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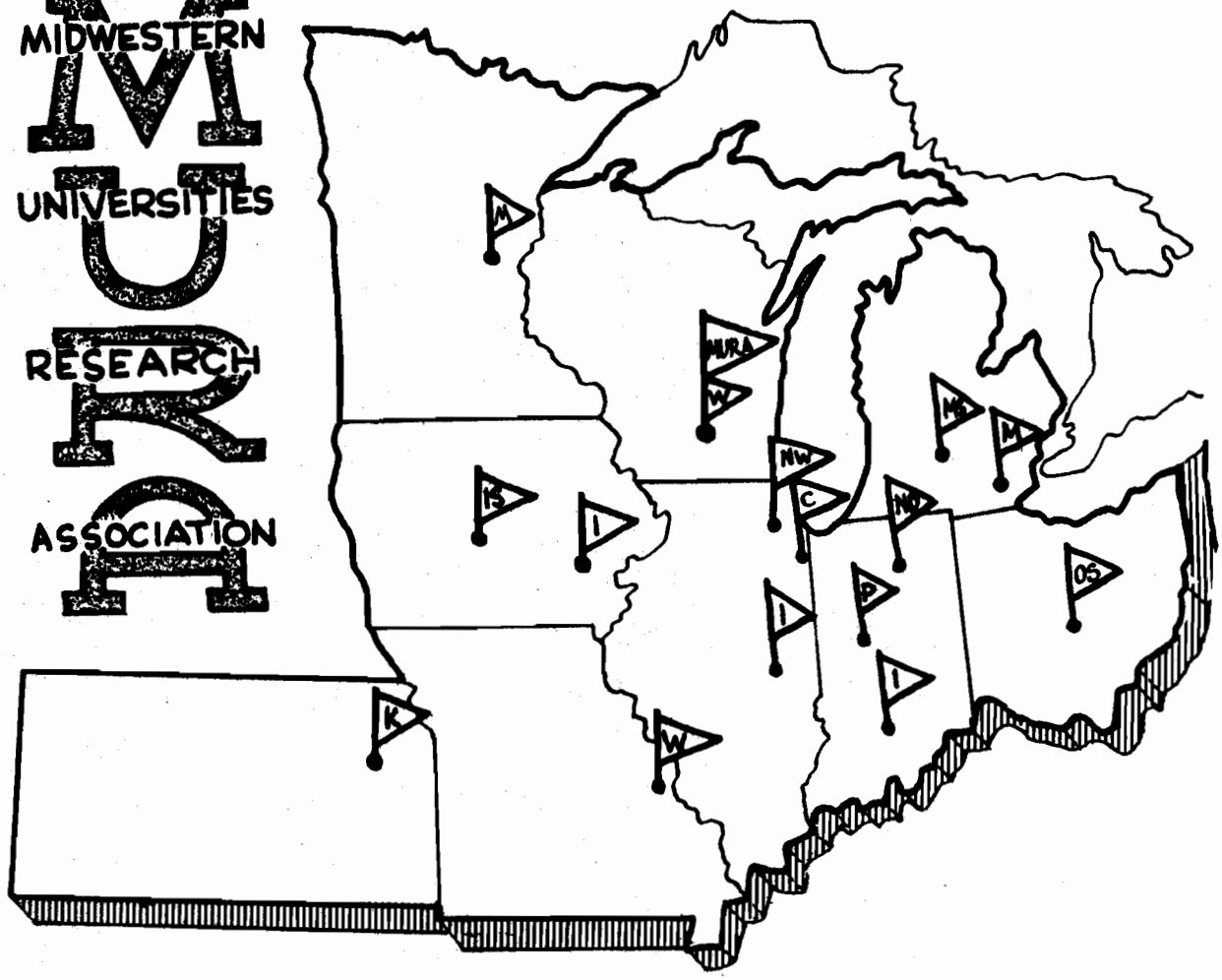


# THE EXCITATION OF A CIRCULAR CYLINDRICAL CAVITY BY A CIRCULAR WAVEGUIDE

C. Bjerke and J. Van Bladel

REPORT NUMBER 644

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# THE EXCITATION OF A CIRCULAR CYLINDRICAL CAVITY BY A CIRCULAR WAVEGUIDE

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July 20, 1962

## ABSTRACT

The problem of determining the pressure in a circular cylindrical cavity of radius $b$ connected to a circular input waveguide of radius $a$ is formulated in terms of an integral equation. Values of the reflection coefficient in the guide and of the pressure in the cavity are given in terms of the frequency and of the dimensionless ratio $a/b$. The problem of the cavity with an input and an output waveguide is also analyzed, and values of the transmission coefficient are obtained.

*AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission Contract No. AT(11-1)-384.

## I. Introduction

The two configurations of concern in this paper are shown in Figs. 1a and 1b. The first configuration is an acoustic resonator with an input waveguide in which a pressure wave of amplitude $P_o$ is incident from the left. The second configuration is the acoustic filter formed by the same cavity provided with an input and an output waveguide. In both cases it is desired to investigate the characteristics of the cavity over a frequency range which extends through the first resonant frequency. We are particularly interested in the pressure in the resonator, the reflection coefficient in the input tube, and the transmission coefficient of the acoustic filter. More specifically, we want to determine the behavior of these various quantities as the radius of the waveguide approaches zero. The observation of this limiting process will give an idea of the accuracy of an approximate method of analysis given elsewhere.[1]

## II. The Integral Equation for the Velocity in the Aperture

In the frequency range of interest the input tube can propagate only one mode, the lowest one, and the other modes are attenuated. The lowest mode is characterized by a constant pressure throughout the cross section of the tube. This property implies that the $\varphi$-independent modes are the only modes to be excited in the waveguide and in the resonant cavity. The general form of the pressure in the input waveguide is therefore

$$\frac{p(r,z)}{P_0} = e^{-jkz} + \underbrace{K e^{jkz}}_{\text{reflected wave}} + \underbrace{\sum_{m=1}^{\infty} B_m J_0\left(x_m \frac{r}{a}\right) e^{\left(\frac{x_m^2}{a^2} - k^2\right)^{\frac{1}{2}} z}}_{\text{distortion around the aperture}} \tag{1}$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ , $K$ is the reflection coefficient in the input tube, and $x_m$ is a zero of $J_1(x)$ . In the aperture:

$$\frac{1}{P_0} \frac{\partial p}{\partial z}\bigg|_0 = -jk(1-K) + \sum_{m=1}^{\infty} B_m \left(\frac{x_m^2}{a^2} - k^2\right)^{\frac{1}{2}} J_0\left(x_m \frac{r}{a}\right) \tag{2}$$

The quantity $\frac{1}{P_0} \frac{\partial p}{\partial z}\Big|_0$ will be denoted by $f(r)$ . It is the main unknown of our problem, and is proportional to the normal component of the velocity in the aperture.[2] In terms of $f(r)$ , the pressure in the aperture is, on the waveguide side,

$$\frac{p(r,0)}{P_0} = 2 + \frac{2}{jka^2}\int_0^a f(r)\, r\, dr + \sum_m J_0\left(x_m \frac{r}{a}\right) \frac{\int_0^a f(r') J_0\left(x_m \frac{r'}{a}\right) r' dr'}{\left(\frac{x_m^2}{a^2} - k^2\right)^{\frac{1}{2}} \int_0^a J_0^2\left(x_m \frac{r'}{a}\right) r' dr'} \tag{3}$$

We now turn our attention to the pressure in the resonator. The $\varphi$ -independent Neumann functions for the resonator are

$$\phi_{ns} = \cos \frac{n\pi z}{L} J_0\left(x_s \frac{r}{b}\right)$$

The corresponding "resonant" wave numbers are given by

$$k_{ns}^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{x_s}{b}\right)^2$$

The lowest mode corresponds to either the $r$ -independent eigenfunction $\phi_{10} = \cos \frac{\pi z}{L}$ (with $k = \frac{\pi}{L}$ ) or the $z$ -independent eigenfunction

$\phi_{01} = J_0(3.8\,\frac{r}{b})$ (with $k = \frac{3.8}{b}$ ). The pressure in the circular cylindrical cavity can be evaluated by the usual waveguide methods, complemented by the requirements that

1. $\frac{\partial p}{\partial z} = f(r)$ for $r \le a$, $\frac{\partial p}{\partial z} = 0$ for $r > a$ at $z = 0$

