i\nu_z\theta} h_3(\theta) + e^{-i\nu_z\theta} h_4(\theta)$$

where h_3 and h_4 are periodic with period $2\pi/N$ and ν_z is a constant. Again the motion is stable only if ν_z is real and $\nu_z \neq \frac{1}{2} n N$ where n is an integer or zero. ν_z is the number of vertical betatron oscillations per revolution and is called the vertical "tune."

When the effects of the nonlinear terms M_x and M_z in equations (5) are considered, it is found that resonances exist which may create unstable motion if ν_x and ν_z satisfy the relation:

$$n_x \nu_x + n_z \nu_z = n N, \tag{8}$$

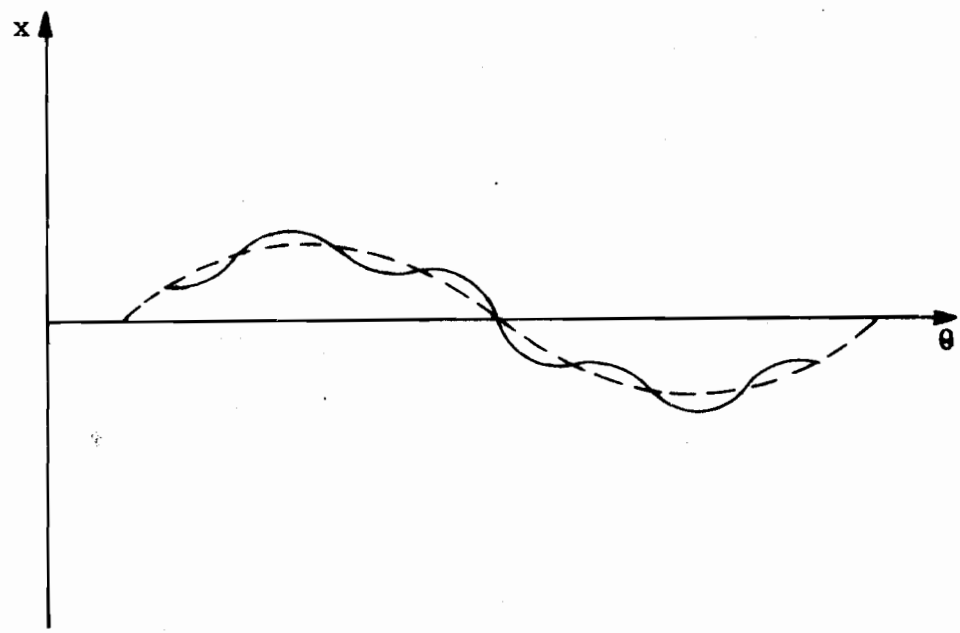


Figure 4. TYPICAL STABLE BETATRON OSCILLATION

where n_x, n_z, and n are zero or positive or negative integers. These resonances are called "essential" resonances because they arise out of the idealized magnetic field assumed for the machine. When $n_z = 0$, relation (8) gives the resonances of the x-motion which are independent of the z-motion. Similarly, when $n_x = 0$, relation (8) gives the resonances of the z-motion which are independent of the x-motion. When n_x and n_z are both nonzero, the resonances given by relation (8) are called "coupling" resonances because they occur when the x-motion and the z-motion are coupled together. Those resonances for which n_x and n_z have the same sign are known as "sum" resonances and those for which n_x and n_z have opposite sign are known as "difference" resonances. Not all of these resonances generate unstable motion. The difference resonances are stable and also, if the amplitude of oscillation is not too large, the resonances are stable if $|n_x| + |n_z| > 4$. Although a particular resonance by itself may be stable, it can cause trouble by making the amplitude of oscillation large enough to drive the particle into the walls of the vacuum chamber or by making the amplitude large enough to put the oscillation on another resonance which is unstable. The latter possibility comes about through the fact that the tune is a function of the amplitude of oscillation when nonlinearities are included.

Resonances can also arise in an accelerator because the actual magnetic field, in practice, cannot be made exactly as desired. These

resonances are known as "imperfection" resonances. Because of the deviations from the desired field, the actual magnetic field, in general, has a periodic structure with a period which is, at most, one revolution around the machine. This is equivalent to saying that the machine has one sector. Thus the possible imperfection resonances occur when ν_x and ν_z satisfy relation (8) with $N = 1$.

Suppose ΔB_z is the deviation from the desired B_z on the median plane. Let ΔB_z be expanded around the equilibrium orbit in a power series in x. Taking the two leading terms gives

$$\Delta B_z = A(\theta) + B(\theta)\, x$$

where $A(\theta)$ and $B(\theta)$ are periodic with period 2π. $A(\theta)$ is referred to as a "field bump" and $B(\theta)$ as a "gradient bump." In the vicinity of the equilibrium orbit the influence of these terms on the radial motion is usually larger than the ignored terms and therefore the resonances which can be excited by them should be avoided. $A(\theta)$ can be expanded in a Fourier series and, if ν_x is equal to the frequency of a nonzero Fourier component, a resonance is excited. The possible resonances excited by the field bump are

$$\nu_x = n; \quad n = 0, 1, 2, \cdots ,$$

and are called "integral" resonances. Similarly, $B(\theta)$ can be expanded in a Fourier series and it can be shown that, if ν_x is equal to half

the frequency of a nonzero Fourier component, a resonance is excited. The possible resonances excited by the gradient bump are

$$\nu_x = \frac{1}{2} n; \quad n = 0, 1, 2, \cdots ,$$

and are called "half-integral" resonances. Similarly, in considering vertical motion with $x = 0$, the deviation ΔB_x in the field B_x from the desired value can be expanded in z and the two leading terms give a field bump and a gradient bump. Again the possible integral resonances

$$\nu_z = n; \quad n = 0, 1, 2, \cdots$$

and the possible half-integral resonances

$$\nu_z = \frac{1}{2} n; \quad n = 0, 1, 2, \cdots$$

are to be avoided. It is to be noted that an integral or half-integral resonance is not excited unless the proper Fourier component of the field or gradient bump is nonzero. The integral and half-integral resonances are imperfection resonances but not the only ones since there are others given by relation (8) with $N = 1$.

The essential and imperfection resonances given by (8) can be plotted on a ν_x - ν_z plane to give a diagram like that in Fig. 5 which shows only some of the resonance lines for $N = 6$. An accelerator is designed so that its tune or operating point on the diagram is well away from the important resonance lines as illustrated by point P in Fig. 5.

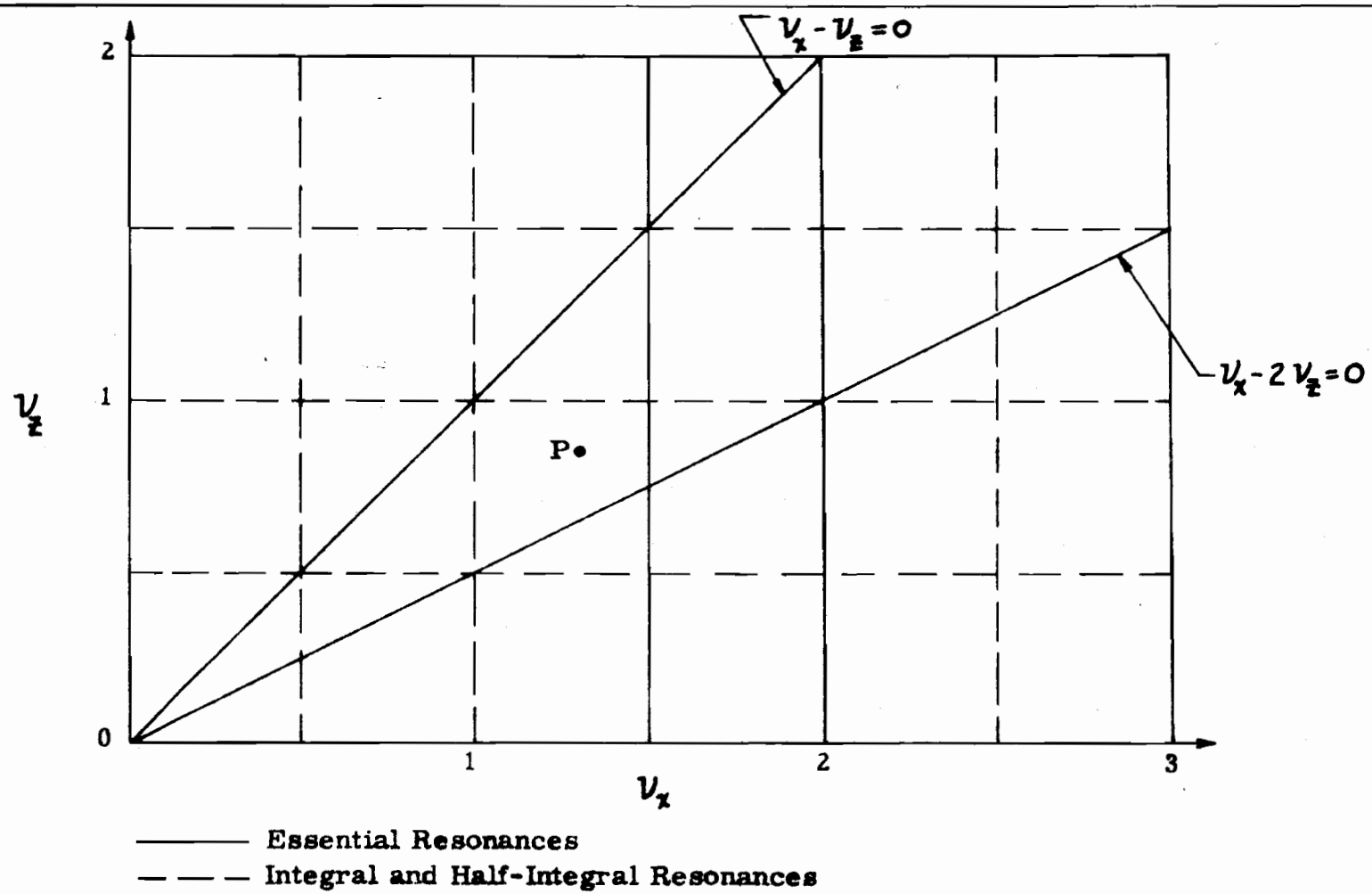


Figure 5. SOME OF THE RESONANCE LINES FOR N = 6

If the tune should vary during the operation of the machine and pass through a resonance line, it must pass through rapidly enough, depending on how serious the resonance is, that too many particles are not lost. Since most AG and FFAG machines are designed so that the tune is relatively constant, the major cause of a change in the tune is the transverse space charge effect.

The above discussion dealt with the transverse motion. The equations of motion also give the longitudinal velocity $v_s = ds/dt$ as a function of t, s, or θ. When a particle is on its equilibrium orbit, v_s equals the velocity v corresponding to its momentum p. If a particle is executing betatron oscillations, v_s varies with the amplitude and ripple of the betatron oscillations but generally v_s is much greater than the transverse velocity and since v is constant, v_s is very nearly equal to v.

B. Elementary Treatment of Transverse Space Charge Effects

The following elementary treatment of transverse space charge effects gives some quantitative insight into their nature. It is based on the method of calculating the space charge force and inserting this into the single-particle equations of motion.

Consider a beam of particles, all having mass m, charge q, momentum p, and velocity v. The particles all have the same equilibrium orbit which is surrounded by the beam. Assume that the cross section of the beam in the transverse dimensions is a circle of radius

a and that the density of particles, n, within the beam is uniform. As an approximation take the longitudinal velocity v_s to be v for each particle.

In an accelerator the average radius of the equilibrium orbit is much greater than the dimensions of the beam cross section and the deviation of the equilibrium orbit from a circle is small enough that to a good approximation the beam can be treated as an infinitely long cylinder of radius a filled uniformly with charge moving with velocity v for the purpose of calculating its electric and magnetic fields. Let $\vec{E}$ and $\vec{B}$ now be the electric and magnetic fields, respectively, produced by the beam:

$$\vec{E} = E_x \hat{e}_x + E_s \hat{e}_s + E_z \hat{e}_z$$

$$\vec{B} = B_x \hat{e}_x + B_s \hat{e}_s + B_z \hat{e}_z$$

where now the coordinate system x, s, and z is treated as a rectangular coordinate system. The cylindrical symmetry and uniform charge distribution give $E_s = B_s = 0$.

To determine the electric field apply Gauss' law to a cylinder coaxial with the beam of unit length and of radius $y < a$. Let E_n be the outward normal electric field on the surface of the cylinder. Then

$$2\pi y E_n = 4\pi \cdot \text{charge in cylinder}$$
$$= 4\pi \cdot \pi y^2 \cdot n q,$$

or

$$E_n = 2\pi q n y.$$

Then

$$E_x = E_n \frac{x}{y} = 2\pi q n x \tag{9}$$

$$E_z = E_n \frac{z}{y} = 2\pi q n z. \tag{10}$$

For the magnetic field apply Ampere's law to a circular path lying in a transverse plane with its center on the central axis of the beam. Let the path have a radius $y < a$ and let B_t be the tangential magnetic field on the path. Assume the longitudinal velocity is positive. Then

$$2\pi y B_t = \frac{4\pi}{c} \cdot \text{current through the area enclosed}$$

$$= \frac{4\pi}{c} \cdot \text{current density} \cdot \text{area}$$

$$= \frac{4\pi}{c} \cdot n q v \cdot \pi y^2$$

where c is the velocity of light; or

$$B_t = 2\pi n q \beta y$$

where $\beta = v/c$. Then

$$B_x = B_t \frac{z}{y} = 2\pi q n \beta z = \beta E_z \tag{11}$$

$$B_z = - B_t \frac{x}{y} = - 2\pi q n \beta x = - \beta E_x. \tag{12}$$

The space charge force $\vec{F}$ acting on a particle in the beam is

$$\vec{F} = q\,\vec{E} + \frac{q}{c}\,\vec{v} \times \vec{B}.$$

From this,

$$F_x = q\,E_x + \frac{q}{c}\,(v_s\,B_z - v_z\,B_s) = q\,E_x + \frac{q}{c}\,v\,B_z$$

$$= q\,(E_x + \beta\,B_z).$$

After substituting equation (12),

$$F_x = q\,(E_x - \beta^2\,E_x) = q\,(1 - \beta^2)\,E_x. \tag{13}$$

Similarly,

$$F_z = q\,E_z + \frac{q}{c}\,(v_x\,B_s - v_s\,B_x) = q\,E_z - \frac{q}{c}\,v\,B_x$$

$$= q\,(E_z - \beta\,B_x).$$

After substituting equation (11),

$$F_z = q\,(E_z - \beta^2\,E_z) = q\,(1 - \beta^2)\,E_z. \tag{14}$$

Equations (9) and (10) put into equations (13) and (14) give

$$F_x = 2\pi\,q^2\,n\,(1 - \beta^2)\,x \tag{15}$$

$$F_z = 2\pi\,q^2\,n\,(1 - \beta^2)\,z. \tag{16}$$

Now consider the linear approximation for the radial motion of a particle ignoring, for the moment, the space charge force F_x. The equation of motion (cf. equation (6)) is

$$\frac{d^2x}{d\theta^2} + K_x(\theta)\, x = 0$$

where $\theta = 2\pi\, s/L$ and L is the length of the equilibrium orbit. In general the oscillations consist of a ripple superimposed on a basic oscillation of frequency ν_x. For an approximation, let the ripple be ignored. Then the oscillation is sinusoidal of frequency ν_x and the equation of motion can be written:

$$\frac{d^2x}{d\theta^2} + \nu_{xo}^2\, x = 0 \tag{17}$$

where ν_x is replaced by ν_{xo} to indicate the radial tune when no space charge forces are present. The space charge force is now taken into account by inserting it into the equation of motion. To see how this modifies equation (17) suppose a force F_o, in general, is acting in the x-direction. The equation of motion would then be

$$\frac{d}{dt}\left(m\gamma \frac{dx}{dt}\right) = F_o$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$.

Assuming that the variation of γ is negligible, this becomes

$$m\gamma \frac{d^2x}{dt^2} = F_o.$$

To a good approximation $v_s = v$ is constant and $s = v\,t$. Transforming from the variable t to the variable s, the equation of motion is

$$m \gamma v^2 \frac{d^2x}{ds^2} = F_o.$$

But $\theta = 2\pi s/L = s/R$ where R is an equivalent radius defined by $R = L/2\pi$. Making a change of variable from s to θ, the equation of motion becomes

$$\frac{m \gamma v^2}{R^2} \frac{d^2x}{d\theta^2} = F_o,$$

or

$$\frac{d^2x}{d\theta^2} = \frac{R^2}{m \gamma v^2} F_o.$$

Thus equation (17) is to be modified to account for the space charge force F_x by inserting

$$\frac{R^2}{m \gamma v^2} F_x = \frac{R^2}{m c^2 \gamma \beta^2} F_x$$

in the right-hand side which gives

$$\frac{d^2x}{d\theta^2} + \nu_{xo}^2 x = \frac{R^2}{m c^2 \gamma \beta^2} F_x.$$

Substituting equation (15) for F_x results in

$$\frac{d^2x}{d\theta^2} + \left[\nu_{xo}^2 - \frac{2\pi q^2 n R^2 (1 - \beta^2)}{m c^2 \gamma \beta^2}\right] x = 0$$

or

$$\frac{d^2x}{d\theta^2} + \nu_x^2 x = 0$$

where

$$\nu_x^2 = \nu_{xo}^2 - \frac{2\pi q^2 n R^2 (1 - \beta^2)}{m c^2 \gamma \beta^2} . \tag{18}$$

Here ν_x is the radial tune under the influence of the space charge force. Instead of measuring the beam intensity by the number of particles per unit time, let it be measured by the beam current I in e.s.u. (i.e., statamperes). Then

$$I = \text{current density} \cdot \text{cross sectional area of beam}$$

$$= n q v \cdot \pi a^2$$

or $n = \dfrac{I}{\pi q a^2 v}$.

Inserting this into equation (18) gives

$$\nu_x^2 = \nu_{xo}^2 - \frac{2 q R^2 I (1 - \beta^2)}{m c^3 a^2 \beta^3 \gamma}$$

$$= \nu_{xo}^2 - \frac{2 q R^2 I}{m c^3 a^2 \beta^3 \gamma^3} . \tag{19}$$

In a similar manner for the vertical motion,

$$\nu_z^2 = \nu_{zo}^2 - \frac{2 q R^2 I (1 - \beta^2)}{m c^3 a^2 \beta^3 \gamma}$$

$$= \nu_{zo}^2 - \frac{2 q R^2 I}{m c^3 a^2 \beta^3 \gamma^3} , \tag{20}$$

where ν_{zo} is the vertical tune when no space charge force is present

and ν_z is its value otherwise.

Several conclusions can be drawn from equations (19) and (20). Both the radial and vertical tunes are lowered by the space charge force. The amount of reduction increases with beam current and is strongly dependent on the energy of the particles. As the energy increases, the factor $1 - \beta^2 \rightarrow 0$ because the magnetic force becomes large enough to cancel the electric force. Consequently the space charge forces which limit the current are usually dominant at the injection energy. According to equations (19) and (20), if the space charge forces are large enough, ν_x^2 or ν_z^2 is negative which would result in an exponentially-growing (unstable) motion for the particles; thus the beam would blow itself apart in this picture. This is an erroneous conclusion physically which arises out of the fact that the above analysis does not allow for the change in the density of particles in time under the influence of the space charge forces. In a later chapter it is shown under suitable assumptions that this type of instability does not exist.

Suppose one of the tunes, say ν_x, has been decreased by the space charge effect to an integer which is an integral resonance being driven by the proper Fourier component of the field bump. The nature of the resonance is such that, at a given longitudinal position, a particle experiences a transverse perturbing force which is always in the same direction on each successive revolution so that its transverse position becomes larger and larger in the direction of the perturbing

force. This occurs regardless of which side of the equilibrium orbit the particle was initially. Then, if the space charge forces create an integral resonance, the above picture implies that the beam as a whole, including its center, continuously moves away from the equilibrium orbit, resulting in an unstable beam.

But a dubious light can be cast on this conclusion. Consider the beam initially when it is centered about the equilibrium orbit and assume one of the particles is going around on the equilibrium orbit. Since this particle is centrally located with respect to the charge distribution, there are no space charge forces acting on it and, therefore, it cannot be put on an integral resonance in this manner. Thus the center of the beam would remain on the equilibrium orbit instead of moving away as indicated earlier. The contradiction suggests that something is wrong with the modified single-particle equation of motion. A closer look at the situation reveals that the difficulty arises from two sources. One has been mentioned already; namely, that no allowance has been made for the change of the beam in time. The other, which is essentially the same, is that the above analysis expresses the space charge force in terms of the particle's displacement from the equilibrium orbit whereas, since the beam may move away from the equilibrium orbit, the force should be in terms of the displacement from a suitable point within the beam which measures the transverse motion of the beam as a whole. Thus there is doubt as to whether or not the space

charge effects can induce beam instability through an integral resonance in the manner described above. This question is examined more thoroughly in a later chapter.