2. $\frac{\partial p}{\partial z} = 0$ at $z = L$

Calculations similar to those performed for the input waveguide yield

$$p(r,z) = a_0 e^{-jkz} + a_0 e^{-2jkL} e^{jkz} + \sum_{m=1}^{\infty} a_m J_0\left(x_m \frac{r}{b}\right)\left[e^{-\sqrt{\frac{x_m^2}{b^2}-k^2}\,z} + e^{\sqrt{\frac{x_m^2}{b^2}-k^2}\,z - 2\sqrt{\frac{x_m^2}{b^2}-k^2}\,L}\right] \tag{4}$$

where

$$a_0 = \frac{2\int_0^a f(r')\,r'\,dr'}{jkb^2\left(e^{-2jkL}-1\right)}$$

$$a_m = \frac{\int_0^a f(r')\,J_0\left(x_m \frac{r'}{b}\right) r'\,dr'}{\sqrt{\frac{x_m^2}{b^2}-k^2}\cdot\left(e^{-2\sqrt{\frac{x_m^2}{b^2}-k^2}\,L}-1\right)\int_0^b J_0^2\left(x_m \frac{r'}{b}\right) r'\,dr'} \tag{5}$$

In the aperture, in particular,

$$p(r,0) = \frac{2\left(e^{-2jkL}+1\right)\int_0^a f(r')r'\,dr'}{\left(e^{-2jkL}-1\right)jkb^2} + \sum_m J_0\left(x_m \frac{r}{b}\right) \frac{\left(e^{-2\sqrt{\frac{x_m^2}{b^2}-k^2}\,L}+1\right)\int_0^a f(r')J_0\left(x_m\frac{r'}{b}\right)r'\,dr'}{\sqrt{\frac{x_m^2}{b^2}-k^2}\left(e^{-2\sqrt{\frac{x_m^2}{b^2}-k^2}\,L}-1\right)\int_0^b J_0^2\left(x_m\frac{r'}{b}\right)r'\,dr'} \tag{6}$$

Equating the right-hand side members of Eqs. (3) and (6) gives the desired integral equation for $f(r)$ .

## III. Some Numerical Results

The integral equation for $f(r)$ has been solved on an IBM 704 computer for a few representative values of the parameters. One of the interesting quantities to plot is the reflection coefficient $K$ in the input tube. From Eq. (3), $K$ is given by the relationship

$$K = 1 + \frac{2}{jka^2}\int_0^a f(r)\,r\,dr = 1 - \frac{\rho_0 c}{P_0}\,v_{av} \tag{7}$$

Figure 2 gives some typical results for a cavity whose ratio $L/b$ is equal to 2. The lowest two modes for this cavity correspond to $x_s = 0$ and to $n=1$ and $n=2$ respectively. The general trend of the results for decreasing values of $a/b$ confirms the predictions of Fig. 4 of I . Curves for other values of the parameters show the same general aspect, and are not reproduced here. Figure 3 shows the frequency variation of the average air velocity in the opening, again for $L/b = 2$ . It is interesting to check how the slope of these curves at $f_{10}$ compares with the value predicted by the "small aperture" approximation. The latter yields

$$\frac{v_{av}}{P_0/\rho_0 c} = \frac{g_{av}}{k_1} = \frac{k-k_1}{k_1} m(\varepsilon) = \frac{f-f_{10}}{f_{10}} 2\pi \frac{b^2}{a^2} \tag{8}$$

for the mode $\phi_{10} = \sqrt{\frac{2}{\pi b^2 L}} \cos \frac{\pi z}{L}$ . The following table presents values calculated from Eq. (8) and actual values obtained from the curves of Fig. 3.

| $a/b$ | Calculated | Curves |
|---|---|---|
| 1 | 6.28 | 5.65 |
| 0.75 | 11.15 | 10.25 |
| 0.5 | 25.12 | 25. |

The validity of the approximation clearly improves as $a/b$ approaches zero, and is already quite good for $a/b = 0.5$ .