Suppose now one of the tunes, say ν_x, has been lowered by the space charge effect to a value which is an half-integral resonance being driven by the proper Fourier component of the gradient bump. The nature of this resonance is such that the strength of the perturbation which a particle experiences is proportional to its distance from the equilibrium orbit so that particles on the equilibrium orbit are unperturbed. Furthermore, consider two particles having the same longitudinal position and the same amplitudes of betatron oscillations but completely out of phase. Then, when one is on one side of the equilibrium orbit, the other is on the other side by the same amount. Their amplitudes of oscillation grow because of the resonance so that when one is making a larger excursion away from the equilibrium orbit than before, the other is also but in the opposite direction. Thus the particles on the equilibrium orbit remain there while those off the equilibrium orbit oscillate with growing amplitudes so that the beam as a whole remains on the equilibrium orbit but steadily spreads out in the transverse dimension. No serious criticism can be made of this conclusion as in the case of an integral resonance.

Ignoring the questions raised above and assuming that increasing the beam current I decreases ν_x and ν_z until one of them is on

a resonance causing particles to be lost, the value of I corresponding to this situation is taken to be the space charge limited current. Let

$$\delta \nu_x = \nu_{xo} - \nu_x$$

$$\delta \nu_z = \nu_{zo} - \nu_z$$

be the change in tunes for current I. Using equation (19) for ν_x, put

$$\delta(\nu_x^2) = \nu_{xo}^2 - \nu_x^2 = \frac{2\,q\,R^2\,I}{m\,c^3\,a^2\,\beta^3\,\gamma^3} .$$

Assuming $\delta\nu_x \ll \nu_{xo}$, $\delta(\nu_x^2)$ is approximately

$$\delta(\nu_x^2) = 2\,\nu_{xo}\,\delta\nu_x = \frac{2\,q\,R^2\,I}{m\,c^3\,a^2\,\beta^3\,\gamma^3} .$$

Solving this for I,

$$I = \frac{m\,c^3\,a^2\,\beta^3\,\gamma^3\,\nu_{xo}\,\delta\nu_x}{q\,R^2} . \tag{21}$$

Similarly, from equation (20) for ν_z,

$$I = \frac{m\,c^3\,a^2\,\beta^3\,\gamma^3\,\nu_{zo}\,\delta\nu_z}{q\,R^2} . \tag{22}$$

The maximum allowable values for $\delta\nu_x$ and $\delta\nu_z$ can be determined from a resonance line diagram and the initial operating point depending on which resonance lines are important. I can then be calculated for $\delta\nu_x$ and $\delta\nu_z$, and the lower value is the space charge limit.

To illustrate the use of the above formulas, consider the 50 Mev

electron FFAG machine at the Midwestern Universities Research Association. At injection at 100 Kev, $\beta = 0.548$, $\gamma = 1.196$, and $R = 128$ cm. The beam is roughly elliptical in cross section with a radial half-width of about 2.5 cm. and a vertical half-width of about 1 cm. As an approximation, take a to be the average of these: $a = 1.75$ cm. Then (21) or (22) gives

$$I = 9.01 \times 10^2 \, \nu_o \, \delta\nu \quad \text{(milliamperes)},$$

where ν_o is either ν_{xo} or ν_{zo} and $\delta\nu$ is either $\delta\nu_x$ or $\delta\nu_z$. The initial operating point is approximately $\nu_{xo} = 4.42$ and $\nu_{zo} = 2.75$. Two resonance lines which are near the operating point are the lines $2\nu_x - \nu_z = 6$ and $2\nu_z - \nu_x = 1$; however, it is not certain how serious these are. On the basis of these resonance lines an estimate for the allowable change in tune is $\delta\nu_x = \delta\nu_z = 0.05$. Since $\nu_{zo}\,\delta\nu_z < \nu_{xo}\,\delta\nu_x$, I is determined by the vertical tune and the result is

$$I = 125 \text{ ma.}$$

The measured space charge limited current is approximately 100 ma. The agreement between the two values is much better than that usually obtained. If the resonances used above are not really serious and other resonances were used for the calculation, the estimated current would be larger by a factor of two or three. In more careful measurements

on the MURA spiral sector electron accelerator the estimated space charge limited current is generally greater than the measured value by a factor of the order of three.[15]

The above elementary treatment exemplifies the usual discussion of transverse space charge effects.[2,5] Refinements can be made such as considering different shapes for the beam cross section, effects of ionization, and longitudinal bunching of the beam; but these are not very important with regard to the following investigation.[13,14]

III. FUNDAMENTAL EQUATIONS AND INITIAL ASSUMPTIONS

A. Preliminary Considerations

The elementary treatment of transverse space charge effects in the preceding chapter raised questions concerning its validity in some situations; the basic difficulty appeared to be that no allowance was made for the change of the beam with time. Conceivably the single-particle approach could be modified accordingly but that would seem to be pushing the method rather hard. Furthermore, since such a modification would require a more detailed description of the beam's behavior, one might as well drop the single-particle approach and start from the beginning with a more fundamental approach based on describing the behavior of the beam; i.e., a many-particle approach. The general Boltzmann equation for the beam is considered first. By using some simplifying assumptions this equation is reduced to a one-dimensional, collisionless Boltzmann equation for the transverse space charge effects. Additional assumptions are made to obtain relatively simple equations for the external force and the space charge force. In Chapter IV the reduced Boltzmann equation together with the force equations are transformed into an infinite set of moment equations. Then, by assuming the forces are linear functions of the transverse position and utilizing a suitable approximation for the space charge force, a finite closed set of moment equations is obtained which describes the behav-

ior of the beam.

Let the beam be described by a distribution function $G(\vec{r}, \vec{p}, t)$ such that $G(\vec{r}, \vec{p}, t)\, d^3\vec{r}\, d^3\vec{p}$ is the number of particles in volume element $d^3\vec{r}$ located at $\vec{r}$ in position space and in volume element $d^3\vec{p}$ located at $\vec{p}$ in momentum space at time t. In other words, G is the density of particles in a single-particle phase space, the $\vec{r}\,\vec{p}$ - space. G must satisfy the conditions:

$$G(\vec{r}, \vec{p}, t) \geq 0,$$

and

$$\int d^3\vec{r}\, d^3\vec{p}\, G(\vec{r}, \vec{p}, t) = N(t),$$

where $N(t)$ is the total number of particles at time t in the system. $N(t)$ is finite except in some idealized situations. Furthermore $G \to 0$ as $|\vec{p}| \to \infty$ and also, when N is finite, $G \to 0$ as $|\vec{r}| \to \infty$. G can be normalized to unity by dividing it by N so that

$$\int d^3\vec{r}\, d^3\vec{p}\, \frac{G}{N} = 1.$$

The function G/N can then be considered a probability distribution function.

With suitable assumptions, some of which are mentioned below, G satisfies the partial differential equation,

$$\frac{\partial G}{\partial t} + \vec{v} \cdot \vec{\nabla} G + \vec{\nabla}_p \cdot (\vec{F} G) = \left(\frac{\partial G}{\partial t}\right)_{coll.}, \qquad (23)$$

known as the Boltzmann equation.[6, 20, 21] Here, $\vec{v}$ is the velocity corresponding to momentum $\vec{p}$, $\vec{\nabla}$ is the gradient operator in $\vec{r}$-space, and $\vec{\nabla}_p$ is the gradient operator in $\vec{p}$-space. It is assumed that the forces acting on a particle can be separated into two parts; one is a macroscopic force representing long-range effects and the other is a force representing short-range effects such as close collisions between particles. $\vec{F}$ in equation (23) is the macroscopic force acting on a particle at the point $\vec{r}\,\vec{p}$ in the phase space at time t. In general, $\vec{F} = \vec{F}(\vec{r}, \vec{p}, t)$. The short-range effects are in the term $\left(\frac{\partial G}{\partial t}\right)_{coll}$ which is defined such that $\left(\frac{\partial G}{\partial t}\right)_{coll} d^3\vec{r}\, d^3\vec{p}\, dt$ is the net increase in the number of particles in $d^3\vec{r}\, d^3\vec{p}$ in time dt due to collisions. More generally this term also includes effects such as the removal of particles from and the addition of particles to the system by absorption processes and injection or creation processes.

In general, $\vec{F}$ and $\left(\frac{\partial G}{\partial t}\right)_{coll}$ are functionally dependent on G and appropriate expressions for them must be obtained. For the present purposes, assume there are no absorption and injection processes and that the beam is in a perfect vacuum. The latter assumption means that collisions with the residual gas are ignored. In addition, assume that collisions between particles of the beam may be ignored. Then the right-hand side of equation (23) is identically zero.

The only forces now acting on the particles are macroscopic forces produced by the external electromagnetic field of the machine

and the internal electromagnetic field created by the particles. One assumes that $\vec{F}$ can be written

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B} \tag{24}$$

where $\vec{E}$ and $\vec{B}$ are the macroscopic electric and magnetic fields, respectively, and are functions of $\vec{r}$ and t, in general. q is the charge of the particles. The term $\vec{\nabla}_p \cdot (\vec{F}\, G)$ in the Boltzmann equation can be written

$$\vec{F} \cdot \vec{\nabla}_p G + G\, \vec{\nabla}_p \cdot \vec{F}$$

and with the assumed form (24) for $\vec{F}$ it is easily shown that $\vec{\nabla}_p \cdot \vec{F} = 0$. So now the Boltzmann equation is

$$\frac{\partial G}{\partial t} + \vec{v} \cdot \vec{\nabla} G + \vec{F} \cdot \vec{\nabla}_p G = 0. \tag{25}$$

Put

$$\vec{F} = \vec{F}_e + \vec{F}_i \tag{26}$$

where $\vec{F}_e$ is the external force and $\vec{F}_i$ is the internal force (i.e., the space charge force). It is assumed that the external electric field is zero and that the external magnetic field is known and constant in time. Then $\vec{F}_e$ is known and does not depend explicitly on t. Hereafter, there will be little need to use symbols to designate the external electric and magnetic fields and unless otherwise stated, $\vec{E}$ and $\vec{B}$ are used to designate the internal electric and magnetic fields created by the beam.

Then

$$\vec{F}_i = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}. \tag{27}$$

The fields $\vec{E}$ and $\vec{B}$ are taken to satisfy Maxwell's equations (in Gaussian units with electric current measured in statcoulombs/sec.):

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \quad , \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} , \\ \vec{\nabla} \cdot \vec{B} &= 0 \quad , \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} , \end{aligned} \tag{28}$$

where ρ is the charge density and $\vec{j}$ is the electric current density in the beam. (c is the speed of light.) ρ and $\vec{j}$ are defined as follows. The integral of G over all $\vec{p}$-space is the density of particles in $\vec{r}$-space; therefore ρ is taken to be

$$\rho = q \int d^3\vec{p}\, G. \tag{29}$$

The integral of $\vec{v}\, G$ over all $\vec{p}$-space is the particle current density; so $\vec{j}$ is taken to be

$$\vec{j} = q \int d^3\vec{p}\, \vec{v}\, G. \tag{30}$$

The set of equations (25) to (30) with $\vec{F}_e$ being known form a closed system which determines the behavior of G. That the forces can be separated as described and the above system of equations defined depend on some tacit assumptions which are not given here but are briefly discussed by Drummond.[22] These equations are far too

complicated to solve in general and additional assumptions are needed to get a simpler system of equations.

B. Simplification of the Boltzmann Equation

The system of equations (25) to (30) is broad enough to include transverse effects, longitudinal effects, and the coupling between both effects. Since this investigation is concerned with transverse effects, simplification is obtained by obtaining a Boltzmann equation for the transverse behavior which is decoupled, in a certain sense, from the longitudinal motion. The decoupling will not be complete because some aspects of the transverse motion depend on the longitudinal motion.

Let the coordinate system be the x, s, z coordinate system defined in Chapter II relative to one particular equilibrium orbit. Let $\hat{e}_x$, $\hat{e}_s$, and $\hat{e}_z$ be the unit vectors corresponding to x, s, and z, and put

$$\vec{r}_\perp = x\,\hat{e}_x + z\,\hat{e}_z$$

for the position vector in a transverse plane. One also has

$$\vec{v} = v_x\,\hat{e}_x + v_s\,\hat{e}_s + v_z\,\hat{e}_z$$

with $v_x = dx/dt$, $v_s = ds/dt$, and $v_z = dz/dt$, and

$$\vec{p} = p_x\,\hat{e}_x + p_s\,\hat{e}_s + p_z\,\hat{e}_z = m\gamma\vec{v}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = \frac{v}{c}$. Put

$$\vec{v}_\perp = v_x\,\hat{e}_x + v_z\,\hat{e}_z \text{ and } \vec{p}_\perp = p_x\,\hat{e}_x + p_z\,\hat{e}_z\,.$$

In addition, write

$$\vec{F}_e = F_{ex}\,\hat{e}_x + F_{es}\,\hat{e}_s + F_{ez}\,\hat{e}_z = \vec{F}_{e\perp} + F_{es}\,\hat{e}_s$$

and similarly for $\vec{F}_i$ and $\vec{F}$.

Let the distribution function be $G = G(\vec{r}_\perp, \vec{p}_\perp, s, p_s, t)$. This is periodic in s with period L (L = the length of the equilibrium orbit used to define the coordinate system) since whenever s is increased by the amount L, the same point in the beam is again obtained. The Boltzmann equation (equation (25)) can now be written

$$\frac{\partial G}{\partial t} + \vec{v}_\perp \cdot \vec{\nabla}_\perp G + v_s \frac{\partial G}{\partial s} + \vec{F}_\perp \cdot \vec{\nabla}_{p\perp} G + F_s \frac{\partial G}{\partial p_s} = 0 \qquad (31)$$

where

$$\vec{\nabla}_\perp \equiv \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\vec{\nabla}_{p\perp} \equiv \hat{e}_x \frac{\partial}{\partial p_x} + \hat{e}_z \frac{\partial}{\partial p_z} .$$

Define

$$G_1 = G_1(\vec{r}_\perp, \vec{p}_\perp, s, t) = \int_{-\infty}^{\infty} dp_s\, G \qquad (32)$$

and integrate equation (31) over all p_s:

$$\frac{\partial G_1}{\partial t} + \int dp_s\, \vec{v}_\perp \cdot \vec{\nabla}_\perp G + \int dp_s\, v_s \frac{\partial G}{\partial s} + \int dp_s\, \vec{F}_\perp \cdot \vec{\nabla}_{p\perp} G$$

$$+ \int dp_s\, F_s \frac{\partial G}{\partial p_s} = 0. \qquad (33)$$

The particles in an accelerator are usually distributed over a narrow range of p_s. For simplicity assume

$$G = G_1 \, \delta (p_s - p_{so})$$

where p_{so} is a definite momentum and δ is the Dirac δ-function. Then the second term in equation (33) may be written

$$\int dp_s \frac{\vec{p}_\perp}{m\gamma} \cdot \vec{\nabla}_\perp G_1 \, \delta (p_s - p_{so}) = \frac{\vec{p}_\perp}{m\gamma_o} \cdot \vec{\nabla}_\perp G_1$$

where γ_o is γ evaluated for p_{so}. γ_o is still dependent on p_x and p_z; however, it is also usually true that $|p_x|$ and $|p_z|$ are much less than $|p_s|$, so assume that the dependence of γ on p_x and p_z may be ignored. Then γ_o is a constant.

The third term in equation (33) becomes

$$\frac{\partial G_1}{\partial s} \int dp_s \frac{p_s}{m\gamma} \, \delta (p_s - p_{so}) = \frac{p_{so}}{m\gamma_o} \frac{\partial G_1}{\partial s} = v_{so} \frac{\partial G_1}{\partial s}$$

where v_{so} is the value of v_s corresponding to p_{so}. The fourth term in equation (33)* is

$$\int dp_s \vec{F}_\perp \, \delta (p_s - p_{so}) \cdot \vec{\nabla}_{p\perp} G_1 = \vec{F}_{\perp o} \cdot \vec{\nabla}_{p\perp} G_1$$

where $\vec{F}_{\perp o}$ is $\vec{F}_\perp$ evaluated for $p_s = p_{so}$. The fifth term in equation (33) is

$$G_1 \int dp_s F_s \frac{\partial}{\partial p_s} \, \delta (p_s - p_{so}) = - G_1 \left(\frac{\partial F_s}{\partial p_s} \right)_{p_s = p_{so}} .$$

The magnetic part of F_s depends on p_s through γ. For example, a typical term of F_s would be

$$F'_s = -\frac{q}{c} v_x B_z = -\frac{q\, p_x B_z}{m\, c} \frac{1}{\gamma}$$

and

$$\frac{\partial F'_s}{\partial p_s} = \frac{q\, p_x B_z}{m\, c} \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial p_s} = -\frac{\partial \gamma}{\partial p_s} \frac{F'_s}{\gamma}$$

Thus, if the longitudinal magnetic force is zero or very small, $\partial F_s / \partial p_s$ may be ignored. The assumption that all particles have $p_s = p_{so}$ carries with it the assumption that $F_s = 0$ for otherwise p_s would vary. Assume therefore that the longitudinal electric and magnetic forces are small enough to ignore their effects. Then

$$\left(\frac{\partial F_s}{\partial p_s}\right)_{p_s = p_{so}} = 0$$

and, putting the above results together, equation (33) becomes

$$\frac{\partial G_1}{\partial t} + \frac{1}{m \gamma_o} \vec{p}_\perp \cdot \vec{\nabla}_\perp G_1 + v_{so} \frac{\partial G_1}{\partial s} + \vec{F}_{\perp o} \cdot \vec{\nabla}_{p\perp} G_1 = 0 . \qquad (34)$$

It is possible to develop an equation a little more general than equation (34) by using definition (32) for G_1 and defining average quantities, $\langle \vec{v}_\perp \rangle$ and $\langle \vec{F}_\perp \rangle$, such that

$$G_1 \langle \vec{v}_\perp \rangle = \int dp_s\, \vec{v}_\perp\, G$$

$$G_1 \langle \vec{F}_\perp \rangle = \int dp_s\, \vec{F}_\perp\, G .$$

Assuming the dependence of F_s on p_s is negligible, equation (33) can then be written

$$\frac{\partial G_1}{\partial t} + \vec{\nabla}_\perp \cdot (\langle \vec{v}_\perp \rangle G_1) + \vec{\nabla}_{p\perp} \cdot (\langle \vec{F}_\perp \rangle G_1) = - \frac{\partial}{\partial s} \int dp_s \, v_s \, G. \quad (35)$$

This is quite a bit more difficult to use than equation (34); one would need some way of estimating the average quantities and the term on the right-hand side. If this could be accomplished, it would give a way of estimating the influence of longitudinal effects on the transverse motion. Suppose the particles are distributed with some spread in p_s. Since F_s is negligible, there is just streaming motion in the longitudinal direction. After a sufficiently long time in which the particles have made many revolutions around the machine, the longitudinal distribution becomes smeared out so that it is uniform. Then G is independent of s so that $\partial G / \partial s = 0$ and the right-hand side of equation (35) vanishes. In addition, if G may be written as a product of G_1 and a function of p_s, equation (35) can be put in a form quite similar to equation (34) but without the term $v_{so} \, \partial G_1 / \partial s$. For simplicity the analysis hereafter is based on equation (34) and the assumptions which go with it.

The x, s, z coordinate system was defined relative to an equilibrium orbit. Let this orbit be the one corresponding to momentum $p = p_o$ with $p_o = p_{so}$. Although this stipulation is not required at the moment, it will be convenient later for calculating the external force.

The particles are all moving along the longitudinal axis with velocity v_{so}. Let the point s move along with the particles; i.e., let $s = v_{so}\, t$. Let G_2 be the value of G_1 as the point s moves along. Then

$$G_2 = G_2(\vec{r}_\perp, \vec{p}_\perp, t) = G_1(\vec{r}_\perp, \vec{p}_\perp, s, t)\Big|_{s = v_{so} t}$$

$$= G_1(\vec{r}_\perp, \vec{p}_\perp, v_{so} t, t)$$

and

$$\frac{\partial G_2}{\partial t} = \frac{\partial G_1}{\partial s}\Big|_{s = v_{so} t} \frac{\partial s}{\partial t} + \frac{\partial G_1}{\partial t}\Big|_{s = v_{so} t}$$

$$= v_{so} \frac{\partial G_1}{\partial s}\Big|_{s = v_{so} t} + \frac{\partial G_1}{\partial t}\Big|_{s = v_{so} t},$$

or

$$\frac{\partial G_1}{\partial t}\Big|_{s = v_{so} t} = \frac{\partial G_2}{\partial t} - v_{so} \frac{\partial G_1}{\partial s}\Big|_{s = v_{so} t}. \tag{36}$$

Evaluating equation (34) at $s = v_{so}\, t$ and inserting equation (36), equation (34) becomes

$$\frac{\partial G_2}{\partial t} + \frac{1}{m \gamma_o} \vec{p}_\perp \cdot \vec{\nabla}_\perp G_2 + \vec{F}_{\perp o} \cdot \vec{\nabla}_{p\perp} G_2 = 0, \tag{37}$$

where $\vec{F}_{\perp o}$ is also evaluated at $s = v_{so}\, t$. This is a Boltzmann equation applied to a transverse slice of longitudinal width ds which is

moving longitudinally with the longitudinal velocity of the particles. Then G_2 ds is the density of particles in the $\vec{r}_\perp \vec{p}_\perp$ - phase space for the particles in the slice. No particles enter or leave the slice so the total number of particles in the slice is constant.

The assumption that all the particles have the same longitudinal velocity so that no particles enter or leave the slice will be referred to as the "slice" approximation. Ignoring the transverse space charge force, there are two factors which influence the validity of the "slice" approximation. One is that the magnitude of the momentum of a particle is constant and therefore, if the particle is performing a betatron oscillation of some amplitude, its longitudinal momentum or velocity will vary. The second factor is that the particles cannot all be injected into an accelerator with the same energy or total momentum and consequently, even if there were no betatron oscillations, the particles have different longitudinal velocities.

To investigate the influence of betatron oscillations assume that the particles have the same total momentum and, for simplicity, that the equilibrium orbit is a circle of radius R. For particles starting on the equilibrium orbit at $\theta = 0$ and $t = 0$, the radial motion is approximately

$$r = R + A \sin \nu_x \theta$$

where r is the radial coordinate and A is the amplitude of the betatron

oscillation with $A \ll R$. Let S be the path length along the particle orbit. The longitudinal velocity is

$$v_s = \frac{ds}{dt} = \frac{d}{dt}(R\,\theta) = R\,\frac{d\theta}{dt} = R\,\frac{\frac{dS}{dt}}{\frac{dS}{d\theta}}$$

$$= \frac{R\,v}{\frac{dS}{d\theta}}$$

where $v = \frac{dS}{dt}$ = the total velocity which is the same for all particles. Here

$$\frac{dS}{d\theta} = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2}$$

$$= \left[R^2 + 2\,A\,R \sin \nu_x \theta + A^2 \sin^2 \nu_x \theta + \nu_x^2 A^2 \cos^2 \nu_x \theta\right]^{1/2} .$$

Expanding by the binomial theorem and keeping the lower order terms gives

$$\frac{1}{\frac{dS}{d\theta}} \cong \frac{1}{R}\left[1 - \frac{A}{R}\sin \nu_x \theta - \frac{1}{2}\frac{A^2}{R^2}\sin^2 \nu_x \theta - \frac{1}{2}\frac{\nu_x^2 A^2}{R^2}\cos^2 \nu_x \theta\right] .$$

Then

$$v_s \cong v\left[1 - \frac{A}{R}\sin \nu_x \theta - \frac{1}{2}\frac{A^2}{R^2}\sin^2 \nu_x \theta - \frac{1}{2}\frac{\nu_x^2 A^2}{R^2}\cos^2 \nu_x \theta\right] .$$

Averaging this over many betatron oscillations, the contribution from

the second term in the brackets is negligible and the long-time average longitudinal velocity $\langle v_s \rangle_1$ is

$$\langle v_s \rangle_1 \cong v \left[1 - \frac{1}{4} \frac{A^2}{R^2} (1 + \nu_x^2) \right] .$$

At time t the particle's position s is

$$s \cong \langle v_s \rangle_1 t .$$

Let Δs be the difference in s for a particle having $A \neq 0$ and a particle having $A = 0$. Comparing Δs with s evaluated for $A = 0$ gives

$$\frac{\Delta s}{s} \cong - \frac{1}{4} \frac{A^2}{R^2} (1 + \nu_x^2).$$

Typical values are $A/R = 5 \times 10^{-3}$ and $\nu_x = 5$ giving

$$\frac{|\Delta s|}{s} \approx 2 \times 10^{-4}.$$

As a criterion for the validity of the "slice" approximation take $|\Delta s| \ll \lambda = \frac{L}{\nu_x}$ = the wavelength of the betatron oscillation where L is the length of the equilibrium orbit.

Then

$$\frac{|\Delta s|}{s} \ll \frac{L}{\nu_x s}$$

or

$$\frac{s}{L} \ll \frac{1}{\nu_x} \frac{s}{|\Delta s|}$$

where s/L is the number of revolutions over which the "slice" approxi-

mation is valid. Using the typical values above gives

$$\frac{s}{L} \ll 10^3 .$$

On this basis the approximation is valid for approximately 100 revolutions.

The above estimate is based on a long-time average; however there is another effect included in the expression for v_s. Over a short time the second term in the bracket is more important than the third and fourth terms in producing variations in v_s since $A \ll R$. Averaging over the interval $0 \leq \theta \leq \pi$ and ignoring the third and fourth terms in the brackets, the short-time average longitudinal velocity $\langle v_s \rangle_s$ is

$$\langle v_s \rangle_s \cong v \left[1 - \frac{2}{\pi \nu_x} \frac{A}{R} \right] .$$

Let Δs be the difference in position of two particles after a half-cycle of oscillation where one has amplitude A and the other has amplitude - A. The two particles take approximately the same time t to make a half-cycle with $t \cong \frac{\lambda}{v}$. Then

$$\Delta s \cong v t \cdot \frac{4}{\pi \nu_x} \frac{A}{R} \cong \frac{4 \lambda}{\pi \nu_x} \frac{A}{R} .$$

As a criterion for the validity of the "slice" approximation take

$$\Delta s \ll \lambda ,$$

Substituting for Δs this becomes

$$\frac{4}{\pi \nu_x} \frac{A}{R} << 1.$$

Using the typical values cited above, the left-hand side of this is approximately 10^{-3} which satisfies the criterion. A similar analysis for the vertical motion gives the same result.

Now consider the effect arising from the particles not having the same total momentum. Assume the particles are on their equilibrium orbits with no betatron oscillations. Consider a particle with velocity v and momentum p. In time t it travels a distance s along the equilibrium orbit given by

$$s = v t = \frac{p t}{m \gamma} .$$

The corresponding value of θ is

$$\theta = \frac{2 \pi s}{L} = \frac{2 \pi p t}{m L \gamma} .$$

A second particle of momentum p + dp with an equilibrium orbit of length L + dL will have "angle" $\theta + d\theta$ in the same time t. Using the above relation for θ one obtains

$$\frac{d\theta}{\theta} = \frac{dp}{p} - \frac{dL}{L} - \frac{d\gamma}{\gamma} = (1 - \frac{1}{\alpha}) \frac{dp}{p} - \frac{d\gamma}{\gamma}$$

where

$$\alpha = \frac{dp}{p} \Big/ \frac{dL}{L} = \text{the momentum compaction factor.}$$

It can be shown that[2]

$$\alpha \cong \nu_x^2 .$$

Using this and the relations between p, γ, and the kinetic energy E_k, the above expression can be reduced to

$$\frac{d\theta}{\theta} \cong \left(\frac{1}{\gamma^2} - \frac{1}{\nu_x^2}\right) \frac{\gamma}{\gamma + 1} \frac{d\,E_k}{E_k} .$$

As the criterion for the validity of the "slice" approximation take

$$R\,|d\theta| = \frac{L}{2\pi}\,|d\theta| \ll \lambda = \frac{L}{\nu_x} .$$

With the above expression for $d\theta/\theta$ this criterion becomes

$$\frac{\theta}{2\pi} \ll \frac{1}{\nu_x} \frac{1}{\left|\frac{1}{\gamma^2} - \frac{1}{\nu_x^2}\right|} \frac{\gamma + 1}{\gamma} \left|\frac{E_k}{d\,E_k}\right|$$

where $\theta/2\pi$ is the number of revolutions over which the "slice" approximation is valid. A typical value for dE_k/E_k at injection is 5×10^{-3}. Suppose $\gamma \simeq 1$ and $\nu_x = 5$. Then the right-hand side of the inequality is approximately 80 indicating that the "slice" approximation is valid for about 10 revolutions.

The above numerical estimates for the situations considered indicate that the "slice" approximation is valid for approximately 10 revolutions and that the energy spread of the beam is the dominant factor. In other situations the approximation may be valid for longer

periods of time and the betatron oscillations may be the dominant factor. Actually, the criteria used above are much more stringent than required. The time rate of change of the number of particles in an element $d^2\vec{r}_\perp \, d^2\vec{p}_\perp$ of the moving slice is determined by

$$\int dp_s \, (v_s - v_{so}) \, \frac{\partial G}{\partial s}$$

where here v_{so} is the velocity of the slice. This rate of change of the number of particles not only depends on the distribution in p_s but also on $\partial G/\partial s$. Thus, if G does not vary much with s, the "slice" approximation will be valid for longer periods of time than estimated by the criteria used above. With these remarks for justification it is assumed that the "slice" approximation is valid and that therefore equation (37) may be utilized.

Further simplification is obtained by making idealizations so that the radial and vertical motion can be studied separately. The beam in an accelerator usually has a radial width greater than its vertical width. To study only the radial motion, assume for a model that the beam has an infinitesimal vertical width, that it lies in the median plane, and that the vertical forces are zero. Take

$$G_2 = G_3 (x, p_x, t) \, \delta(z) \, \delta(p_z) \, . \tag{38}$$

Insert this into equation (37) and integrate over all z and p_z. The result is

$$\frac{\partial G_3}{\partial t} + \frac{1}{m\,\gamma_o}\, p_x \frac{\partial G_3}{\partial x} + F_x \frac{\partial G_3}{\partial p_x} = 0, \tag{39}$$

where F_x is the x-component of $\vec{F}_{\perp o}$ evaluated for $z = p_z = 0$. Let

$$N_3 = \int dx\, dp_x\, G_3. \tag{40}$$

N_3 is the number of particles per unit of longitudinal beam length in the moving transverse slice; it is constant in time since no particles enter or leave the slice. Define a probability distribution function $f_x(x, p_x, t)$ by

$$G_3 = N_3 f_x$$

so that

$$\int dx\, dp_x\, f_x(x, p_x, t) = 1.$$

Substituting the above expression for G_3 into equation (39) and dividing by N_3 gives

$$\frac{\partial f_x}{\partial f} + \frac{1}{m\,\gamma_o}\, p_x \frac{\partial f_x}{\partial x} + F_x \frac{\partial f_x}{\partial p_x} = 0. \tag{41}$$

To investigate the vertical motion separately assume the beam has some vertical width which is small compared to its radial width. Let

$$G_4 = G_4(x, z, p_z, t) = \int_{-\infty}^{\infty} dp_x\, G_2. \tag{42}$$

Write equation (37) in terms of components where F_x and F_z are the x- and z-components of $\vec{F}_{\perp o}$:

$$\frac{\partial G_2}{\partial t} + \frac{1}{m\gamma_o} p_x \frac{\partial G_2}{\partial x} + \frac{1}{m\gamma_o} p_z \frac{\partial G_2}{\partial z} + F_x \frac{\partial G_2}{\partial p_x} + F_z \frac{\partial G_2}{\partial p_z} = 0. \quad (43)$$

Integrate this over all p_x. The first term gives $\partial G_4 / \partial t$. The second term gives

$$\frac{1}{m\gamma_o} \frac{\partial}{\partial x} \int dp_x \, p_x \, G_2.$$

Define $\langle p_x \rangle$ such that

$$G_4 \langle p_x \rangle = \int dp_x \, p_x \, G_2. \quad (44)$$

Then the second term can be written

$$\frac{1}{m\gamma_o} \frac{\partial}{\partial x} (G_4 \langle p_x \rangle).$$

The third term gives

$$\frac{1}{m\gamma_o} p_z \frac{\partial}{\partial z} \int dp_x \, G_2 = \frac{1}{m\gamma_o} p_z \frac{\partial G_4}{\partial z}.$$

Consider the integral of the fourth term. F_x has a dependence on p_x through γ, in general, but it was assumed that the dependence of on p_x is negligible. So F_x can be taken outside of the integral and one gets

$$F_x \int_{-\infty}^{\infty} dp_x \frac{\partial G_2}{\partial p_x} = F_x \Big[G_2 \Big]_{p_x = -\infty}^{p_x = \infty} = 0,$$

since $G_2 \rightarrow 0$ as $|p_x| \rightarrow \infty$. Similarly, for the last term, F_z can be taken outside the integral with the result:

$$F_z \frac{\partial}{\partial p_z} \int dp_x \, G_2 = F_z \frac{\partial G_4}{\partial p_z}.$$

Thus equation (43), after integration, becomes

$$\frac{\partial G_4}{\partial t} + \frac{1}{m \gamma_0} \frac{\partial}{\partial x} (G_4 \langle p_x \rangle) + \frac{1}{m \gamma_0} p_z \frac{\partial G_4}{\partial z} + F_z \frac{\partial G_4}{\partial p_z} = 0. \qquad (45)$$

Assume

$$\frac{\partial}{\partial x} (G_4 \langle p_x \rangle) = \frac{\partial G_4}{\partial x} \langle p_x \rangle + G_4 \frac{\partial \langle p_x \rangle}{\partial x} = 0. \qquad (46)$$

One way of accomplishing this is to assume G_4 is independent of x inside the beam. Then $\partial G_4 / \partial x = 0$. This and definition (42) imply $\partial G_2 / \partial x = 0$ since $G_2 \geq 0$. Hence, by definition (44):

$$\frac{\partial \langle p_x \rangle}{\partial x} = \frac{1}{G_4} \int dp_x \, p_x \frac{\partial G_2}{\partial x} = 0.$$

So relation (46) is satisfied and equation (45) becomes

$$\frac{\partial G_4}{\partial t} + \frac{1}{m \gamma_0} p_z \frac{\partial G_4}{\partial z} + F_z \frac{\partial G_4}{\partial p_z} = 0. \qquad (47)$$

Since G_4 is assumed to be independent of x, let the variable x be suppressed and write $G_4 = G_4 (z, p_z, t)$. Let

$$N_4 = \int dz \, dp_z \, G_4. \qquad (48)$$

N_4 is the number of particles per unit of longitudinal beam length and per unit of radial beam width; it is constant in time. Define a probability distribution function $f_z(z, p_z, t)$ by

$$G_4 = N_4 f_z$$

so that

$$\int dz\, dp_z\, f_z = 1.$$

Substituting the above expression for G_4 into equation (47) and dividing by N_4 gives

$$\frac{\partial f_z}{\partial t} + \frac{1}{m\gamma_o} p_z \frac{\partial f_z}{\partial z} + F_z \frac{\partial f_z}{\partial p_z} = 0. \tag{49}$$

Equations (41) and (49) are the basic Boltzmann equations for the remainder of the investigation.

C. The Space Charge Force

It is now necessary to obtain suitable expressions for the space charge forces to be able to get information out of the reduced Boltzmann equations developed in the last section. These forces are determined by the fields $\vec{E}$ and $\vec{B}$ satisfying Maxwell's equations (28). Assume the propagation time for the fields is negligible; i.e., that the fields may be considered to be quasistatic. Then equations (28) become

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad , \quad \vec{\nabla} \times \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}, \tag{50}$$

with

$$\rho = q \int d^3 \vec{p}\, G \tag{51}$$

$$\vec{j} = q \int d^3 \vec{p}\, \vec{v}\, G. \tag{52}$$

Assume also that the effects of charges and currents induced in the vacuum chamber walls are negligible; i.e., that there is no vacuum chamber and the beam is in free space.

Consider first the case for radial motion only. Let F_{ix} be the space charge part of F_x. From equation (27), the force equation is

$$F_{ix} = q\, E_x + \frac{q}{c} (v_s B_z - v_z B_s).$$

By the definition of F_x, it is evaluated for $v_s = v_{so}$, and it was assumed that the beam is flat in the median plane so that $v_z = 0$. Then

$$F_{ix} = q\, E_x + \frac{q}{c} v_o B_z, \tag{53}$$

where, for brevity, v_o replaces v_{so} since $p_{so} = p_o$. Also F_x is to be evaluated with the change of variables $s = v_o t$; hence, this same change is made in evaluating E_x and B_z.

E_x and B_z are to be evaluated on the median plane at the longitudinal position of the moving transverse slice. Assume that the general curvature of the equilibrium orbit in bending around the machine can be ignored and that the variation in curvature caused by sectors can be ignored. Then the coordinate system can be treated as a rectangular coordinate system and the beam as an infinitely long beam in this coordinate system. Assume the number of particles per unit of longitudinal beam length is constant. Then the variation of particle density is caused by variations in the transverse distribution with longitudinal position and time. Since a large part of the fields at the slice is generated by the particles in the longitudinal vicinity of the slice, assume that the variation of the transverse distribution with longitudinal position is small enough that it can be ignored for calculating the fields.

Then the charge density ρ in the slice is

$$\rho = q \int d^3\vec{p}\, G = q \int d^3\vec{p}\, G_3 (x, p_x, t)\, \delta(z)\, \delta(p_z)\, \delta(p_s - p_{so})$$

$$= q\, \delta(z) \int dp_x\, G_3 (x, p_x, t) = q\, N_3\, \delta(z) \int dp_x\, f_x (x, p_x, t).$$

Let

$$D_x (x, t) = \int dp_x\, f_x (x, p_x, t). \tag{54}$$

Then

$$\rho = q\, N_3\, D_x (x, t)\, \delta(z). \tag{55}$$

The current density $\vec{j}$ is

$$\vec{j} = q \int d^3\vec{p}\,\vec{v}\,G = \frac{q}{m}\int d^3\vec{p}\,\frac{\vec{p}}{\gamma}\,G$$

$$= \frac{q}{m}\int d^3\vec{p}\,\frac{1}{\gamma}\,(p_x\,\hat{e}_x + p_s\,\hat{e}_s + p_z\,\hat{e}_z)\,G_3(x, p_x, t)\,\delta(z)\,\delta(p_z)\,\delta(p_s - p_{so})$$

$$= \frac{q}{m\,\gamma_o}\,\delta(z)\left[\hat{e}_x \int dp_x\,p_x\,G_3 + N_3\,p_{so}\,D_x(x, t)\,\hat{e}_s\right]$$

$$= j_x\,\hat{e}_x + j_s\,\hat{e}_s\,.$$

Since $|p_x|$ is small compared to $|p_{so}|$ for the particles, the contribution of j_x to B_z is assumed to be negligible; so take as an approximation

$$\vec{j} = j_s\,\hat{e}_s = q\;N_3\,v_o\,D_x(x, t)\;\delta(z)\,\hat{e}_s\,. \tag{56}$$

The beam may now be considered as a distribution in x of infinitely long line charges and line currents. The line charge density per unit length at x' in dx' is

$$\tau(x') = q\;N_3\,D_x(x', t)\,dx' \tag{57}$$

and the line current at x' in dx' is

$$i(x') = q\;N_3\,v_o\,D_x(x', t)\,dx'\,. \tag{58}$$

The contribution to E_x at x in the median plane from $\tau(x')$ is

$$dE_x = \frac{2\,\tau(x')}{x - x'} = 2\,q\;N_3\,\frac{D_x(x', t)}{x - x'}\,dx'\,.$$

Integrating over all x':

$$E_x = 2\,q\,N_3 \int_{-\infty}^{\infty} dx' \frac{D_x(x', t)}{x - x'} . \tag{59}$$

Let

$$h_x(x, t) = \int_{-\infty}^{\infty} dx' \frac{D_x(x', t)}{x - x'} . \tag{60}$$

Then

$$E_x = 2\,q\,N_3\,h_x(x, t). \tag{61}$$

The contribution to B_z at x in the median plane from i (x') is

$$dB_z = -\frac{2\,i(x')}{c\,(x - x')} = -\frac{2\,q\,N_3\,v_o}{c} \frac{D_x(x', t)}{x - x'}\,dx'$$

and

$$B_z = -\frac{2\,q\,N_3\,v_o}{c}\,h_x(x, t) . \tag{62}$$

Then, by equation (53),

$$F_{ix} = 2\,q^2\,N_3\,h_x - \frac{2\,q^2\,N_3\,v_o^2}{c^2}\,h_x$$

$$= 2\,q^2\,N_3\,(1 - \beta_o^2)\,h_x(x, t) \tag{63}$$

where $\beta_o = v_o/c$. Let I be the beam current:

$$I = q\,N_3\,v_o = q\,N_3\,c\,\beta_o . \tag{64}$$

F_{ix} can now be written

$$F_{ix} = C_x h_x (x, t) \tag{65}$$

where

$$C_x = 2 q^2 N_3 (1 - \beta_o^2) = \frac{2 q I (1 - \beta_o^2)}{c \beta_o} = \frac{2 q I}{c \beta_o \gamma_o^2} . \tag{66}$$

Now consider the case for vertical motion. Let F_{iz} be the space charge part of F_z. From equation (27), the force equation is

$$F_{iz} = q E_z + \frac{q}{c} (v_x B_s - v_s B_x).$$

This is to be evaluated for $v_s = v_o$. Since it was assumed that $|p_x|$ and $|p_z|$ are much smaller than $|p_{so}|$, B_s and v_x are small so that the contribution from $v_x B_s$ is negligible. Then

$$F_{iz} = q E_z - \frac{q}{c} v_o B_x . \tag{67}$$

Again let the same assumptions be made as for the case of radial motion to calculate the fields; i.e., the beam is treated as an infinitely long beam in a rectangular coordinate system, the number of particles per unit longitudinal length is constant, and longitudinal variations in the transverse distribution at a given time are ignored. In addition, assume the beam is much wider radially than vertically such that, at a point x well inside the beam, the beam can be treated as being infinitely wide radially. Recalling the definition of G_4 and that

it was assumed to be independent of x inside the beam, the charge density is

$$\rho = q \int d^3\vec{p}\, G = q \int dp_x\, dp_s\, dp_z\, G_2\, \delta(p_s - p_o)$$

$$= q \int dp_z\, G_4(z, p_z, t) = q\, N_4 \int dp_z\, f_z(z, p_z, t) .$$

Let

$$D_z(z, t) = \int dp_z\, f_z(z, p_z, t) . \tag{68}$$

Then

$$\rho = q\, N_4\, D_z(z, t) . \tag{69}$$

The current density is

$$\vec{j} = q \int d^3\vec{p}\, \vec{v}\, G = \frac{q}{m} \int d^3\vec{p}\, \frac{\vec{p}}{\gamma}\, G$$

$$= \frac{q}{m} \int d^3\vec{p}\, \frac{1}{\gamma} (p_x\, \hat{e}_x + p_s\, \hat{e}_s + p_z\, \hat{e}_z)\, G_2\, \delta(p_s - p_o)$$

$$= \frac{q}{m\gamma_o} \int dp_z\, G_4 \langle p_x \rangle\, \hat{e}_x + \frac{q}{m\gamma_o} p_o \int dp_z\, G_4\, \hat{e}_s + \frac{q}{m\gamma_o} \int dp_z\, p_z\, G_4\, \hat{e}_z$$

$$= j_x\, \hat{e}_x + j_s\, \hat{e}_s + j_z\, \hat{e}_z .$$

Since $|p_x|$ and $|p_z|$ are relatively small, the main contribution is j_s. As an approximation take

$$\vec{j} = j_s \hat{e}_s = q\, v_o \int dp_z\, G_4\, \hat{e}_s = q\ N_4\, v_o\, D_z\,(z,\ t)\, \hat{e}_s\ . \qquad (70)$$