Another important property which should be investigated is the frequency variation of the pressure in the resonator, and particularly the possibility of resonance phenomena. In Fig. 4 the pressure at point $P$ (see Fig. 1) is plotted for $L/b = 2$ and for several values of the ratio $a/b$ . The apparent $Q$ of the cavity is difficult to read from the figure for $a/b = 0.1$ , but for $a/b = 0.75$ and $0.5$ the curves indicate $Q = 2.2$ and $5.7$ respectively. The "small aperture" approximation predicts a value of $\pi b^2/2a^2$ , that is, 2.8 and 6.28 for $a/b = 0.75$ and $a/b = 0.5$ respectively. In Fig. 5 the pressure is plotted at three points on the axis for $a/b = 0.5$ . We notice that the resonance phenomenon does not occur at

$\xi = L/2$ . The reason is obvious: the "resonant" eigenfunction has a node at that point.

## IV. Transmission Properties of a Filter

The filter of concern here is the "expansion chamber" structure shown in Fig. 1b. We are particularly interested in the transmission coefficient $T$ of that filter.[3] It is shown in the Appendix that

$$T = \frac{1}{2}(K' - K) \tag{9}$$

where $K$ is the reflection coefficient at $\xi = 0$ in the presence of a <u>rigid</u> wall at $\xi = L$ (boundary condition $\partial p/\partial \xi = 0$ ) and where $K'$ is the reflection coefficient at $\xi = 0$ in the presence of a <u>soft</u> wall at $\xi = L$ (boundary condition $p = 0$ ). The method of calculation of $K$ has been given above. The evaluation of $K'$ proceeds along similar lines. The only difference is in the form of Eq. (4), which must now be modified to take the new boundary condition into account. Both reflection coefficients in Eq. (9) have unit magnitude, and can therefore be written as

$$K = e^{j\theta}$$
$$K' = e^{j\theta'}$$

from which we deduce that

$$|T| = \left[\frac{1}{2} - \frac{1}{2}\cos(\theta - \theta')\right]^{\frac{1}{2}}$$

Curves for $\theta$ are given in Fig. 2. Corresponding results for $\theta'$ are given in Fig. 6. We notice that the "soft wall" cavity has a resonance at $f_{10}/2$ , that is, at half the lowest resonant frequency of the "rigid wall"

cavity of length $L$ . Curves for the transmission coefficient are given in Fig. 7. These curves are for $L/b = 2$ , but the results for $L/b = 1$ and $L/b = 10$ are practically the same. The mode which resonates at $f_{10}/2$ is characterized by the eigenfunction $\phi = (\pi L b^2)^{-\frac{1}{2}} \cos \frac{\pi z}{L}$

Equation (20) of Paper I then predicts the following small aperture limit of $T$:

$$T = \frac{1}{1 + j \frac{\pi}{2} \frac{b^2}{a^2} \frac{k - k_1}{k_1}}$$

For $a/b = 0.1$ , for example, the formula gives $\frac{\Delta k}{k_1} = \frac{12}{100\pi} = 0.038$ for the ratio of the $10\,db$ bandwidth to the resonant frequency. The curves, (which are rather difficult to read) give $0.0366$ for that ratio.

## APPENDIX

A wave $A e^{-jkz}$ incident from the left (Fig. 1) gives rise to a pressure $A e^{-jkz} + A K'' e^{jkz}$ in the left-hand guide, and to a pressure $A T e^{-jkz_0}$ in the right-hand guide. Similarly, a wave $B e^{jkz_0}$ incident from the right produces a pressure $B e^{jkz_0} + B K'' e^{-jkz_0}$ in the right-hand guide, and a pressure $B T e^{jkz}$ in the left-hand guide. In the presence of both incident waves, the pressures are

$$A e^{-jkz} + A K'' e^{jkz} + B T e^{jkz} \quad \text{in the left arm} \tag{10}$$

$$A T e^{-jkz_0} + B e^{jkz_0} + B K'' e^{-jkz_0} \quad \text{in the right arm.} \tag{11}$$

If $A = B$, the pressure is symmetric with respect to the median plane $z = L$. It follows that $\partial p / \partial z = 0$ in that plane. Under these circumstances, the reflection coefficient in the left-hand tube is $K'$. But Eq. (10) implies that this reflection coefficient is equal to $K'' + T$. In consequence,

$$K'' + T = K' \tag{12}$$

If $A = -B$, the pressure is anti-symmetric with respect to the median plane $z = L$, hence $p = 0$ there, and the reflection coefficient in the left-hand tube $K$. From Eq. (10), therefore,

$$K'' - T = K \tag{13}$$

Subtraction of corresponding members of Eqs. (12) and (13) gives the desired relationship, Eq. (9).