Using

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_s}{\partial s} + \frac{\partial E_z}{\partial z} = 4\pi\rho$$

and that $\partial E_x / \partial x = \partial E_s / \partial s = 0$ by symmetry, one has

$$\frac{\partial E_z}{\partial z} = 4\pi\rho = 4\pi q\ N_4\, D_z\,(z,\ t).$$

Integrating gives

$$E_z = 4\pi q\ N_4 \int_{-\infty}^{z} dz'\, D_z\,(z',\ t) + \text{const.}$$

Assuming the vertical width of the beam is finite, symmetry requires E_z at $z = -\infty$ to be the negative of E_z at $z = +\infty$. Also using the fact that

$$\int_{-\infty}^{\infty} dz'\, D_z\,(z',\ t) = \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dp_z'\, f_z\,(z',\ p_z',\ t) = 1\ ,$$

one obtains

$$E_z = -2\pi q\ N_4 + 4\pi q\ N_4 \int_{-\infty}^{z} dz'\, D_z\,(z',\ t)\ .$$

This can be written

$$E_z = 2\pi q\ N_4 \int_{-\infty}^{\infty} dz'\, S\,(z - z')\, D_z\,(z',\ t) \qquad (71)$$

where $S(z - z')$ is a step function such that

$$S(z - z') = 1, \quad \text{for } z - z' > 0$$

$$= -1, \quad \text{for } z - z' < 0 .$$

Let

$$h_z(z, t) = \int_{-\infty}^{\infty} dz' \, S(z - z') \, D_z(z', t) . \tag{72}$$

Then

$$E_z = 2\pi q \, N_4 \, h_z(z, t) . \tag{73}$$

From the equation

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

and symmetry, there results

$$\frac{\partial B_x}{\partial z} = \frac{4\pi}{c} j_s = \frac{4\pi}{c} q \, N_4 \, v_o \, D_z(z, t) .$$

Integrating and proceeding in the same way as for E_z, one obtains

$$B_x = 2\pi q \, N_4 \, \beta_o \, h_z(z, t) . \tag{74}$$

Substituting equations (73) and (74) into equation (67) gives

$$F_{iz} = 2\pi q^2 \, N_4 \, h_z - 2\pi q^2 \, N_4 \, \beta_o^2 \, h_z$$

$$= 2\pi q^2 \, N_4 \, (1 - \beta_o^2) \, h_z(z, t) . \tag{75}$$

Letting W be the radial width of the beam, the beam current is

$$I = q\ N_4\ v_o\ W = q\ W\ N_4\ c\ \beta_o\ . \tag{76}$$

F_{iz} can then be written

$$F_{iz} = C_z\ h_z\ (z,\ t) \tag{77}$$

where

$$C_z = 2\pi q^2\ N_4\ (1 - \beta_o^2) = \frac{2\pi q\ I\ (1 - \beta_o^2)}{W\ c\ \beta_o} = \frac{2\pi q\ I}{W\ c\ \beta_o\ \gamma_o^2}\ . \tag{78}$$

D. The External Force

Besides the space charge forces, suitable expressions for the external force produced by the accelerator's magnetic field must be employed. Let F_{ex} and F_{ez} be the x- and z-components of the external part of F_x and F_z, respectively, for the two cases being considered. It is assumed that F_{ex} is independent of the z-motion and that F_{ez} is independent of the x-motion. It is also assumed that F_{ex} is independent of p_x and that F_{ez} is independent of p_z. Then $F_{ex} = F_{ex}\ (x,\ \theta)$ and $F_{ez} = F_{ez}\ (z,\ \theta)$ where $\theta = 2\pi s/L$. In part of the discussion in the next chapter these forces will be considered as general functions of x or z and θ, but later it will be assumed that it is sufficient to use the linear approximations.

Consider F_{ex}. It was seen in the elementary treatment of Chapter II that a force F_o is related to its corresponding term in the

equations of motion after changing from the variable t to the variable θ by a factor $m \gamma_o v_o^2/R^2$; i.e.,

$$F_o = \frac{m \gamma_o v_o^2}{R^2} \times \text{corresponding term.}$$

R is the equivalent radius of the equilibrium orbit. It will be convenient to keep this factor separate because it will cancel out later. For the idealized magnetic field of the machine, the linear approximation is

$$F_{ex} = - \frac{m \gamma_o v_o^2}{R^2} K_x(\theta) x$$

where $K_x(\theta)$ was defined in the discussion of betatron oscillations in Chapter II. When field and gradient bumps are also present, the linear approximation becomes

$$F_{ex} = \frac{m \gamma_o v_o^2}{R^2} \left\{ b_x(\theta) + \left[- K_x(\theta) + g_x(\theta) \right] x \right\} \qquad (79)$$

where $b_x(\theta)$ and $g_x(\theta)$ are due to the field bump and the gradient bump, respectively, and are periodic in θ with period 2π. b_x and g_x are usually small. As was done in Chapter II for an approximation, it will be assumed when desired that the ripple in the betatron oscillations may be ignored and therefore that $K_x(\theta)$ may be replaced by an effective constant, namely, ν_x^2.

Similarly, the linear approximation for F_{ez} is

$$F_{ez} = \frac{m \gamma_o v_o^2}{R^2} \left\{ b_z(\theta) + \left[- K_z(\theta) + g_z(\theta) \right] z \right\} \qquad (80)$$

where b_z and g_z are due to the field bump and the gradient bump, respectively. $K_z(\theta)$ will be replaced by an effective constant ν_z^2 as an approximation when desired.

E. Summary of Assumptions

Various assumptions were made in the previous sections which are now summarized. In the preliminary considerations it was assumed that

1) the beam can be described by a distribution function which satisfies the Boltzmann equation;
2) absorption, injection, and collision processes may be ignored;
3) the fields and forces satisfy Maxwell's equations and the Lorentz force law with the charge and current densities suitably defined in terms of the distribution function;
4) the external electric field is zero and the external magnetic field is constant in time.

To obtain a simpler Boltzmann equation it was assumed that

1) the longitudinal forces are negligible and the longitudinal momentum of the particles is constant;
2) $|p_x| << |p_{so}|$, $|p_z| << |p_{so}|$, and the dependence of γ on p_x and p_z is negligible;
3) for studying radial motion, there are no vertical forces on the beam and the beam is flat in the median plane;
4) for studying vertical motion, the beam's vertical width is

much less than its radial width and the distribution is uniform in x.

For calculating the space charge forces the assumptions were that

1) the propagation time for the electromagnetic field is negligible and the effects of charge and current induced in the vacuum chamber walls are negligible;
2) the coordinate system may be approximated by a rectangular one;
3) the longitudinal variation of the number of particles per unit of longitudinal beam length is negligible;
4) variations of the transverse distribution with longitudinal position at a given time may be ignored;
5) for studying vertical motion, the radial width of the beam is much larger than the vertical width.

For calculating the external forces it was assumed that

1) the force in a given transverse direction is independent of the motion in the other transverse direction and of the momentum in its own direction;
2) the linear approximation may be used when desired;
3) $K_x(\theta)$ and $K_z(\theta)$ may be replaced by ν_x^2 and ν_z^2, respectively, when desired.

F. Summary of Equations

A set of basic equations was obtained in the preceding sections for radial and vertical motions separately. The radial-motion equations and the vertical-motion equations are structurally similar. For brevity the two motions will no longer be distinguished except when necessary and the same set of equations will be used where possible for either motion. Let x now represent either the radial coordinate or the vertical coordinate, as the case may be, and p represent the corresponding momentum. Let the x- and z-subscripts be dropped; it being understood which coordinate motion is under investigation.

A transverse distribution function G (x, p, t) (not to be confused with the original G) was obtained such that, for radial motion, G is the density of particles in the x p-space per unit of longitudinal beam length and, for vertical motion, it is the density of particles in x p-space per unit of longitudinal beam length and of radial width. A probability distribution function f (x, p, t) in x and p was defined such that

$$G = N f \tag{81}$$

and

$$\int dx\, dp\, f = 1 \tag{82}$$

where

N = number of particles/unit of longitudinal beam length, for the case of radial motion

= number of particles/unit of longitudinal beam length/unit of radial beam width, for the case of vertical motion.

Each of the reduced Boltzmann equations, (41) and (49), can now be written

$$\frac{\partial f}{\partial t} + \frac{p}{m \gamma_o} \frac{\partial f}{\partial x} + F \frac{\partial f}{\partial p} = 0 \tag{83}$$

where F is the transverse force in the direction represented by x and

$$F = F_e + F_i \tag{84}$$

with F_e and F_i being, respectively, the external and internal (or space charge) parts of F.

The internal force F_i is given by

$$F_i = C\, h(x, t) \tag{85}$$

where

$$h(x, t) = \int_{-\infty}^{\infty} dx'\, Q(x, x')\, D(x', t) \tag{86}$$

and D, Q, and C are defined as follows:

$$D(x, t) = \int dp\, f(x, p, t) \tag{87}$$

and, by relation (82), D has the property

$$\int dx\, D = 1 . \tag{88}$$

D (x, t) is the probability distribution in x. Q (x, x') is

$$Q(x, x') = \frac{1}{x - x'} , \quad \text{for the radial motion} \tag{89}$$

$$= S(x - x') , \text{ for the vertical motion} \tag{90}$$

where S is the step function:

$$S(x - x') = 1 , \quad \text{for } x - x' > 0$$

$$= -1, \quad \text{for } x - x' < 0 . \tag{91}$$

The constant C is

$$C = 2 q^2 N (1 - \beta_o^2) = \frac{2 q I}{c\, \beta_o \gamma_o^2} , \text{ for radial motion} \tag{92}$$

$$= 2\pi q^2 N (1 - \beta_o^2) = \frac{2\pi q I}{W c\, \beta_o \gamma_o^2} , \text{ for vertical motion} \tag{93}$$

where W is the radial beam width in the latter case and I is the beam current. q is the charge of the particles and

$$\beta_o = \frac{v_o}{c} , \quad \gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}} \tag{94}$$

where v_o is the longitudinal velocity of the particles.

The external force $F_e = F_e(x, \theta)$ is, in general, a function of x and θ. In the linear approximation:

$$F_e = \frac{m \gamma_o v_o^2}{R^2} \left\{ b(\theta) + \left[- K(\theta) + g(\theta) \right] x \right\} \tag{95}$$

where $K(\theta)$ is due to the ideal magnetic field of the machine, $b(\theta)$ is due to a field bump, and $g(\theta)$ is due to a gradient bump. m is the mass of the particles and $R = L/2\pi$ is the equivalent radius where L is the length of the equilibrium orbit. As an additional approximation, when desired, the ripple in the motion produced by $K(\theta)$ will be ignored and then $K(\theta)$ will be replaced by an effective constant ν_o^2.

It will be convenient later to compare the beam behavior with the betatron oscillations and, therefore, to use the variable θ instead of t. In all the functions above which depend on t, change from t to θ by the relations $s = v_o t = R\theta$. Let the same symbols be used to represent the new functions after the change of variable as before; e.g., $f(x, p, t)$ becomes $f(x, p, \theta)$, $h(x, t)$ becomes $h(x, \theta)$, etc. The Boltzmann equation (83) becomes

$$\frac{\partial f}{\partial \theta} + \frac{R}{p_o} p \frac{\partial f}{\partial x} + \frac{R}{v_o} F \frac{\partial f}{\partial p} = 0 \tag{96}$$

where $p_o = m \gamma_o v_o$.

This last equation and the equations giving F_i in terms of f form the basic set of equations to be solved for f or from which in-

formation concerning f is to be obtained if f cannot be obtained exactly. A technique for doing the latter is developed in the next chapter.

IV. THE MOMENT EQUATIONS

A. The Method of Moments

The problem has been reduced to one of solving the equation

$$\frac{\partial f}{\partial \theta} + \frac{R}{p_o} p \frac{\partial f}{\partial x} + \frac{R}{v_o} F \frac{\partial f}{\partial p} = 0 \tag{97}$$

for $f(x, p, \theta)$ with

$$F = F_e + F_i \tag{98}$$

where F_i is related to f by

$$F_i = C\, h(x, \theta), \tag{99}$$

$$h(x, \theta) = \int_{-\infty}^{\infty} dx'\, Q(x, x')\, D(x', \theta), \tag{100}$$

$$D(x, \theta) = \int_{-\infty}^{\infty} dp\, f(x, p, \theta), \tag{101}$$

and where F_e and Q are known functions as described at the end of Chapter III. A method is now presented which allows information concerning f to be obtained even though an exact solution for f cannot be obtained; it is based on what is known as the "method of moments." An infinite set of moment equations is obtained which is equivalent to the Boltzmann equation. These equations are in terms of "moments" which are defined later. A finite number of the equations do not, in general,

form a closed set which can be solved for the unknown "moments." A finite closed set of moment equations is obtained by means of two approximations. The first approximation is the assumption that the forces are linear. The second approximation is a "best" linear approximation for the space charge force which is shown to depend on only two of the "moments." The resulting closed set of moment equations forms the basis for the analysis of the beam behavior which is presented in the next chapter.

The "method of moments" is to multiply the Boltzmann equation (97) by $x^j p^k$ ($j, k = 0, 1, 2, \cdots$) and integrate over all x and p. If F can be expressed as a power series in x, the integrations can be performed and there results, for various values of j and k, a sequence of differential equations for the "moments" which are now defined. Let $f(x)$ be a function of x (for the present f is any function; it need not be the f in equation (97)). The quantity M_k, defined by

$$M_k = \int_{-\infty}^{\infty} dx\, x^k f(x), \quad k = 0, 1, 2, \cdots, \tag{102}$$

is called the "k^{th} moment" of f with respect to x. If $f(x)$ is such that

$$\int_{-\infty}^{x} dx'\, f(x')$$

is a never decreasing function of x and is bounded for all x and if $f(x)$ satisfies other suitable conditions, then all the moments M_k, if they

exist, completely determine f (x); i.e., if all the M_k exist and are known, then one can determine f (x) in principle.[23] It is assumed in what follows that the functions for which moments are used are determined by their moments. If f (x) is also a function of another variable, say θ, then M_k is a function of θ, in general.

Suppose f (x) is a probability distribution function such that $f(x) \geq 0$, $f(\pm\infty) = 0$, and

$$\int_{-\infty}^{\infty} dx\, f(x) = 1 .$$

Then $M_o = 1$, M_1 is the average or mean value of x, M_2 is the mean value of x^2, etc.

In general, the value of M_k depends on the location of the origin of the coordinate system. The central moments M_k^c are defined by

$$M_k^c = \int_{-\infty}^{\infty} dx\, (x - M_1)^k f(x). \tag{103}$$

If the origin of the coordinate system is such that $M_1 = 0$, then $M_k^c = M_k$. In particular, no matter where the origin is located,

$$M_1^c = 0 \quad \text{and} \quad M_2^c = M_2 - M_1^2 . \tag{104}$$

The above concept of moments can be extended to functions of two variables. Let f (x, y) be a function of x and y. The quantity

$$M_{jk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, x^j y^k f(x, y); \quad j, k = 0, 1, 2, \cdots, \tag{105}$$

is the jk^{th} moment of f with respect to x and y. If f also depends on a parameter θ, then M_{jk} is a function of θ in general. With suitable conditions, f is completely determined by a knowledge of all M_{jk} and this is assumed to be true in what follows.

In particular now, let f be the probability distribution function $f(x, p, \theta)$ in equation (97) and define its moments with respect to x and p by

$$M_{jk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dp\, x^j p^k f(x, p, \theta); \quad j, k = 0, 1, 2, \cdots. \tag{106}$$

Consider the first few moments.

$$M_{o,o} = \int dx\, dp\, f = 1 \tag{107}$$

by equation (82).

$$M_{1,0} = \int_{-\infty}^{\infty} dx\, x \int_{-\infty}^{\infty} dp\, f = \int_{-\infty}^{\infty} dx\, x\, D(x, \theta) = \text{mean value of } x. \tag{108}$$

$$= \bar{x}(\theta)$$

$\bar{x}(\theta)$ gives the mean position of the beam as a function of θ.

$$M_{2,0} = \int_{-\infty}^{\infty} dx\, x^2 \int_{-\infty}^{\infty} dp\, f = \int_{-\infty}^{\infty} dx\, x^2\, D$$

$$= \text{the mean value of } x^2 = \overline{x^2}\,(\theta) \tag{109}$$

$$M_{0,1} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, p\, f = \int_{-\infty}^{\infty} dp\, p \int_{-\infty}^{\infty} dx\, f\,.$$

Since $\int_{-\infty}^{\infty} dx\, f$ is the probability distribution in p,

$$M_{0,1} = \text{mean value of } p = \overline{p}\,(\theta)\,. \tag{110}$$

$\overline{p}\,(\theta)$ is the mean transverse momentum in the beam as a function of θ. Similarly,

$$M_{0,2} = \text{mean value of } p^2 = \overline{p^2}\,(\theta)\,, \tag{111}$$

and

$$M_{1,1} = \text{mean value of } xp = \overline{xp}\,(\theta)\,. \tag{112}$$

Let central moments with respect to x be defined by

$$M_k^c = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, (x - M_{1,0})^k\, f = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, (x - \overline{x})^k\, f$$

$$= \int_{-\infty}^{\infty} dx\, (x - \overline{x})^k\, D\,(x, \theta)\,, \tag{113}$$

having used equation (101). Central moments with respect to p will

not be needed. In particular, $M_1^c = 0$ and

$$M_2^c = M_{2,0} - M_{1,0}^2 = \overline{x^2} - \overline{x}^2 . \tag{114}$$

M_2^c gives a measure of the spread in x about $\overline{x}$ for the beam. $\sqrt{M_2^c}$ is the root-mean-square width of the beam in the x-dimension; for brevity hereafter, it will be referred to as the "width" w of the beam. Then

$$w = \sqrt{M_2^c} = \sqrt{\overline{x^2} - \overline{x}^2} . \tag{115}$$

Since $f(x, p, \theta) \geq 0$, it follows that

$$M_{jk} \geq 0 , \quad \text{for both } j, k = \text{even integers} , \tag{116}$$

and

$$M_k^c \geq 0 , \quad \text{for } k = \text{even integer} . \tag{117}$$

A relation which will be needed later is now derived. Consider

$$J(\alpha) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \, (x + \alpha p)^2 f$$

where α is real. Since $(x + \alpha p)^2 \geq 0$ and $f \geq 0$, $J(\alpha) \geq 0$. Expanding the integrand:

$$J = \int dx \int dp \, (x^2 + 2\alpha x p + \alpha^2 p^2) f$$

$$= M_{2,0} + 2\alpha M_{1,1} + \alpha^2 M_{0,2} \tag{118}$$

The value of α which minimizes J is given by

$$\frac{dJ}{d\alpha} = 2\, M_{1,1} + 2\, \alpha\, M_{0,2} = 0 ,$$

or

$$\alpha = - \frac{M_{1,1}}{M_{0,2}} .$$

Substituting this into equation (118), the minimum value of J is

$$J_{min} = M_{2,0} - \frac{M_{1,1}^2}{M_{0,2}} \geq 0 \quad \text{since } J(\alpha) \geq 0 .$$

Thus

$$M_{1,1}^2 \leq M_{2,0}\, M_{0,2} . \tag{119}$$

This is the desired relation. Using equation (114), this relation can be written

$$M_{1,1}^2 \leq M_{0,2}\, (M_2^c + M_{1,0}^2)$$

or

$$\overline{xp}^2 \leq \overline{x^2}\, \overline{p^2} = \overline{p^2}\, (w^2 + \overline{x}^2) . \tag{120}$$

As stated before, the method of moments is to multiply the Boltzmann equation (97) by $x^j\, p^k$ and integrate over all x and p. If F is known as a power series in x, the integrations can be carried out and one obtains, for various values of j and k, a sequence of differ-

ential equations in θ for the moments M_{jk}; i.e., a set of "moment" equations, which is equivalent to the Boltzmann equation. If the set of differential equations can be solved, then the M_{jk} are known and, hence, in principle, f can be determined. Even if the set cannot be solved, if information concerning the mean position $\bar{x}$ and the width w as functions of θ can be obtained, then this tells one quite a bit about the behavior of the beam.

In using the method of moments, integrals of the form

$$I_{jk} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^j p^k \frac{\partial f}{\partial x} \tag{121}$$

and

$$J_{jk} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^j p^k \frac{\partial f}{\partial p} \tag{122}$$

need to be evaluated. From the definition of the probability distribution function f, it has the property that $f \to 0$ as $|x| \to \infty$ or $|p| \to \infty$. Assume, in addition, that $f \to 0$ as $|x| \to \infty$ or $|p| \to \infty$ faster than any negative power of x or p; i.e., $x^k f \to 0$ as $|x| \to \infty$ and $p^k f \to 0$ as $|p| \to \infty$ for all $k \geq 0$. Also assume that the order of integration may be interchanged. Then, integrating by parts,

$$I_{jk} = \int_{-\infty}^{\infty} dp\, p^k \int_{-\infty}^{\infty} dx\, x^j \frac{\partial f}{\partial x}$$

$$= \int_{-\infty}^{\infty} dp\, p^k \left\{ \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} (x^j f) - j \int_{-\infty}^{\infty} dx\, x^{j-1} f \right\}$$

$$= \int_{-\infty}^{\infty} dp\, p^k \left\{ \left[x^j f \right]_{-\infty}^{\infty} - j \int_{-\infty}^{\infty} dx\, x^{j-1} f \right\}$$

$$= -j \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^{j-1} p^k f$$

$$I_{jk} = j\, M_{j-1,k} \,. \tag{123}$$

Similarly,

$$J_{jk} = \int_{-\infty}^{\infty} dx\, x^j \int_{-\infty}^{\infty} dp\, p^k \frac{\partial f}{\partial p}$$

$$= \int_{-\infty}^{\infty} dx\, x^j \left\{ \int_{-\infty}^{\infty} dp \frac{\partial}{\partial p} (p^k f) - k \int_{-\infty}^{\infty} dp\, p^{k-1} f \right\}$$

$$= \int_{-\infty}^{\infty} dx\, x^j \left\{ \left[p^k f \right]_{-\infty}^{\infty} - k \int_{-\infty}^{\infty} dp\, p^{k-1} f \right\}$$

$$= -k \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^j p^{k-1} f$$

$$J_{jk} = -k\, M_{j,\,k-1} \,. \tag{124}$$

B. General Moment Equations

The method of moments will now be used on the Boltzmann equation to obtain a set of moment equations. Put

$$K_1 = \frac{R}{m \gamma_o v_o} = \frac{R}{p_o} . \tag{125}$$

Using this and relation (98), the Boltzmann equation (97) becomes

$$\frac{\partial f}{\partial \theta} + K_1 p \frac{\partial f}{\partial x} + \frac{R}{v_o} (F_e + F_i) \frac{\partial f}{\partial p} = 0 . \tag{126}$$

F_e is a function of x and θ. Assume it can be expanded in a power series in x about the equilibrium orbit and write

$$\frac{R}{v_o} F_e = \sum_{n=o}^{\infty} A_n x^n \tag{127}$$

where the A_n are known functions of θ. By equation (99), F_i is a function of x and θ. Assume it can be expanded in a power series in x about some point x_o which may be a function of θ; i.e., $x_o = x_o(\theta)$. Write

$$\frac{R}{v_o} F_i = \sum_{n=o}^{\infty} B_n (x - x_o)^n = \sum_{n=o}^{\infty} \sum_{i=o}^{n} B_n C_i^n (-1)^{n-i} x_o^{n-i} x^i , \tag{128}$$

where the B_n are functions of θ and the C_i^n are the usual binomial expansion coefficients. Consider equation (99) more closely. It contains $h(x, \theta)$ which depends on $D(x, \theta)$. $D(x, \theta)$ is completely determined by its moments with respect to x. But, by equation (101) and the definition of M_{jk}, the moments of D are the $M_{j,o}$. Therefore,

the B_n are completely determined by the $M_{j,o}$; i.e., each B_n is, in general, a function of the $M_{j,o}$ with no explicit dependence on θ. The θ-dependence of the B_n arises from the dependence of the $M_{j,o}$ on θ. Substituting equations (127) and (128) in equation (126), multiplying by $x^j p^k$, and integrating over all x and p, gives

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \left\{ x^j p^k \frac{\partial f}{\partial \theta} + K_1 x^j p^{k+1} \frac{\partial f}{\partial x} + \sum_n A_n x^{j+n} p^k \frac{\partial f}{\partial p} \right.$$

$$\left. + \sum_{n,i} B_n C_i^n (-1)^{n-i} x_o^{n-i} x^{j+i} p^k \frac{\partial f}{\partial p} \right\} = 0 , \qquad (129)$$

where it is understood that the sum on n is for all $n \geq 0$, and that the sum on n and i is first on i from 0 to n and then on all $n \geq 0$. The first term on the left is

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^j p^k \frac{\partial f}{\partial \theta} = \frac{d}{d\theta} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^j p^k f = M'_{jk}$$

where the prime indicates differentiation with respect to θ. Using this and definitions (121) and (122), equation (129) becomes

$$M'_{jk} + K_1 I_{j,k+1} + \sum_n A_n J_{j+n,k}$$

$$+ \sum_{n,i} B_n C_i^n (-1)^{n-i} x_o^{n-i} J_{j+i,k} = 0 . \qquad (130)$$

Substituting relations (123) and (124), the above equation can be written

$$M'_{jk} - j\,K_1\,M_{j-1,k+1} - k\sum_n A_n\,M_{j+n,k-1}$$

$$- k\sum_{n,i} B_n\,C_i^n\,(-1)^{n-i}\,x_o^{n-i}\,M_{j+i,k-1} = 0\,. \tag{131}$$

For the various values of j and k, this gives the set of moment equations.

For $j = k = 0$, the above equation reduces to $M'_{o,o} = 0$ which is to be expected since $M_{o,o} = 1$. Therefore, this particular equation can be ignored hereafter.

Consider $j = 1$ and $k = 0$; equation (131) then gives

$$M'_{1,0} - K_1\,M_{0,1} = 0\,,$$

or

$$\bar{x}' = K_1\,\bar{p} = \frac{R}{m\,\gamma_o\,v_o}\,\bar{p}\,. \tag{132}$$

If one makes the transformation back to t by the relation $v_o\,t = R\theta$, this becomes

$$m\,\gamma_o\,\frac{d\bar{x}}{dt} = \bar{p}\,. \tag{133}$$

Thus, if $\bar{x}$ is considered as the motion of a particle, then $\bar{p}$ is its momentum, keeping in mind that this result is dependent on the approximations which have been made. A closer look shows that it depends on the assumption that the dependence of γ on p may be ignored; in general,

$\overline{p}$ is not the momentum of a particle with motion $\overline{x}$.

Now consider the case $j = 0$ and $k = 1$; equation (131) becomes

$$M'_{0,1} - \sum_n A_n M_{n,o} - \sum_{n,i} B_n C_i^n (-1)^{n-i} x_o^{n-i} M_{i,o} = 0, \quad (134)$$

or

$$\overline{p}' = \sum_n A_n \overline{x^n} + \sum_{n,i} B_n C_i^n (-1)^{n-i} x_o^{n-i} \overline{x^i}, \quad (135)$$

where

$$\overline{x^k} = M_{k,o} = \text{the mean value of } x^k. \quad (136)$$

Suppose $\overline{x}$ is the motion of a particle with momentum $\overline{p}$. Then equation (135) is the equation of motion for the particle and the structure of the equation shows that it depends on the detailed behavior of the beam. Suppose there were no space charge forces (i.e., all $B_n = 0$). Then equation (135), after transforming back to t, is

$$\frac{d\overline{p}}{dt} = \frac{v_o}{R} \sum_n A_n \overline{x^n}.$$

If it were true that $\overline{x^n} = \overline{x}^n$ for all n, then by equation (127) this could be written

$$\frac{d\overline{p}}{dt} = F_e(\overline{x}, \theta) \quad (137)$$

and one could say that the motion of $\overline{x}$ is that of a particle under the force F_e. But $\overline{x^n} = \overline{x}^n$ is true only for $n = 0, 1$. Thus, if F_e is linear

in x, equation (137) holds when the space charge force is ignored. Now let F_e be linear and include the space charge force. Again, since $\overline{x^n} \neq \bar{x}^n$ in general, one has

$$\frac{d\bar{p}}{dt} \neq F_e(\bar{x}, \theta) + \frac{v_o}{R} \sum_n B_n (\bar{x} - x_o)^n ;$$

but, if the space charge force is linear (i.e., $B_n = 0$ for $n > 1$), then

$$\frac{d\bar{p}}{dt} = F_e(\bar{x}, \theta) + \frac{v_o}{R} \left[B_o + B_1 (\bar{x} - x_o) \right] .$$

x_o is an arbitrary function of θ; let $x_o = \bar{x}$. If $B_o = 0$, then

$$\frac{d\bar{p}}{dt} = F_e(\bar{x}, \theta) .$$

Thus, if F_e and F_i are linear in x and F_i is zero at $\bar{x}$ ($B_o = 0$), then $\bar{x}$ behaves like the motion of a particle under the external force F_e independently of the space charge force F_i.

In general, the moment equations (131) are coupled; an equation for M'_{jk} is usually coupled to one for higher values of j and k. Some approximation is needed to obtain a closed finite set of equations so that they can be solved for the lower-order moments. The number of equations must be small if they are to be handled analytically. Assume F_e and F_i are linear in x ($A_n = B_n = 0$ for $n > 1$) and put $x_o = \bar{x} = M_{1,0}$. Equations (131) become

$$M'_{jk} - j K_1 M_{j-1,k+1} - k \Big[(A_o + B_o - B_1 M_{1,0}) M_{j,k-1} + (A_1 + B_1) M_{j+1,k-1} \Big] = 0 . \tag{138}$$

Consider the set of equations (138) for j and k satisfying $1 \leq j + k \leq 2$. Using the fact that $M_{o,o} = 1$, one obtains the following system of equations.

$$M'_{1,0} - K_1 M_{0,1} = 0 \tag{139a}$$

$$M'_{0,1} - A_o - B_o - A_1 M_{1,0} = 0 \tag{139b}$$

$$M'_{1,1} - K_1 M_{0,2} - (A_o + B_o - B_1 M_{1,0}) M_{1,0} - (A_1 + B_1) M_{2,0} = 0 \tag{139c}$$

$$M'_{2,0} - 2 K_1 M_{1,1} = 0 \tag{139d}$$

$$M'_{0,2} - 2 (A_o + B_o - B_1) M_{1,0} - 2 (A_1 + B_1) M_{1,1} = 0 . \tag{139e}$$

The A_n are known functions of θ and the B_n are functions of the $M_{k,o}$. If B_o and B_1 were functions only of $M_{1,0}$ and $M_{2,0}$, then the system (139) would be closed. In particular, $M_{1,0} = \overline{x}$ and $M_{2,0} = \overline{x^2}$ could be obtained, thus giving considerable information about the beam behavior. Also, if B_o were small enough to ignore, equations (139a) and (139b) would form a closed system which could be solved for $M_{1,0}$ and $M_{0,1}$, and these could then be used as known functions in the remaining equations of the set (139).

Intuitively one feels, if the distribution of particles in position x has a certain amount of symmetry about $x = \overline{x}$, that B_o would be small. Furthermore it would seem reasonable that B_1 would be determined essentially by the width of the beam; i.e., to a good approximation as some function of $w = \sqrt{M_{2,0} - M_{1,0}^2} = \sqrt{\overline{x^2} - \overline{x}^2}$. Therefore an investigation is made of a linear approximation for the space charge force F_i and the dependence of the corresponding B_o and B_1 on the moments $M_{k,o}$.

C. Linear Approximation for the Space Charge Force

A method is now developed for approximating the space charge force F_i by a polynomial of finite degree in x and then applied for a linear approximation. It will be found that the linear approximation depends almost entirely on $M_{1,0}$ and $M_{2,0}$ and thus, to a good approximation, the set (139) becomes a closed set. The pertinent relations needed are equations (99), (100), and (101). Let the variable θ be suppressed for now and write for these equations:

$$F_i = C\, h(x), \tag{140}$$

$$h(x) = \int_{-\infty}^{\infty} dx'\, Q(x, x')\, D(x'), \tag{141}$$

$$D(x) = \int_{-\infty}^{\infty} dp\, f(x, p), \tag{142}$$

where $Q(x, x')$ is given at the end of Chapter III. $D(x)$ is the probability distribution function in x for the particles. Instead of dealing directly with F_i, let $h(x)$ be the function which is approximated since C is just a constant.

Let $h_a(x)$ be the function which is to be used as an approximation in place of $h(x)$ and let it be a polynomial of degree n:

$$h_a(x) = \sum_{j=o}^{n} h_j x^j \tag{143}$$

where the h_j are the coefficients of the polynomial. The criterion for choosing the h_j to give the "best" approximation is taken to be that the h_j make the quantity

$$J = \int_{-\infty}^{\infty} dx \left[h(x) - h_a(x)\right]^2 D(x) \tag{144}$$

a minimum. Since $D(x) \geq 0$, $J \geq 0$. J is zero only if $h_a(x) = h(x)$. Hence, the smaller J is, the better is the approximation. Using $D(x)$ as a weight function makes $h_a(x)$ a better approximation in the regions where there are more particles than in the regions having fewer particles. This seems to be a reasonable procedure to use for studying the over-all behavior of the particles. A figure of merit η for the quality of the approximation is $\eta = (J/\overline{h^2})^{1/2}$ where $\overline{h^2}$ is the average value of $h^2(x)$ with respect to $D(x)$. η is a sort of fractional error of the approximation. Since h_a approximates h, let h be replaced by h_a

in the definition of η for the sake of simplicity; i.e., take $\eta = (J/\overline{h_a^2})^{1/2}$ where

$$\overline{h_a^2} = \int dx\, h_a^2 (x)\, D(x) .$$

J is a function of the h_j; the condition for a minimum is

$$\frac{\partial J}{\partial h_k} = 0 ; \quad k = 0, 1, 2, \cdots, n . \tag{145}$$

Substituting equation (144) into equation (145) gives

$$\int_{-\infty}^{\infty} dx \left[h(x) - h_a(x) \right] \frac{\partial h_a}{\partial h_k} D(x) = 0 . \tag{146}$$

From equation (143), $\partial h_a / \partial h_k = x^k$. Putting this and equation (143) into equation (146) results in

$$\int_{-\infty}^{\infty} dx \left[h(x) - \sum_j h_j x^j \right] x^k D(x) = 0$$

$$\int_{-\infty}^{\infty} dx\, x^k h(x) D(x) - \sum_j h_j \int_{-\infty}^{\infty} dx\, x^{j+k} D(x) = 0 . \tag{147}$$

But by the definition (142) of D (x) and the definition of the moments,

$$\int_{-\infty}^{\infty} dx\, x^{j+k} D(x) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, x^{j+k} f(x, p) = M_{j+k,o} .$$

Rearranging terms and using this last result, equation (147) can be written

$$\sum_j h_j M_{j+k,o} = \int_{-\infty}^{\infty} dx\, x^k h(x) D(x)$$

or

$$\sum_{j=o}^{n} h_j M_{j+k,o} = H_k; \quad k = 0, 1, 2, \cdots, n \tag{148}$$

where, using equation (141),

$$H_k = \int_{-\infty}^{\infty} dx\, x^k h(x) D(x) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, x^k Q(x, x') D(x) D(x'). \tag{149}$$

With suitable restrictions, an arbitrary D (x) can be written as a universal function of its moments in terms of Hermite functions.[24] It follows then that for a given Q (x, x'), H_k is a universal function of the moments for arbitrary D (x). If the H_k can be found, then the equations (148) give the h_j as universal functions of the moments for a given Q (x, x').

From Chapter III, equations (89) and (90),

$$Q(x, x') = \frac{1}{x - x'}, \quad \text{for radial motion} \tag{150}$$

$$= S(x - x'), \quad \text{for vertical motion} \tag{151}$$

where

$$S(x - x') = 1 , \qquad \text{for } x - x' > 0$$
$$= -1 , \qquad \text{for } x - x' < 0 . \tag{152}$$

For either type of motion $Q(x, x')$ has the property:

$$Q(x, x') = -Q(x', x) . \tag{153}$$

By equation (149),

$$H_o = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, Q(x, x') \, D(x) \, D(x') .$$

Interchanging x and x' :

$$H_o = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \, Q(x', x) \, D(x') \, D(x) .$$

Inserting relation (153):

$$H_o = -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \, Q(x, x') \, D(x') \, D(x)$$
$$= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, Q(x, x') \, D(x) \, D(x') = -H_o .$$

Thus $H_o = 0$.

Consider H_k for radial motion and $k > 0$. Equations (149) and (150) give

$$H_k = \int dx \int dx' \frac{x^k D(x) D(x')}{x - x'} .$$

Put

$$x^k = \left[(x - x') + x'\right]^k = \sum_{i=o}^{k} C_i^k (x - x')^{k-i} x'^i$$

where the C_i^k are the binomial expansion coefficients. Then

$$H_k = \sum_{i=o}^{k} C_i^k \int dx \int dx' \frac{(x - x')^{k-i} x'^i D(x) D(x')}{x - x'}$$

$$= \sum_{i=o}^{k-1} C_i^k \int dx \int dx' (x - x')^{k-i-1} x'^i D(x) D(x')$$

$$+ \int dx \int dx' \frac{x'^k D(x) D(x')}{x - x'} .$$

By interchanging x and x' it is seen that the last integral is $- H_k$.

Solving the above equation for H_k and putting

$$(x - x')^{k-i-1} = \sum_{m=o}^{k-i-1} C_m^{k-i-1} (-1)^{k-i-m-1} x^m x'^{k-i-m-1}$$

gives

$$H_k = \frac{1}{2} \sum_{i=o}^{k-1} C_i^k \sum_{m=o}^{k-i-1} C_m^{k-i-1} (-1)^{k-i-m-1} \int dx \int dx' x^m x'^{k-m-1} D(x) D(x')$$

$$= \frac{1}{2} \sum_{i=o}^{k-1} C_i^k \sum_{m=o}^{k-i-1} C_m^{k-i-1} (-1)^{k-i-m-1} M_{m,o} M_{k-m-1,o} . \qquad (154)$$

In this case it has been possible to work out a general expression for the H_k in terms of the moments. In particular, for radial motion,

$$H_o = 0, \quad H_1 = \frac{1}{2}, \quad \text{and} \quad H_2 = M_{1,0} . \tag{155}$$

Consider the linear approximation. $n = 1$ and equations (148) give

$$h_o + h_1 M_{1,0} = 0$$

$$h_o M_{1,0} + h_1 M_{2,0} = \frac{1}{2} .$$

If the origin of x is shifted so that $\bar{x} = M_{1,0} = 0$, then the moments in these equations are central moments M_k^c (cf. equation (113)) and the equations become

$$h_o = 0$$

$$h_1 M_2^c = \frac{1}{2} \quad \text{or} \quad h_1 = \frac{1}{2 M_2^c}$$

Shifting the origin of x back to its original position, one obtains

$$h_a (x) = \frac{1}{2 M_2^c} (x - \bar{x}) . \tag{156}$$

In general, in a power series for h_a in non-negative powers of $x - \bar{x}$, the leading term is not zero except in the linear approximation.

As an example, suppose D (x) is a parabola in the interval $-d < x < d$ and zero outside this interval. Let

$$D(x) = \frac{3}{4d}\left(1 - \frac{x^2}{d^2}\right) \quad \text{for} \quad -d < x < d$$

$$= 0 \qquad\qquad \text{for } |x| > d .$$

This has $M_{0,0} = 1$, $M_{1,0} = 0$, $M_{2,0} = M_2^c = \frac{1}{5} d^2$. Then

$$h_a(x) = \frac{5}{2d^2} x = \frac{5}{2d^2} (x - \bar{x}) \text{ since } \bar{x} = 0 .$$

Also,

$$h(x) = \frac{3}{4d}\left[\left(\frac{x^2}{d^2} - 1\right) \ln\left(\frac{1 - \frac{x}{d}}{1 + \frac{x}{d}}\right) + 2\frac{x}{d}\right] .$$

J cannot be determined exactly but an estimate can be made and it is found that $J \approx 9 \times 10^{-3}/d^2$ and $\eta \approx 8.5 \times 10^{-2}$.

Consider now the case of vertical motion. Equations (149) and (151) give

$$H_k = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, x^k \, S(x - x') \, D(x) \, D(x') . \tag{157}$$

It does not appear to be possible to develop a general expression for H_k in terms of the moments as in the previous case. Examples have been worked out to determine the approximate dependence of h_o and h_1

on the moments. Consider the linear approximation with $n = 1$. Equations (148) are

$$h_o + h_1 M_{1,0} = 0$$

$$h_o M_{1,0} + h_1 M_{2,0} = H_1 .$$

Shifting the origin so that $\bar{x} = M_{1,0} = 0$, these equations become

$$h_o = 0$$

$$h_1 M_2^c = H_1 \quad \text{or} \quad h_1 = \frac{H_1}{M_2^c} . \tag{158}$$

Shifting the origin back to its original position, one has

$$h_a(x) = h_1 (x - \bar{x}) = \frac{H_1}{M_2^c} (x - \bar{x}) . \tag{159}$$

Shifting the origin in the above manner is permissible because H_1 is invariant under a change of the origin.

Consider the linear approximation for some examples now. The details of the calculations are not given; only the results are quoted. In each example H_1 is calculated and then expressed in terms of M_2^c. This is an approximation since H_1 depends on all moments in general. Let

$$D(x) = \frac{1}{2d} \quad \text{for } -d < x < d$$

$$= 0 \quad \text{for } |x| > d .$$

This is a uniform distribution inside $-d < x < d$ and has

$$M_{0,0} = 1, \quad M_{1,0} = 0, \quad M_{2,0} = \frac{1}{3} d^2 .$$

Inside the interval, $h(x)$ is exactly linear:

$$h(x) = \frac{1}{d} x .$$

In this case the linear approximation is the same as $h(x)$. One finds

$$h_a(x) = \frac{1}{d} x = \frac{1}{\sqrt{3}\sqrt{M_2^c}} x .$$

$$= \frac{1}{\sqrt{3}\sqrt{M_2^c}} (x - \bar{x}) . \tag{160}$$

Also, $J = 0$.

For another example, let $D(x)$ be a parabola as used before:

$$D(x) = \frac{3}{4d}\left(1 - \frac{x^2}{d^2}\right) \quad \text{for } -d < x < d$$

$$= 0 \qquad \text{for } |x| > d .$$

It is found that

$$h_a(x) = \frac{9}{7\sqrt{5}\sqrt{M_2^c}} (x - \bar{x}) , \tag{161}$$

$J \cong 2.7 \times 10^{-3}$, and $\eta \cong 9 \times 10^{-2}$.

Finally, let $D(x)$ be a triangular distribution:

$$D(x) = \frac{2}{d^2}\, x \quad \text{for} \quad 0 < x < d$$

$$= 0 \quad \text{for} \quad x < 0 \quad \text{or} \quad x > d\,.$$

It is found that

$$h_a(x) = \frac{2\sqrt{2}}{5\sqrt{M_2^c}}\,(x - \bar{x})\,, \tag{162}$$

$J = 1.36 \times 10^{-2}$, and $\eta \cong 2.1 \times 10^{-1}$.

From equations (160), (161), and (162), the constants of proportionality are

$$\frac{1}{\sqrt{3}\sqrt{M_2^c}}\,, \qquad \frac{9}{7\sqrt{5}\sqrt{M_2^c}}\,, \qquad \frac{2\sqrt{2}}{5\sqrt{M_2^c}}$$

or

$$\frac{0.578}{\sqrt{M_2^c}}\,, \qquad \frac{0.576}{\sqrt{M_2^c}}\,, \qquad \frac{0.566}{\sqrt{M_2^c}}\,.$$

These differ from one another due to the fact that the constant of proportionality actually depends on higher moments as well as M_2^c. But the examples show that the dependence on higher moments is rather insensitive and the constant depends essentially on M_2^c. As an approximation for the vertical motion, take

$$h_a(x) = \frac{0.57}{\sqrt{M_2^c}}\,(x - \bar{x})\,. \tag{163}$$

Equations (156) and (163) will be used now for linear approximations for h (x). Replacing h (x) in equation (140) by its linear approximation, F_i can be written

$$F_i = \frac{C\,k}{w^n}\,(x - \bar{x}) \tag{164}$$

where

$$w = \sqrt{M_2^c} = \text{the width of the beam,}$$

and

$$k = 0.5, \quad n = 2 \qquad \text{for radial motion}$$

$$k = 0.57, \quad n = 1 \qquad \text{for vertical motion.}$$

Comparing equations (128) and (164) with $x_o = \bar{x}$,

$$B_o = 0 \quad \text{and} \quad B_1 = \frac{v_o\,C\,k}{R\,w^n} = \frac{K_2}{w^n} \tag{165}$$

where

$$K_2 = \frac{v_o\,C\,k}{R}\,. \tag{166}$$

Relations (165) will be put in the moment equations (139) for the linear approximation.

D. Approximate Moment Equations

A closed set of approximate moment equations is obtained from the set (139) by using the linear approximation for F_i developed in the

preceding section and the linear approximation (95) for F_e. The quantities B_o and B_1 have just been determined. Comparing equations (95) and (127),

$$A_o = K_3 \, b(\theta) \tag{167}$$

and

$$A_1 = K_3 \left[- K(\theta) + g(\theta) \right] \tag{168}$$

where

$$K_3 = \frac{p_o}{R} = \frac{1}{K_1} \quad \text{(cf. (125))} . \tag{169}$$

To eliminate using the double subscript notation for the moments, let the following notation be adopted.

$$\begin{aligned} &\overline{x} = M_{1,0} , \quad \overline{p} = M_{0,1} \\ &u = \overline{xp} = M_{1,1} , \quad v = \overline{p^2} = M_{0,2} \\ &\overline{x^2} = M_{2,0} = M_{1,0}^2 + M_2^c = \overline{x}^2 + w^2 . \end{aligned} \tag{170}$$

Using this notation and relations (165), (167), and (168), a set of approximate moment equations is obtained from the set (139) which can be written:

$$\bar{x}' - K_1 \bar{p} = 0 \tag{171a}$$

$$\bar{p}' + K_3 (K - g) \bar{x} - K_3 b = 0 \tag{171b}$$

$$u' - K_1 v + K_3 (K - g) (\bar{x}^2 + w^2) - K_2 w^{2-n} - K_3 b \bar{x} = 0 \tag{171c}$$

$$(w^2 + \bar{x}^2)' - 2 K_1 u = 0 \tag{171d}$$

$$v' + 2 K_3 (K - g) u - 2 K_2 \frac{u}{w^n} + 2 K_2 \frac{\bar{x}\bar{p}}{w^n} - 2 K_3 b \bar{p} = 0 \tag{171e}$$

where K, b, and g are functions of θ.

This is a closed system of equations in the variables $\bar{x}$, $\bar{p}$, $\overline{xp}$, $\overline{p^2}$, and w where the prime denotes $d/d\theta$. The first two equations are closed and give the motion $\bar{x}$ as the motion of a particle under the action of the external force alone as discussed earlier. These approximate moment equations form the basis for an analysis of the behavior of the beam presented in the next chapter.

V. BEHAVIOR OF THE BEAM

A. Stationary Solution

The approximate moment equations (171) obtained in the last chapter will be used now to determine the behavior of the beam. They are

$$\bar{x}' - K_1 \bar{p} = 0 \tag{171a}$$

$$\bar{p}' + K_3 (K - g) \bar{x} - K_3 b = 0 \tag{171b}$$

$$u' - K_1 v + K_3 (K - g) (\bar{x}^2 + w^2) - K_2 w^{2-n} - K_3 b \bar{x} = 0 \tag{171c}$$

$$(w^2 + \bar{x}^2)' - 2 K_1 u = 0 \tag{171d}$$

$$v' + 2 K_3 (K - g) u - 2 K_2 \frac{u}{w^n} + 2 K_2 \frac{\bar{x} \bar{p}}{w^n} - 2 K_3 b \bar{p} = 0 \tag{171e}$$

where K, b, and g are functions of θ and a prime indicates $d/d\theta$. K is the focusing term of the external magnetic field, b is the field bump, and g is the gradient bump, as defined in Chapter III. $\bar{x}$ is the mean position, $\bar{p}$ is the mean momentum, $u = \overline{xp}$, $v = \overline{p^2}$, and $w^2 = \overline{x^2} - \bar{x}^2$. The constants are

$$K_1 = \frac{R}{p_o}, \quad K_2 = \frac{v_o C k}{R}, \quad K_3 = \frac{1}{K_1},$$

where C is given by (92) or (93) in Chapter III. Also

$k = 0.5, \quad n = 2$ for radial motion

$k = 0.57, \quad n = 1$ for vertical motion.

A stationary solution of the Boltzmann equation is a solution for which $\partial f/\partial t$ is identically zero. In the present case this corresponds to having $\partial f/\partial \theta$ identically zero, or equivalently, to having all M'_{jk} vanish identically. Putting all derivatives equal to zero in equations (171) will then give the necessary conditions for a stationary solution. With $\bar{x}' = 0$, equation (171a) gives $\bar{p} = 0$. From equation (171b) with $\bar{p}' = 0$, one obtains

$$(K - g)\bar{x} = b$$

which cannot be satisfied by a constant $\bar{x}$, in general, when $b \not\equiv 0$. So assume $b \equiv 0$. Then $\bar{x} = 0$ satisfies the above condition. Putting the derivatives equal to zero in equation (171d) gives $u = 0$. $v' = 0$ in equation (171e) is consistent with $u = \bar{x} = \bar{p} = 0$. The conditions so far obtained and equation (171c) give, after multiplying by K_1 and using $K_1 K_3 = 1$,

$$K_1^2 v = (K - g) w^2 - K_1 K_2 w^{2-n} .$$

Since K and g are functions of θ, this relation cannot be satisfied by constant v and w. Even with $g \equiv 0$, a stationary solution is not possible.

Assuming $g \equiv 0$, an approximately stationary solution exists if one assumes that the ripple in the betatron oscillations may be ignored. In this case K is replaced by an effective constant ν_o^2 as described in Chapter III which for brevity will be denoted by K_o; i.e., $K_o = \nu_o^2$ where ν_o is ν_x or ν_z, as the case may be. Then, in this approximation, the above relation becomes

$$K_1^2 v = \left(K_o - \frac{K_s}{w^n}\right) w^2 \tag{172}$$

where

$$K_s = K_1 K_2 . \tag{173}$$

v and w must satisfy this relation to have a stationary solution. Reversing the arguments show that these conditions are also sufficient in the sense of the approximations. Hereafter, these conditions will be used for the stationary solution with the approximation being understood. Summarizing, the conditions for a stationary solution are b, g, $\bar{x}$, and u vanish identically, and

$$K_1^2 v = \left(K_o - \frac{K_s}{w^n}\right) w^2 \tag{174}$$

where $K_o = \nu_o^2$.

Since w^2 is the mean value of $(x - \bar{x})^2$, it follows that $w^2 \geq 0$ and $w = \sqrt{w^2} \geq 0$. Similarly, $v \geq 0$ since v is the mean value of p^2. The quantity $K_s = K_1 K_2$, since it contains C by definition (166), is a

measure of the space charge force or, more precisely, the space charge effect for a fixed w. It is directly proportional to the beam current or the number of particles. In the limit in which space charge effects are negligible, $K_s = 0$ and equation (174) gives a non-zero value for v. Since $\bar{p} = 0$, v gives a measure of the spread in the transverse momentum. As the beam current is increased while holding w fixed, K_s increases and (174) shows that v must be decreased to maintain the condition for a stationary solution. If K_s is made large enough, (174) would give $v < 0$. But $v < 0$ is not possible, hence (174) can no longer be satisfied. The limiting value of K_s for a stationary solution is the value which makes $v = 0$ in (174):

$$K_s = K_o w^n . \tag{175}$$

For values of K_s greater than this, no stationary solution is possible. The limiting value of K_s for a stationary solution occurs when the space charge defocusing force just cancels the effective focusing force due to K_o resulting in a net transverse force of zero.

The maximum value of v for a stationary solution occurs when $K_s = 0$ and is given by

$$v_{max} = \frac{K_o w^2}{K_1^2} = \frac{v_o^2 w^2}{K_1^2} .$$

Comparing $v_{max}^{1/2}$ with the longitudinal momentum,

$$\frac{v_{max}^{1/2}}{p_o} = \frac{\nu_o\, w}{K_1\, p_o} = \frac{\nu_o\, w}{R} \,. \tag{176}$$

For the MURA 50 Mev electron accelerator and for radial motion at injection, $\nu_x = 4.4$, $w \approx 1.5$ cm., and $R = 128$ cm. which gives

$$\frac{v_{max}^{1/2}}{p_o} \approx 5 \times 10^{-2} \,.$$

B. Reduction of Equations

The set of equations (171) can be reduced to two differential equations in $\bar{x}$ and w. Differentiating (171a), substituting (171b), and using $K_1 K_3 = 1$, gives

$$\bar{x}'' + (K - g)\,\bar{x} = b \,. \tag{177}$$

Although not necessary for obtaining an equation for w, it is assumed for simplicity now that K may be replaced by K_o. Then by differentiating (171d), eliminating u' with (171c), differentiating again, eliminating v' with (171e), eliminating u with (171d), and finally eliminating $\bar{p}$ with (171a), one obtains

$$(w^2)''' + 4\,(K_o - g)\,(w^2)' - K_s\,(4 - n)\,\frac{(w^2)'}{w^n} - 2\,g'\,w^2$$

$$= -\,(\bar{x}^2)''' - 4(K_o - g)\,(\bar{x}^2)' + 2\,g'\,\bar{x}^2 + 6\,b\,\bar{x}' + 2\,b'\,\bar{x} \,. \tag{178}$$

The initial conditions are specified by giving the initial values of $\bar{x}$, $\bar{x}'$, w, w' and w''. From equations (171), the last two quanti-

ties can be written

$$w' = \frac{K_1 u - \bar{x}\,\bar{x}'}{w} \tag{179}$$

$$w'' = \frac{K_1^2 v}{w} - \frac{K_1^2 u^2}{w^3} + K_s w^{1-n} - (K_o - g)\, w$$

$$+ \frac{2 K_1 u \bar{x}\,\bar{x}'}{w^3} - \frac{(\bar{x}')^2}{w} - \frac{(\bar{x}\,\bar{x}')^2}{w^3} \tag{180}$$

in terms of $\bar{x}$, $\bar{x}'$, w, u, v, and g.

C. Motion of the Center of the Beam

By definition let $\bar{x}$ be called the "center" of the beam. The equation of motion for the center is equation (177):

$$\bar{x}'' + (K - g)\,\bar{x} = b\,. \tag{177}$$

Here K is the focusing term of the external magnetic field, g is the gradient bump, and b is the field bump. This equation shows that the motion of the center is independent of the space charge effects and is the same as the motion of a single particle under the influence of only the external forces. This result has been previously conjectured but now the above derivation brings out the assumptions that are needed. These assumptions are that the forces are linear in x and that the space charge force depends only on $\bar{x}$ and w as described in Chapter IV. Since equation (177) can be handled by the usual methods of the orbit theory for betatron oscillations, there is no need to discuss it in detail; the behavior of $\bar{x}$ is just like that for betatron oscillations.

b and g are functions of θ of period 2π and can be written as Fourier series. Furthermore, in the construction of an accelerator, b and g are made as small as possible. Suppose g is zero and write

$$b = \sum_m b_m e^{im\theta} ; \quad m = 0, \pm 1, \pm 2, \cdots .$$

For a particular value of m, if $b_m \neq 0$, equation (177) has a resonance if $\nu_0 = m$. Thus the possible resonances are integral resonances which are not influenced by space charge effects; i.e., the tune for the motion of the center cannot be put on an integral resonance by space charge effects. Since the operating point of an accelerator is designed to be non-integral, such resonances do not occur to create difficulty. Then the effect of the bump b is to produce a small variation in $\bar{x}$ which is usually ignorable since b is small.

Suppose now b is zero but g is not necessarily zero. If the beam is launched so that $\bar{x} = \bar{x}' = 0$, then equation (177) shows that $\bar{x}$ remains zero thereafter. Since generally this condition cannot be satisfied exactly, there exists the possibility of half-integral resonances which are independent of space charge effects. Since these resonances are also avoided in design and g is small, such resonances do not occur and the result is to produce a small variation in $\bar{x}$ which is usually ignorable.

Thus, in summary, it is seen with regard to the motion of the <u>center</u> of the beam that transverse space charge effects do not produce

integral or half-integral resonances.

D. Elimination of the Field Bump

The field bump b can be eliminated from the equations in the following manner. Ignoring the space charge force, the equation of motion for a single particle can be written

$$x'' + K(\theta)\, x = b(\theta) + g(\theta)\, x + A(\theta)\, x^2 + B(\theta)\, x^3 + \cdots \qquad (181)$$

where nonlinear terms Ax^2, Bx^3, $\cdots$ have been included. There exists a periodic solution $x_o(\theta)$ of equation (181) with period 2π. Let this orbit represent a new equilibrium orbit and define a new transverse coordinate y with respect to this orbit by $x = x_o(\theta) + y$. Substituting this for x in equation (181), using the fact that $x_o(\theta)$ satisfies equation (181), and dropping the nonlinear terms in y gives

$$y'' + Ky = (g + 2Ax_o + 3Bx_o^2 + \cdots)\, y$$

for the linear equation of motion in the new coordinate system.

The new equation of motion does not contain a field bump but does contain a new gradient bump, $g + 2Ax_o + 3Bx_o^2 + \cdots$, which is the original gradient bump modified by additional terms. Replacing the symbol y by x and letting g represent the modified gradient bump, the entire analysis giving the final moment equations can be repeated with respect to the new equilibrium orbit with the result that the field bump b is removed from the equations. With the understanding that

this has been done, the field bump b is now dropped from all equations (i.e., $b = 0$) and g is taken to represent the modified gradient bump. Equations (177) and (178) become

$$\bar{x}'' + (K - g)\,\bar{x} = 0 \tag{182}$$

and

$$(w^2)''' + 4\,(K_o - g)\,(w^2)' - K_s\,(4 - n)\,\frac{(w^2)'}{w^n} - 2\,g'\,w^2$$

$$= -\,(\bar{x}^2)''' - 4\,(K_o - g)\,(\bar{x}^2)' + 2\,g'\,\bar{x}^2\,. \tag{183}$$

E. Linear Theory of Free-Beam Behavior

Equation (182) shows that, if the beam is launched so that initially $\bar{x} = \bar{x}' = 0$, then $\bar{x}$ remains zero. Generally, this condition cannot be satisfied exactly and thus $\bar{x}$ contributes a driving term in the right-hand side of equation (183) for w. Furthermore the presence of g in the left-hand side of equation (183) makes the behavior of w more complicated. To consider the simplest behavior of w, assume that there is no gradient bump (i.e., $g = 0$) and that the beam is launched so that $\bar{x}$ vanishes identically. In this situation let the beam and its behavior be described as "free." For the free behavior of the beam, equation (183) becomes

$$(w^2)''' + 4\,K_o\,(w^2)' - K_s\,(4 - n)\,\frac{(w^2)'}{w^n} = 0\,. \tag{184}$$

Equations (179) and (180) become

$$w' = \frac{K_1 u}{w} \tag{185}$$

$$w'' = \frac{1}{w}\left[K_1^2 v - \left(K_o - \frac{K_s}{w^n}\right) w^2\right] - \frac{K_1^2 u^2}{w^3} . \tag{186}$$

A possible solution of equation (184) is $(w^2)' = 0$ or $w = a$ constant which requires $w' = w'' = 0$ and hence, by equations (185) and (186),

$$u = 0$$

and

$$K_1 v^2 - \left(K_o - \frac{K_s}{w^n}\right) w^2 = 0 .$$

These relations just state that u and v must satisfy the conditions for a stationary solution (cf. equation (174)). If K_s is larger than the limiting value given by equation (175) for a fixed w, then these conditions cannot be satisfied and w = a constant is not the solution. The limiting value of K_s for $w = 0$ is $K_s = 0$ and therefore, since $K_s > 0$, $w \equiv 0$ cannot be a solution.

The differential equation for w is nonlinear. Let

$$y = w^2 \quad \text{or} \quad w = y^{1/2} . \tag{187}$$

Equation (184) can now be written

$$y''' + 4\,K_o\,y' - K_s\,(4 - n)\,\frac{y'}{y^{n/2}} = 0\,. \tag{188}$$

For small perturbations about a stationary solution, a linearized approximation can be developed as follows. Let

$$y_o = w_o^2 = \text{a constant}$$

be a stationary solution and put

$$y = y_o + e$$

with $|e| \ll y_o$. Substituting for y in $y^{-n/2}$, expanding this in a power series in e about y_o, and keeping terms only up to first order in e, gives

$$\frac{1}{y^{n/2}} = \frac{1}{y_o^{n/2}} - \frac{n}{2}\,\frac{y_o^2}{y_o^{n+2}}\,e\,.$$

Inserting these relations into equation (188) and retaining terms only up to first order, one obtains

$$e''' + 4\,K_o\,e' - \frac{K_s\,(4 - n)}{y_o^{n/2}}\,e' = 0$$

or

$$e''' + \Omega^2\,e' = 0 \tag{189}$$

where

$$\Omega^2 = 4 K_o - \frac{K_s (4 - n)}{y_o^{n/2}} \quad . \tag{190}$$

Integrating equation (189) once,

$$e'' + \Omega^2 e = c = \text{a constant} \tag{191}$$

where c is determined by the initial conditions. If $\Omega^2 > 0$, (191) is the equation for a simple oscillator driven by the constant c. For $\Omega^2 < 0$, the solution is exponential. The solution can be written

$$e = \frac{c}{\Omega^2} + E \sin (\Omega\theta + a) \tag{192}$$

where E and a are constants. Then w is given by

$$w = (y_o + e)^{1/2} = y_o^{1/2} (1 + \frac{e}{y_o})^{1/2}$$

and, to first order in e,

$$w = y_o^{1/2} + \frac{1}{2} \frac{e}{y_o^{1/2}} = w_o + \frac{1}{2} \frac{e}{w_o}$$

$$= w_o + \frac{c}{2 \Omega^2 w_o} + \frac{E}{2 w_o} \sin (\Omega\theta + a) \, . \tag{193}$$

Assuming $\Omega^2 > 0$, Ω is the frequency of oscillation of the beam width w. With $K_o = \nu_o^2$, equation (190) gives

$$\Omega = 2 \nu_o \left[1 - \frac{K_s (4 - n)}{4 \nu_o^2 w_o^n} \right]^{1/2} . \tag{194}$$

If the space charge effect is negligible ($K_s = 0$), the frequency Ω is $2\,\nu_o$; i.e., twice the betatron oscillation frequency. As K_s increases, Ω decreases towards zero and, for large enough K_s, Ω becomes imaginary giving an unstable solution for w. However the value of K_s which decreases Ω to zero is greater than the limiting value of K_s for a stationary solution, as given by (175), by a factor of $4/(4 - n)$. This means that the situation is far from a stationary solution and so (194) is applicable only for K_s less than the value given by (175). If the limiting value of K_s for a stationary solution is inserted in (192), the result is $\Omega = n^{1/2}\,\nu_o$, or $\Omega = \sqrt{2}\,\nu_o$ for radial motion and $\Omega = \nu_o$ for vertical motion.

The above result points out an aspect of the beam behavior which did not show up in the elementary treatment of Chapter II; namely, that the beam width can oscillate. Let equation (193) be written

$$w = w_o + \Delta_1 + \Delta_2 \sin(\Omega\theta + a) .$$

Differentiating twice and evaluating the various expressions at $\theta = 0$ gives

$$w_i = w_o + \Delta_1 + \Delta_2 \sin a$$

$$w_i' = \Omega\Delta_2 \cos a$$

$$w_i'' = -\Omega^2 \Delta_2 \sin a$$

where the subscript i indicates the initial values of the variables. Solving these for Δ_1 and Δ_2:

$$\Delta_1 = w_i - w_o + \frac{1}{\Omega^2} w_i'' \tag{195}$$

$$\Delta_2^2 = \frac{1}{\Omega^4} \left[\Omega^2 (w_i')^2 + (w_i'')^2 \right] . \tag{196}$$

Suppose that before $\theta = 0$ the solution is stationary and the perturbation is applied at $\theta = 0$. The values of w_i' and w_i'' are given by (185) and (186) with u and v equal to their new values, u_i and v_i, just after the perturbation is applied. Assume $w_i = w_o$ and $\Omega \approx 2\,\nu_o$. Before the perturbation, $u = 0$ and v is given approximately by (176). Suppose the perturbation is such that u is unchanged and v is doubled so that the change Δv is (using (176)):

$$\Delta v \approx \frac{p_o^2\, \nu_o^2\, w_o^2}{R^2} .$$

Then from (186),

$$w_i'' \approx \frac{K_1^2\, \Delta v}{w_o} .$$

Now (195) gives

$$\frac{\Delta_1}{w_o} \approx \frac{1}{4\,\nu_o^2} \frac{K_1^2}{w_o^2} \frac{p_o^2\, \nu_o^2 w_o^2}{R^2}$$

and, using $K_1 = R/p_o$,

$$\frac{\Delta_1}{w_o} \approx \frac{1}{4} .$$

Similarly Δ_2 is about the same. In this case then w oscillates with an amplitude of about 25% of w_o. Equation (196) indicates that when Ω is decreased by space charge effects, a given change in u or v induces a still larger amplitude of oscillation.

F. Nonlinear Theory of Free-Beam Behavior

A more complete theory of the free behavior of the beam than the linear theory is developed in this section. After substituting (187):

$$y = w^2 , \tag{187}$$

equation (184) for w transformed into equation (188):

$$y''' + 4 K_o y' - K_s (4 - n) \frac{y'}{y^{n/2}} = 0 . \tag{188}$$

Define

$$Y = y' \tag{197}$$

and

$$Z = Y' . \tag{198}$$

Then $Z' = y'''$. Substituting these relations into (188), the resulting equation and the definitions (197) and (198) form the following set of three first-order equations:

$$y' = Y \tag{199a}$$

$$Y' = Z \tag{199b}$$

$$Z' + 4\,K_o\,Y - K_s\,(4 - n)\,\frac{Y}{y^{n/2}} = 0\,. \tag{199c}$$

Using (199a), one can write

$$\frac{d}{d\theta} = \frac{dy}{d\theta}\,\frac{d}{dy} = y'\,\frac{d}{dy} = Y\,\frac{d}{dy}\,. \tag{200}$$

Applying (200) to (199b) gives

$$Y\,\frac{dY}{dy} = Z$$

or

$$\frac{1}{2}\,\frac{d}{dy}\,(Y^2) = Z\,. \tag{201a}$$

Doing the same to (199c) gives

$$Y\,\frac{dZ}{dy} + 4\,K_o\,Y - K_s\,(4 - n)\,\frac{Y}{y^{n/2}} = 0$$

or, after dividing by Y,

$$\frac{dZ}{dy} + 4\,K_o - K_s\,(4 - n)\,\frac{1}{y^{n/2}} = 0\,. \tag{201b}$$

Rewriting and summarizing, one has from equations (201a) and (201b) the following two equations:

$$\frac{1}{2}\frac{d}{dy}(Y^2) = Z \tag{202a}$$

$$\frac{dZ}{dy} = K_s (4 - n) \frac{1}{y^{n/2}} - 4 K_o . \tag{202b}$$

Equation (202b) can be integrated to give Z as a function of y. After substituting this for Z in (202a), a second integration can be made to obtain Y^2 as a function of y. In integrating $y^{-n/2}$, the case of $n = 2$ is different than when n has any other value; so the cases of $n = 1$ and $n = 2$ must be treated separately.

Consider first the case for vertical motion with $n = 1$. Equation (202b) becomes

$$\frac{dZ}{dy} = \frac{3 K_s}{y^{1/2}} - 4 K_o . \tag{203}$$

Integration with respect to y gives

$$Z = D_1 + 6 K_s y^{1/2} - 4 K_o y \tag{204}$$

where D_1 is a constant of integration. Substituting this into (202a) results in

$$\frac{1}{2}\frac{d}{dy}(Y^2) = D_1 + 6 K_s y^{1/2} - 4 K_o y , \tag{205}$$

and integrating, one has

$$P = \frac{1}{2} Y^2 = E_1 + D_1 y + 4 K_s y^{3/2} - 2 K_o y^2 \tag{206}$$

where E_1 is a constant of integration and, by definition,

$$P = \frac{1}{2} Y^2 . \tag{207}$$

The constants of integration, D_1 and E_1, can be found by evaluating equations (204) and (206), respectively, in terms of the values of the variables at any point θ.

In analogy with particle mechanics, consider y as the position of a fictitious particle, $Y = y'$ as its momentum, and $P = Y^2/2$ as its kinetic energy. Then equation (206) can be written

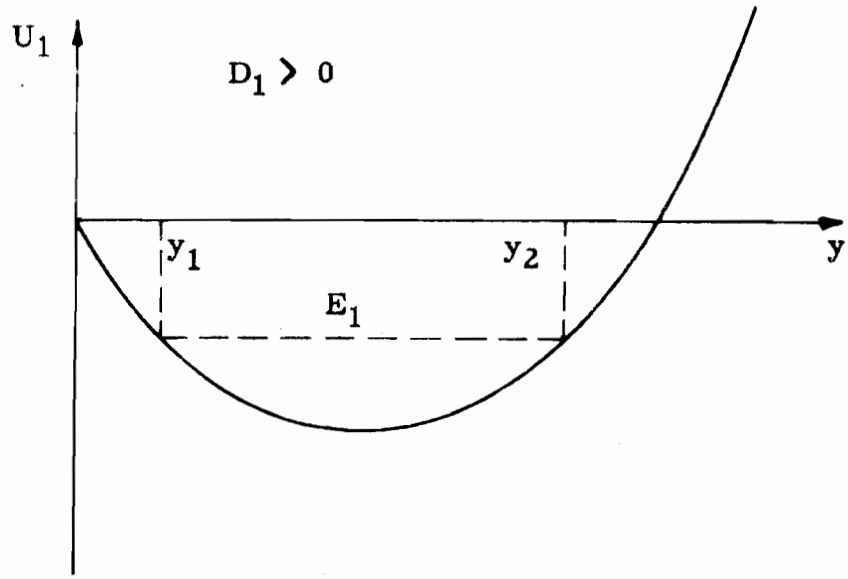
$$E_1 = P + U_1 \tag{208}$$

with

$$U_1 = - D_1 y - 4 K_s y^{3/2} + 2 K_o y^2 . \tag{209}$$

Equation (208) can be interpreted as an energy equation of a particle moving in a potential energy well U_1 with total energy E_1. Since $K_s > 0$ and $K_o \geq 0$, U_1 can have the forms shown in Fig. 6 depending on the sign of D_1. (209) applies only for $y \geq 0$ since physically $y = w^2 \geq 0$. For the situations shown in Fig. 6, y oscillates between the values y_1 and y_2. The stationary solutions occur when E_1 equals the minimum value of U_1 so that y_1 and y_2 coincide.

The allowable values of D_1 and E_1 are restricted as will now be shown. From the definition of Y and (185),



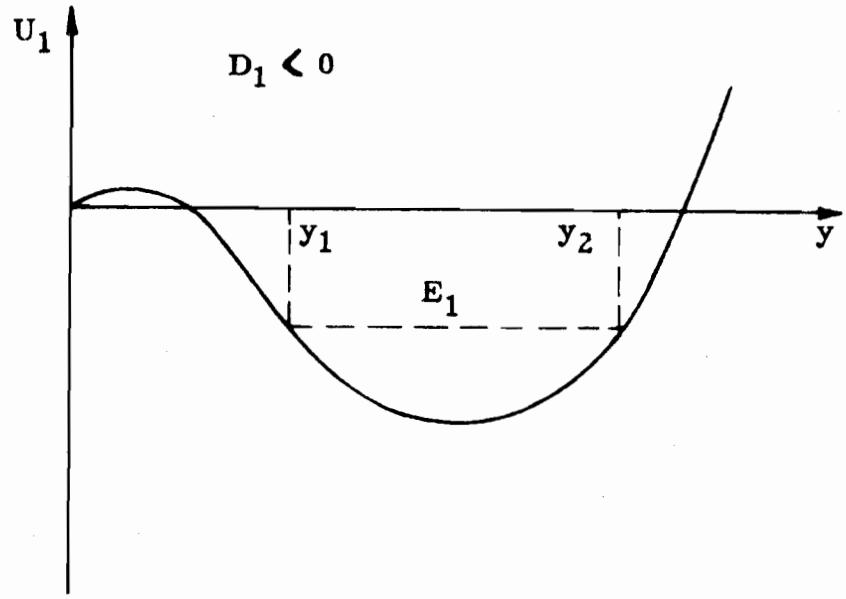


Figure 6. CURVES OF U_1

$$Y = y' = (w^2)' = 2\,w\,w' = 2\,K_1\,u$$

$$Y = 2\,K_1\,u\,. \tag{210}$$

Similarly, from the definition of Z, (185) and (186),

$$Z = Y' = y'' = 2\,(w\,w')' = 2\,w\,w'' + 2\,(w')^2$$

$$Z = 2\left[K_1^2\,v - \left(K_o - \frac{K_s}{w}\right) w^2\right]. \tag{211}$$

From (204) and (206),

$$D_1 = Z - 6\,K_s\,y^{1/2} + 4\,K_o\,y \tag{212}$$

$$E_1 = \frac{1}{2}\,Y^2 - D_1\,y - 4\,K_s\,y^{3/2} + 2\,K_o\,y^2 \tag{213}$$

Substituting (212) in (213) gives

$$E_1 = \frac{1}{2}\,Y^2 - y\,Z + 2\,K_s\,y^{3/2} - 2\,K_o\,y^2$$

$$= \frac{1}{2}\,Y^2 - w^2\,Z + 2\,K_s\,w^3 - 2\,K_o\,w^4\,. \tag{214}$$

Substituting (210) and (211) in this gives

$$E_1 = 2\,K_1^2\,(u^2 - v\,w^2)\,. \tag{215}$$

From Chapter IV, the inequality (120) with $\overline{x} = 0$ is

$$u^2 \leq v\,w^2$$

or

$$u^2 - v\,w^2 \leq 0 .$$

This and (215) imply

$$E_1 \leq 0 . \tag{216}$$

Thus the behavior of y is as shown in Fig. 6 with E_1 on or below the y-axis.

Let U_{1min} be the minimum value of U_1. Then, since $Y^2 \geq 0$, $U_{1min} \leq E_1 \leq 0$ or

$$U_{1min} \leq 0 .$$

Putting (208) in this gives

$$- D_1\, y_o - 4\, K_s\, y_o^{3/2} + 2\, K_o\, y_o^2 \leq 0$$

where y_o is the value of y corresponding to U_{1min}. Since $y_o > 0$, the above inequality can be divided by y_o and rearranged to give

$$D_1 \geq 2\, K_o\, y_o - 4\, K_s\, y_o^{1/2} .$$

The minimum value of the right-hand side of this relation occurs at $y_o^{1/2} = K_s/K_o$ and consequently

$$D_1 \geq - \frac{2\, K_s^2}{K_o} . \tag{217}$$

By differentiating equation (209) for U_1 it is easily verified that the above value of y_o minimizes U_1 when D_1 is given by the equality in (217); hence the equality in (217) can actually occur.

The values of D_1 and E_1 for a stationary solution y_o can be evaluated from (212) and (214) by using the fact that $Y = y' = 0$ and $Z = y'' = 0$. Then

$$D_1 = 4 K_o y_o - 6 K_s y_o^{1/2} \tag{218}$$

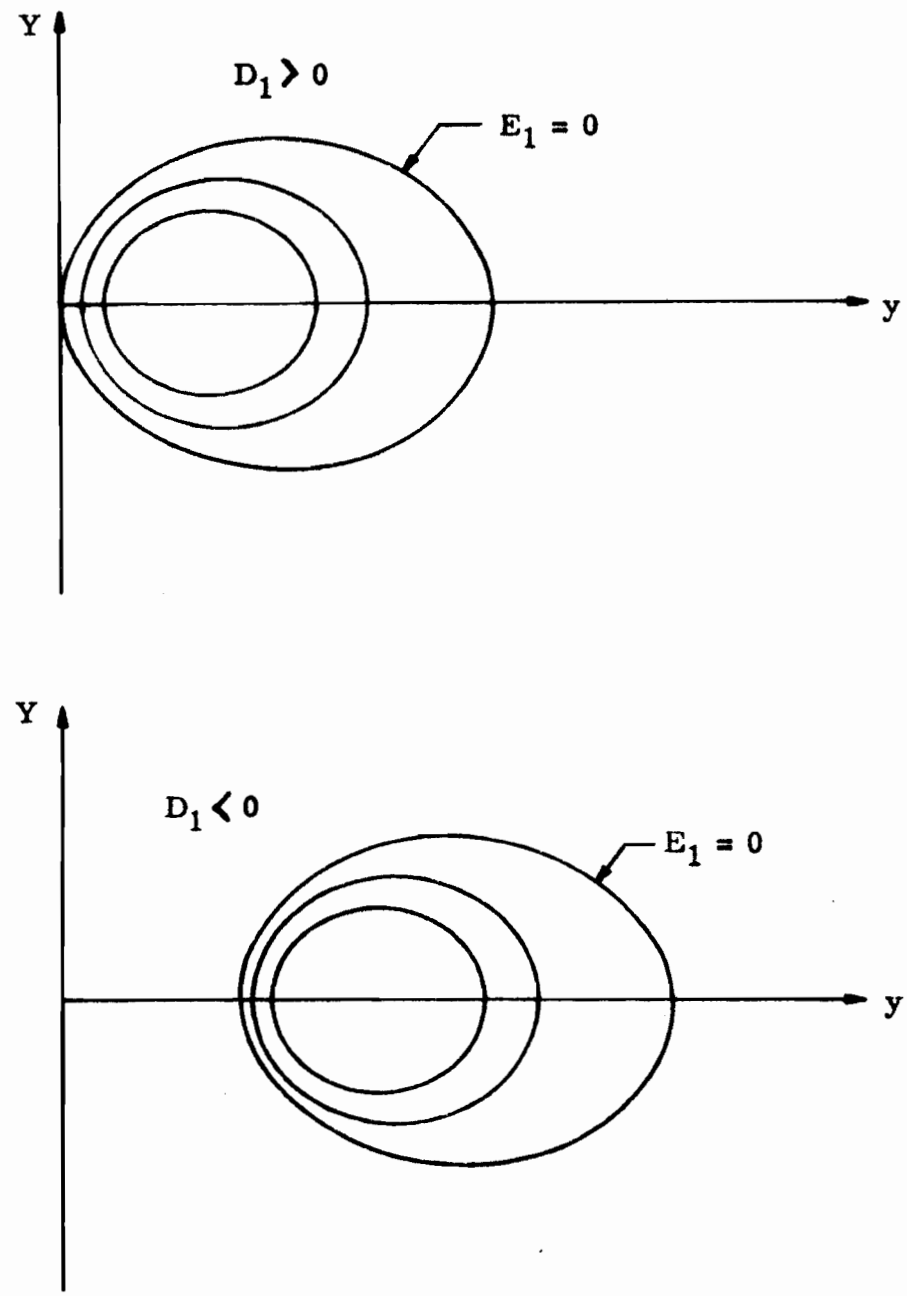
$$E_1 = 2 K_s y_o^{3/2} - 2 K_o y_o^2 \tag{219}$$

for a stationary solution y_o.

The behavior of y can be described by trajectories in the phase plane of $Y = y'$ versus y as shown in Fig. 7. Y as a function of y can be obtained from (206). The trajectories are symmetric about the y-axis. The expression for U_1, (209), shows that the only effect of increasing K_s is to lower the curve of U_1 as y increases but eventually the y^2 term dominates and U_1 becomes positive. Hence the motion is always bounded since $E_1 \leq 0$ and the beam never blows itself completely apart no matter how large the space charge effect is.

Since $Y = y' = dy/d\theta$,

$$d\theta = \frac{dy}{Y} = \pm \frac{1}{\sqrt{2}} \frac{dy}{\left[E_1 + D_1 y + 4 K_s y^{3/2} - K_o y^2\right]^{1/2}}$$



In each case the physical region corresponding to $E_1 \leq 0$ lies on or interior to the curve for $E_1 = 0$.

Figure 7. PHASE PATHS

from (206). Let $\Delta\theta$ be the change in θ as y goes through one oscillation. Integrating the above expression and using the fact that the trajectories in the phase plane are symmetric about the y-axis,

$$\Delta\theta = \sqrt{2} \int_{y_1}^{y_2} \frac{dy}{\left[E_1 + D_1 y + 4 K_s y^{3/2} - 2 K_o y^2\right]^{1/2}} . \quad (220)$$

The frequency of oscillation Ω is then

$$\Omega = \frac{2\pi}{\Delta\theta} . \quad (221)$$

It is possible to express $\Delta\theta$ in terms of elliptic integrals of the first and third kind but these are of little use for the present purposes. For the case where $D_1 < 0$ and $E_1 = 0$, $\Delta\theta$ can be evaluated exactly with the result $\Omega = \nu_o$.

Now consider the case for radial motion with n = 2; the treatment is similar to the above. Equation (202b) becomes

$$\frac{dZ}{dy} = \frac{2 K_s}{y} - 4 K_o$$

and integration gives

$$Z = D_2 + 2 K_s \ln y - 4 K_o y \quad (222)$$

where D_2 is the constant of integration. With this, (202a) becomes

$$\frac{1}{2} \frac{d}{dy} (Y^2) = D_2 + 2 K_s \ln y - 4 K_o y$$

and integration gives

$$P = \frac{1}{2} Y^2 = E_2 + (D_2 - 2 K_s) y + 2 K_s y \ln y - 2 K_o y^2 \tag{223}$$

where E_2 is the constant of integration. Considering E_2 as the total energy and P as the kinetic energy of a fictitious particle, this can be written

$$E_2 = P + U_2 \tag{224}$$

where

$$U_2 = - (D_2 - 2 K_s) y - 2 K_s y \ln y + 2 K_o y^2 \tag{225}$$

is the potential energy. The shape of the U_2 - curves have the same general characteristics as in the previous case. In the same manner as the previous case it can be shown that

$$E_2 \leq 0 \tag{226}$$

and

$$D_2 \geq 4 K_s - 2 K_s \ln \left(\frac{K_s}{K_o} \right) . \tag{227}$$

The motion is again always bounded. For a stationary solution y_o, the values of D_2 and E_2 are

$$D_2 = 4 K_o y_o - 2 K_s \ln y_o \tag{228}$$

$$E_2 = 2 K_s y_o - 2 K_o y_o^2 . \tag{229}$$

An integral for $\Delta\theta$ can be written but it appears to be nonintegrable.

The above nonlinear theory for the free-beam behavior shows that the beam width oscillates if the conditions for a stationary solution are not satisfied. Furthermore, the oscillations are always bounded. It is also apparent that the amplitude of oscillation may be quite large. This could result in the loss of particles either by making the amplitude of oscillation for single particles large enough (i.e., exceed their stability limits) that their betatron oscillations are unstable or by directly driving the particles into the vacuum chamber walls.

G. Linear Theory of Resonances

A linear theory of beam resonances driven by the gradient bump is now developed. For simplicity assume the beam is launched so that $\bar{x} \equiv 0$. After putting

$$y = w^2 , \tag{230}$$

equation (183) becomes

$$y''' + 4 (K_o - g) y' - K_s (4 - n) \frac{y'}{y^{n/2}} - 2 g' y = 0 . \tag{231}$$

Now linearize the above equation by considering small perturbations about a stationary solution. Assume that

$$y = y_o + e \tag{232}$$

where $y_o = w_o^2$ = a constant is a stationary solution and $|e| << y_o$, and also assume g and g' are small of first order. Substituting (232) in (231), using the first-order approximation for $y^{-n/2}$ given in Section E, and keeping only terms up to first order, one obtains

$$e''' + \Omega^2 e' = 2 y_o g' \tag{233}$$

where

$$\Omega = 2 \nu_o \left[1 - \frac{K_s (4 - n)}{4 \nu_o^2 w_o^n} \right]^{1/2} \tag{234}$$

is the free-beam oscillation frequency as obtained in the linear theory of Section E. Integration of (233) gives

$$e'' + \Omega^2 e = 2 y_o g + c \tag{235}$$

where c is a constant of integration. This is the equation of a driven simple oscillator. The gradient bump g can be written as a Fourier series:

$$g = \sum_m g_m e^{i m \theta} ; \quad m = 0, \pm 1, \pm 2, \cdots . \tag{236}$$

Then, if Ω is an integer m and the corresponding $g_m \neq 0$, the oscillator is being driven on a resonance and e will contain a sinusoidal term having an amplitude which grows linearly with θ. Thus resonances which cause the beam width to grow can occur when Ω is equal to an integer:

$$\Omega = m ; \quad m = 0, 1, 2, \cdots . \tag{237}$$

When the space charge effect is negligible ($K_s = 0$), (234) gives $\Omega = 2\nu_o$ and (237) becomes $2\nu_o = m$. This is the usual relation for half-integral resonances obtained from the single-particle theory when space charge effects are ignored. When K_s is retained in (234), then Ω decreases as the beam current is increased and a resonance is obtained when Ω has been decreased to an integral value. This result agrees qualitatively with that of the elementary treatment of Chapter II where the effect was to put each particle onto a half-integral resonance.

In this linear theory the beam grows indefinitely on the resonance because Ω is constant. Eventually it must stop growing because, by loss of particles, the beam current is reduced thus taking Ω off the resonance. Another mechanism for stopping the growth of the beam width may exist. Expression (234) for Ω is based on the linear theory but, in the nonlinear theory of the free beam, a preliminary examination of Ω in terms of integrals such as (220) indicates that Ω is a function of the amplitude of oscillation. Therefore it is possible that after a certain amount of growth Ω will be removed from the resonance thus stopping further growth. A nonlinear theory based on equation (231) is required to determine how serious the resonances are. An attempt in this direction has been made but with no success. In the next

chapter an approach for a nonlinear theory of resonances is sketched out which may be successful in the future.

H. Summary

The following results have been obtained for the behavior of the beam. The motion of the center of the beam is independent of the space charge forces as previously conjectured; it is the same as the betatron oscillations of a single particle under the influence of only the external forces. Space charge effects therefore cannot put the center of the beam on an integral or half-integral resonance.

An approximate stationary solution exists for the free-beam behavior. If the conditions for a stationary solution are not satisfied under free-beam conditions, the beam width oscillates; however, no matter how strong the space charge effects may be, the oscillations are stable (i.e., bounded). The amplitude of oscillation may be large enough under appropriate conditions to cause the loss of particles.

A linear theory for resonances driven by a gradient bump indicates that resonances exist in qualitative agreement with the half-integral resonances predicted by the elementary treatment of Chapter II. The seriousness of these resonances cannot be determined from the linear theory and the solution of this problem awaits the development of a more complete nonlinear theory.

VI. CONCLUSION

A. Limitations of the Theory

A theory of transverse space charge effects in particle accelerators has been developed in the preceding chapters which is based on several assumptions. Therefore, there are limitations on the theory depending on how closely the assumptions are satisfied in actual accelerators. Some of the limitations are not discussed at all or in detail because to do so would require the solution of problems which have not yet been solved. The theory may be extended by dropping assumptions in order to remove some of the limitations but it may not be clear how to obtain an adequate solution. Some of the possible extensions of the theory are briefly discussed in this section but the more immediate extensions are sketched out in more detail in the next section.

In the last chapter a linear theory of resonances driven by a gradient bump was developed. Since it applies only to small perturbations about a stationary solution, it does not tell whether the beam width continues to grow indefinitely or not. To remove this difficulty a better method is needed to investigate the solutions of equation (231). An approach which may be useful is discussed in the next section.

Consider the assumption that the longitudinal variation of the transverse distribution at a given time is negligible for the calculation of the internal force. If the transverse distribution in every transverse

slice is initially the same, then the free behavior of each slice is identical; each slice oscillates in synchronism and there is no longitudinal variation in the transverse distribution. But, if the initial transverse distribution varies longitudinally in the same manner as the transverse distribution varies in one slice while the slice is moving longitudinally, then the assumption requires $\Delta w \ll \lambda$ where Δw is the variation in w and

$$\lambda = \frac{2\pi}{\Omega} R = \text{the wavelength of the beam oscillation.}$$

Using $\Omega = 2\nu_o$ for the maximum value of Ω, the requirement becomes

$$\frac{\Delta w}{R} \ll \frac{\pi}{\nu_o} .$$

For values of Δw of interest, this criterion is usually satisfied.

Another assumption for the calculation of the internal force was that the longitudinal variation of the longitudinal distribution at a given time is negligible. Suppose the beam is bunched longitudinally by the accelerating electric field. Let ℓ be the longitudinal length of a bunch. Then, for a transverse slice well away from the ends of the bunch, the above assumption requires $w \ll \ell$. In many cases this will be satisfied but then the space charge factor K_s must be calculated on the basis of the local beam current, not the time-averaged beam current.

A third assumption for the calculation of the internal force was that the propagation time for the electromagnetic field is negligible. If the fields created only by the charge near the slice are considered to be important, then the assumption requires that the time to propagate the field over the width of the beam be much less than the time for one oscillation of the beam width; i.e.,

$$\frac{w}{c} \ll \frac{\lambda}{v_o} = \frac{2\pi R}{\Omega v_o}$$

or, putting $\Omega = 2\nu_o$,

$$\frac{w}{R} \ll \frac{\pi}{\nu_o \beta_o} .$$

Even for $\beta_o = 1$ this is usually satisfied. However, if fields produced by the charge as far away as λ from the slice are important, then the propagation time is comparable to the time for one oscillation unless $\beta_o \ll 1$. So for relativistic energies there may be important effects which have been ignored.

Now consider the assumption that the external force is linear in x for one-dimensional motion. The single-particle theory of betatron oscillations (ignoring space charge forces) indicates that, if the non-linear forces are included, the oscillations are unstable when their amplitudes exceed the stability limit. Hence a criterion for the linear assumption is that the beam width be much less than the stability limit which, in many cases of interest, is not satisfied. One way of possibly

removing this limitation approximately is to assume that the particles are instantly lost from the beam when their amplitudes exceed the stability limit. This would result in dealing with a Boltzmann equation in which the number of particles is not constant and is discussed in the next section. A more exact approach would be to include the nonlinear forces in the force term of the Boltzmann equation and, similarly, higher order terms in the power series expansion for the space charge force could be included.

Two idealized models of the beam were assumed for studying the radial and vertical motion separately. Each model is equivalent to assuming that the vertical extent of the beam is much less than the radial extent. In general, this assumption is poorly satisfied. One way of approximately removing this limitation would be to use the two-dimensional Boltzmann equation (37) and define average quantities by integrating over the position and momentum of one transverse direction to obtain a one-dimensional equation for the other transverse direction. Some method of approximating the average quantities would then be needed. A better way would be to use moments with respect to the position and momentum of both transverse directions and apply the method of moments to the two-dimensional Boltzmann equation.

The validity of the "slice" approximation has been discussed in Chapter III. The removal of this limitation would require considerable modification of the theory. The general Boltzmann equation (25) would

have to be used. If suitable approximations could be made, some simplification might be attained by reducing the general Boltzmann equation to two coupled Boltzmann equations, one for transverse motion and the other for longitudinal motion. The first equation would be obtained by integrating the general Boltzmann equation over the longitudinal momentum. The second equation would result from integrating the general Boltzmann equation over the transverse position and momentum. Without carrying the program through in detail, it is difficult to tell how far this approach would get.

Finally, consider the assumption that the number of particles is constant. Let N be the density of particles as defined in Section F, Chapter III, and let ΔN be the change in N in the time of one turn or revolution around the machine. The longitudinal rate of change of N is $\Delta N/L$ where L is the length of the equilibrium orbit. A criterion for satisfying the above assumption is that

$$\frac{\Delta N}{L} \ll \frac{N}{\lambda} = \frac{N\Omega}{L}$$

or

$$\frac{\Delta N}{N} \ll \Omega \quad .$$

During the time of a multiturn injection process, this criterion will not be satisfied, at least not during the earlier turns of injection. Consequently, the oscillations of the beam will be influenced by the rate of

injection. This aspect of the beam behavior can be included in the theory by modifying the one-dimensional Boltzmann equation (83) to allow for changes in the number of particles.

B. Some Extensions of the Theory

In this section some extensions of the theory are sketched out which, in the future, may allow the investigation to be carried out farther than it has been. In Chapter V a linear theory of resonances was obtained by linearizing equation (231):

$$y''' + 4\,(K_o - g)\,y' - K_s\,(4 - n)\,\frac{y'}{y^{n/2}} - 2\,g'\,y = 0 \tag{231}$$

where $y = w^2$ and g is the gradient bump. Instead of linearizing, a different approach is needed to investigate how serious the resonances are. With the definitions $Y = y'$ and $Z = Y'$, equation (231) is equivalent to the following system of first-order equations:

$$y' = Y \tag{238a}$$

$$Y' = Z \tag{238b}$$

$$Z' = 2\,g'\,y + K_s\,(4 - n)\,\frac{Y}{y^{n/2}} - 4\,(K_o - g)\,Y\,. \tag{238c}$$

For $g \equiv 0$ equations (238) are the same as equations (199) for the free-beam behavior and, for this case, it was possible to find two integrals of the system. For the moment consider the case of vertical motion with $n = 1$. The two integrals are equations (204) and (206)

which contain the constants D_1 and E_1. Let D_1 and E_1 be replaced, respectively, by variables β and α in equations (204) and (206) and consider these equations as the definitions of the new variables β and α :

$$Z = \beta + 6 K_s y^{1/2} - 4 K_o y \tag{239}$$

$$\frac{1}{2} Y^2 = \alpha + \beta y + 4 K_s y^{3/2} - 2 K_o y^2 . \tag{240}$$

Transforming from the variables y, Y, Z to the variables y, α , β equations (238) with n = 1 become

$$y' = Y \tag{241a}$$

$$\alpha' = - y \beta' \tag{241b}$$

$$\beta' = 4 Y g + 2 y g' , \tag{241c}$$

where Y is a function of α, β , and y by equation (240). If $g \equiv 0$, equations (241) give $\alpha' = \beta' = 0$ or α and β are constants of the motion corresponding to the constants E_1 and D_1. Equation (240) for various fixed values of α and β specifies a family of trajectories in the y - Y phase plane. A point in the phase plane can be specified by giving the values of α , β , and y. With the perturbation g applied, α and β will vary with θ as determined by equations (241).

Similarly, for the case of radial motion with $n = 2$, the integrals (222) and (223) can be used to define new variables α and β by replacing E_2 and D_2 by α and β, respectively. Equations (238) then transform into equations (241) as before, the only difference being the functional dependence of Y on α, β, and y through equation (223) with E_2 and D_2 replaced by α and β. In either case Y is a function of α, β, and y: $Y = Y(\alpha, \beta, y)$.

An "angle" variable γ is now defined as follows. Let

$$\Theta = \Theta(\alpha, \beta, y) = \int^{y} \frac{d\xi}{Y(\alpha, \beta, \xi)} \tag{242}$$

and

$$\Omega = \Omega(\alpha, \beta) = 2\pi \left[\int \frac{d\xi}{Y(\alpha, \beta, \xi)} \right]^{-1} \tag{243}$$

where the latter integral is taken over one complete circuit of the trajectory in the phase plane specified by α and β. Then γ is defined by

$$\gamma = \Omega\Theta . \tag{244}$$

This relation can, in principle, be solved for y as a function of α, β, and γ which can be used to eliminate y from equations (241) to obtain equations for α, β, and γ. These new equations can be

written in the form

$$\alpha' = A(\alpha, \beta, \gamma, \theta) \tag{245a}$$

$$\beta' = B(\alpha, \beta, \gamma, \theta) \tag{245b}$$

$$\gamma' = \Omega(\alpha, \beta) + C(\alpha, \beta, \gamma, \theta) \tag{245c}$$

where A, B, and C are periodic with period 2π in both γ and θ. These equations can then be written

$$\alpha' = \sum_{m,n} A_{mn} e^{i(m\gamma - n\theta)} \tag{246a}$$

$$\beta' = \sum_{m,n} B_{mn} e^{i(m\gamma - n\theta)} \tag{246b}$$

$$\gamma' = \Omega + \sum_{m,n} C_{mn} e^{i(m\gamma - n\theta)} \tag{246c}$$

where the A_{mn}, B_{mn}, and C_{mn} are functions of α and β. Suppose the system is near a resonance being driven by the Fourier components

$$e^{\pm i(m\gamma - n\theta)}$$

where m and n are fixed. Transforming to a new variable

$$\underline{\gamma} = \gamma - \frac{n}{m}\theta,$$

the Fourier components which are not driving the resonance become

rapidly fluctuating terms compared to those driving the resonance. It is assumed as an approximation that the rapidly fluctuating terms average to zero and may be ignored. Then, after transforming to $\underline{\gamma}$ and dropping the rapidly fluctuating terms, equations (246) take the form:

$$\alpha' = a \cos (m \underline{\gamma} + \delta_1) \tag{247a}$$

$$\beta' = b \cos (m \underline{\gamma} + \delta_2) \tag{247b}$$

$$\underline{\gamma}' = \Omega - \frac{m}{n} + c \cos (m \underline{\gamma} + \delta_3) , \tag{247c}$$

where a, b, c, and Ω are functions of α and β. δ_1, δ_2, and δ_3 are constants.

The system (247) is to be investigated for stability. It may be possible to do this by an extension of the Moser technique.[25] Moser's method applies only to Hamiltonian systems. To illustrate, suppose there were just two equations:

$$\alpha' = a \cos (m \underline{\gamma} + \delta_1)$$

$$\underline{\gamma}' = \Omega - \frac{m}{n} + c \cos (m \underline{\gamma} + \delta_3) ,$$

where a, c, and Ω are functions only of α. If $\delta_1 = \delta_3 - \frac{\pi}{2}$ and $c = - \frac{1}{m} \frac{\partial a}{\partial \alpha}$, then the system is Hamiltonian with a Hamiltonian function of the form

$$H (\alpha , \underline{\gamma}) = F (\alpha) + G (\alpha) \cos (m \underline{\gamma} + \delta_3)$$

which is a constant of the motion; i.e., independent of θ. Using polar coordinates with α being the radius and $\underline{\gamma}$ being the angle, constant values of $H(\alpha, \underline{\gamma})$ define trajectories of the system. By means of these trajectories and transforming back to the original variables it is possible to determine the stability of the system. Moser's method consists of finding the transformation from the original variables to the new variables which results in a Hamiltonian function which is a constant of the motion. The transformations from equations (238) to equations (247) is, in fact, Moser's method except that a constant Hamiltonian is obtained.

The equations (247) are not Hamiltonian. An extension of Moser's method would be to find a function of α, β, and $\underline{\gamma}$ which is a constant of the motion. If this could be accomplished, then trajectories of the system could be obtained with which to determine the stability. If a constant of the motion cannot be found, it may be possible to investigate the stability by a topological study of equations (247). From these equations one could determine trajectories in the $\alpha\beta\underline{\gamma}$-space along which $\alpha' = 0$, $\beta' = 0$, or $\underline{\gamma}' = 0$. From a study of these trajectories and equations (247) it may be possible by topological arguments to show when system trajectories are stable. If none of the above methods work out, one could investigate solutions of equation (231) numerically with computer studies.

Now consider the extensions of the theory which are obtained when nonlinear forces are allowed in the one-dimensional Boltzmann equation with all the other assumptions still retained or by using the two-dimensional transverse Boltzmann equation. If the space charge force is allowed to be nonlinear, an approximation for its dependence on some of the higher moments may be needed. If the two-dimensional Boltzmann equation is used, then moments with respect to position and momentum in both transverse directions are needed. In all these cases a closed set of moment equations could be obtained if appropriate approximations could be made but there would then be a larger number of moment equations than before. This set of equations would probably be too difficult to handle analytically but a numerical investigation with a computer could be made.

Finally, consider the situation which results when the assumptions in the previous chapters are retained except that the number of particles is allowed to vary. Equation (83) is then replaced by the following one-dimensional Boltzmann equation:

$$\frac{\partial G}{\partial t} + \frac{p}{m \gamma_o} \frac{\partial G}{\partial x} + F \frac{\partial G}{\partial p} = H \tag{248}$$

where G (x, p, t) is the distribution function and H (x, p, t) determines the time rate of change of G due to the injection or loss of particles; i.e., H (x, p, t) dx dp dt is the number of particles added to the system in time dt in the element dx dp of the phase space at the

point x, p. Let

$$G(x, p, t) = N(t)\, f(x, p, t) \tag{249}$$

where

$$N(t) = \int dx\, dp\, G$$

and

$$\int dx\, dp\, f = 1 .$$

N (t) is the same as defined in Chapter III but now it may vary in time and f is again a probability distribution function. Substituting (249) in (248), dividing by N, and rearranging terms gives

$$\frac{\partial f}{\partial t} + \frac{p}{m\gamma_o}\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial p} = \frac{1}{N} H - \frac{1}{N}\frac{dN}{dt} f .$$

Changing from t to θ by the relation $v_o t = R\theta$ and using the same symbols f, N, and H after the change, the above equation becomes

$$\frac{\partial f}{\partial \theta} + K_1 p \frac{\partial f}{\partial x} + \frac{R}{v_o} F \frac{\partial f}{\partial p} = \frac{R}{v_o}\frac{1}{N} H - \frac{N'}{N} f \tag{250}$$

where $N' = dN/d\theta$. This equation is the same as equation (126) from which the moment equations were obtained in Chapter IV except that now there are additional terms on the right-hand side of the equation.

So the moment equations will be the same as before except there will be additional terms on the right-hand side and the factor K_2 (cf. (166)) for the space charge force will be a function of θ through the dependence of K_2 on $N(\theta)$. Let L_{jk} be the jk^{th} moment of the right-hand side of equation (250):

$$L_{jk} = \frac{R}{v_o} \frac{1}{N} \int dx\, dp\, x^j\, p^k\, H - \frac{N'}{N} M_{jk}$$

$$= \frac{1}{N} H_{jk} - \frac{N'}{N} M_{jk} \tag{251}$$

where

$$H_{jk} = \frac{R}{v_o} \int dx\, dp\, x^j\, p^k\, H\,. \tag{252}$$

The moment equations (171) are now modified by placing L_{jk} with the proper values of j and k on the right-hand side. There must be added to these the equation for $j = k = 0$ which is

$$M'_{o,o} = \frac{1}{N} H_{o,o} - \frac{N'}{N} M_{o,o}$$

or, since $M_{o,o} = 1$,

$$N' = H_{o,o}\,.$$

The new set of moment equations can be used to investigate the

effects of the injection process. In this case $H(x, p, \theta)$ is a specified function. Equation (253) can then be solved for $N(\theta)$ which can then be inserted in the remaining moment equations. The quantities H_{jk} are known and the modified moment equations still form a closed set. This set of equations would probably be too difficult to handle analytically but computer studies could be made.

The new moment equations perhaps could be used also to investigate the effect of the loss of particles when their amplitude of betatron oscillations exceeds the stability limit. As an approximation assume a particle is lost instantaneously when its position x becomes equal to X_1 or X_2 where X_1 and X_2 are fixed positions on each side of the equilibrium orbit. It can be shown then that

$$H(x, p, \theta) = \frac{2}{m \gamma_o} N p f \left[\delta(x - X_1) + \delta(x - X_2) \right] \tag{254}$$

and that consequently

$$H_{jk} = 2 K_1 N(\theta) \sum_i X_i^j \int dp\, p^{k+1} f(X_i, p, \theta) . \tag{255}$$

In general, the H_{jk} are functions of all the moments and hence the new moment equations do not form a closed set. Perhaps it may be possible to approximate the H_{jk} by assuming a definite form for f in (255) which depends on only the first few moments so that the set of moment equations becomes closed. This would certainly be crude but might give useful qualitative results.

Some insight into transverse space charge effects in particle accelerators has been obtained in this investigation but there are several limitations which were pointed out above. Thus there is a strong need for the theory to be extended and it is hoped that this is accomplished in the future.

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Report #649 in file.

MURA-649

ADDENDA TO MURA REPORT 649

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

TRANSVERSE SPACE CHARGE EFFECTS IN PARTICLE ACCELERATORS

Dornis C. Morin, Jr.
August 10, 1962

The following errors in MURA Report 649 should be noted.

1. p. 105: equations (165) should be

$$B_o = 0 \quad \text{and} \quad B_1 = \frac{R\,C\,k}{v_o\,w^n} = \frac{K_2}{w^n} .$$

2. p. 105: equation (166) should be

$$K_2 = \frac{R\,C\,k}{v_o} .$$

3. p. 108, line 14: the expression for K_2 should be

$$K_2 = \frac{R\,C\,k}{v_o} .$$

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