## REFERENCES

1. J. Van Bladel, "The Excitation of a Resonant Cavity by Waveguides of Small Cross-Sectional Dimensions," MURA Report 637. This paper will henceforth be referred to as I.

2. The actual relationship is $v_z = -\frac{1}{jk\rho_0 c}\frac{\partial p}{\partial z}$, where $\rho_0$ is the specific mass of air, and $c$ is the velocity of sound.

3. If a unit pressure wave $e^{-jkz}$ is incident in the left-hand tube, $T$ is a complex number such that $T e^{-jkz_0}$ is the transmitted wave in the right-hand tube.

## LIST OF CAPTIONS

Fig. 1. Acoustic resonator and acoustic filter.

Fig. 2. Argument of the reflection coefficient in the input waveguide for a "rigid wall" resonator.

Fig. 3. Average air velocity in the coupling aperture.

Fig. 4. Pressure at a point in the resonator. Resonance phenomena.

Fig. 5. Pressure at various points along the axis of the resonator.

Fig. 6. Argument of the reflection coefficient in the input waveguide for a "soft wall" resonator.

Fig. 7. Magnitude of the transmission coefficient of an acoustic filter.

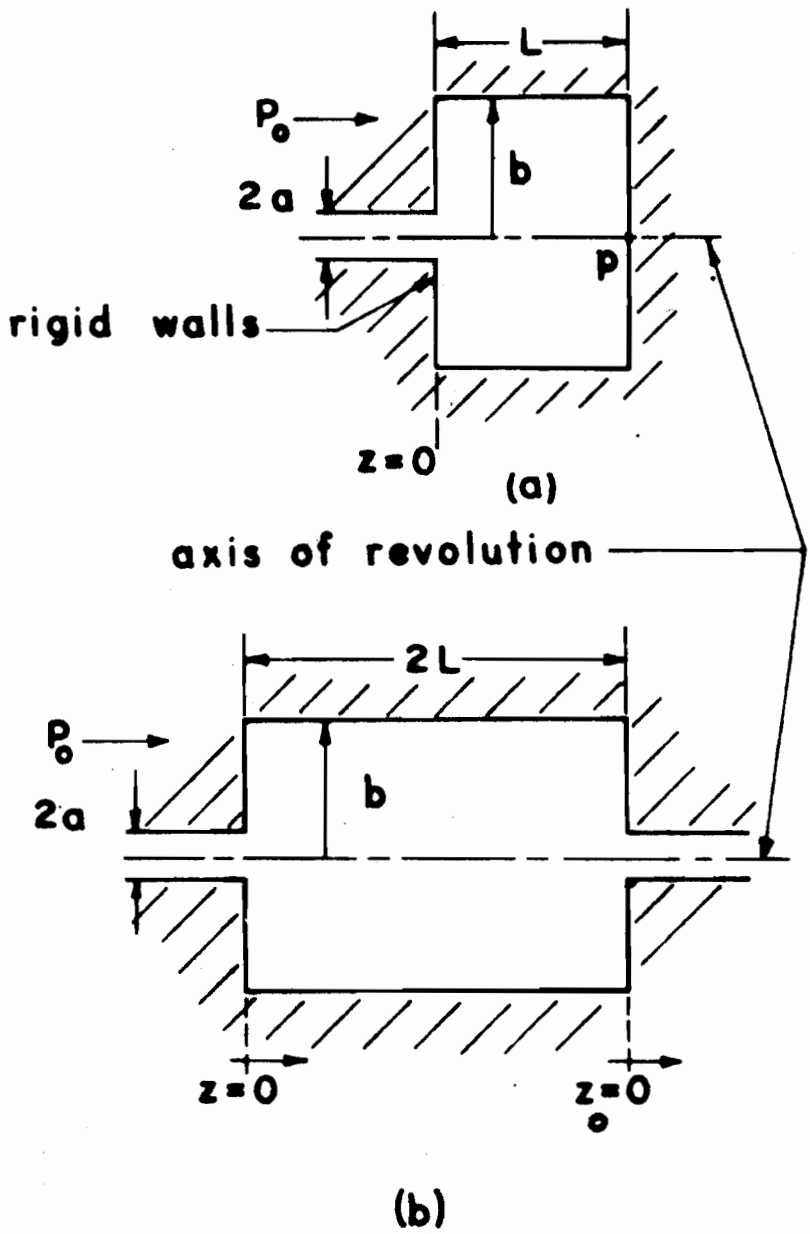
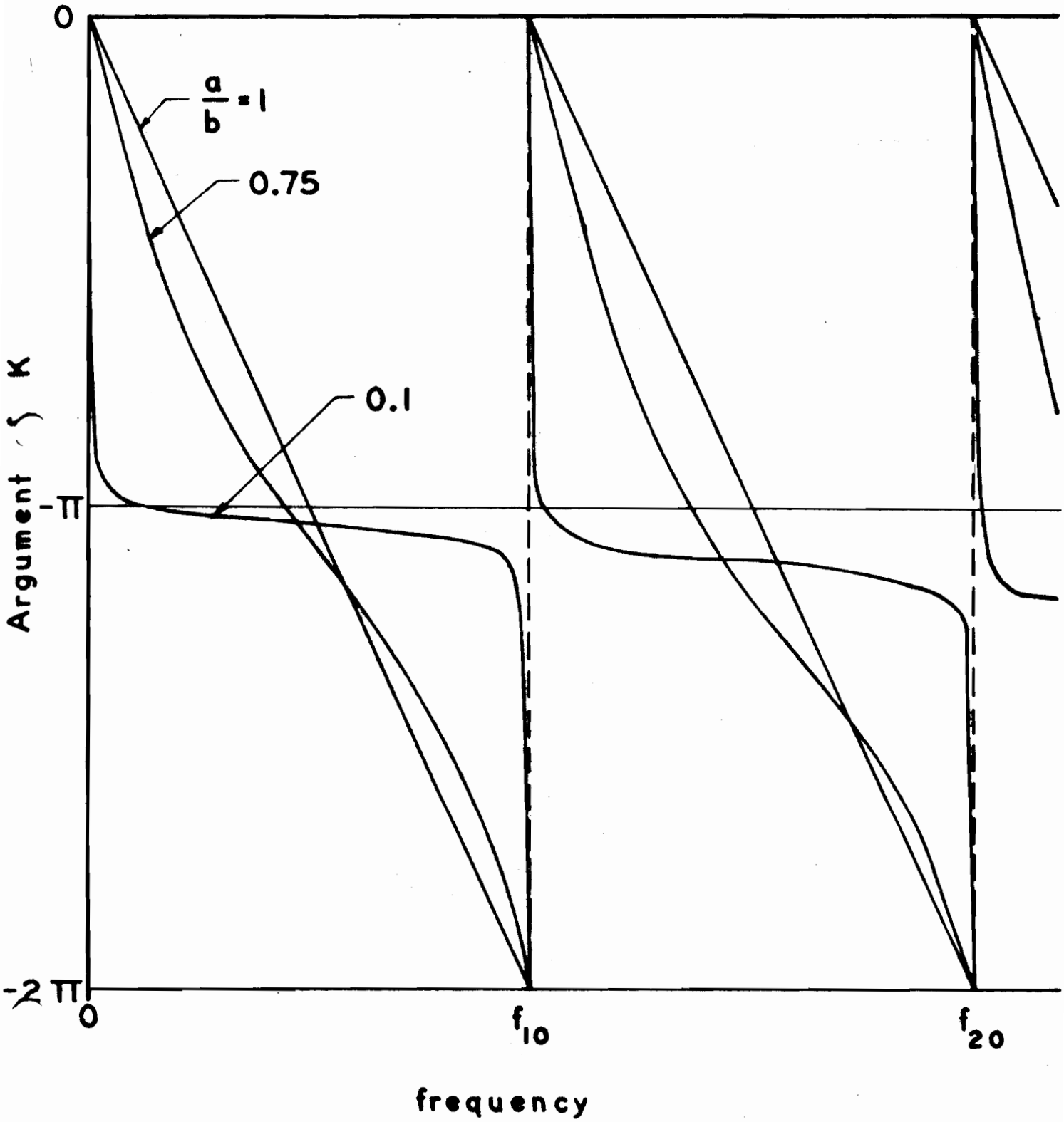
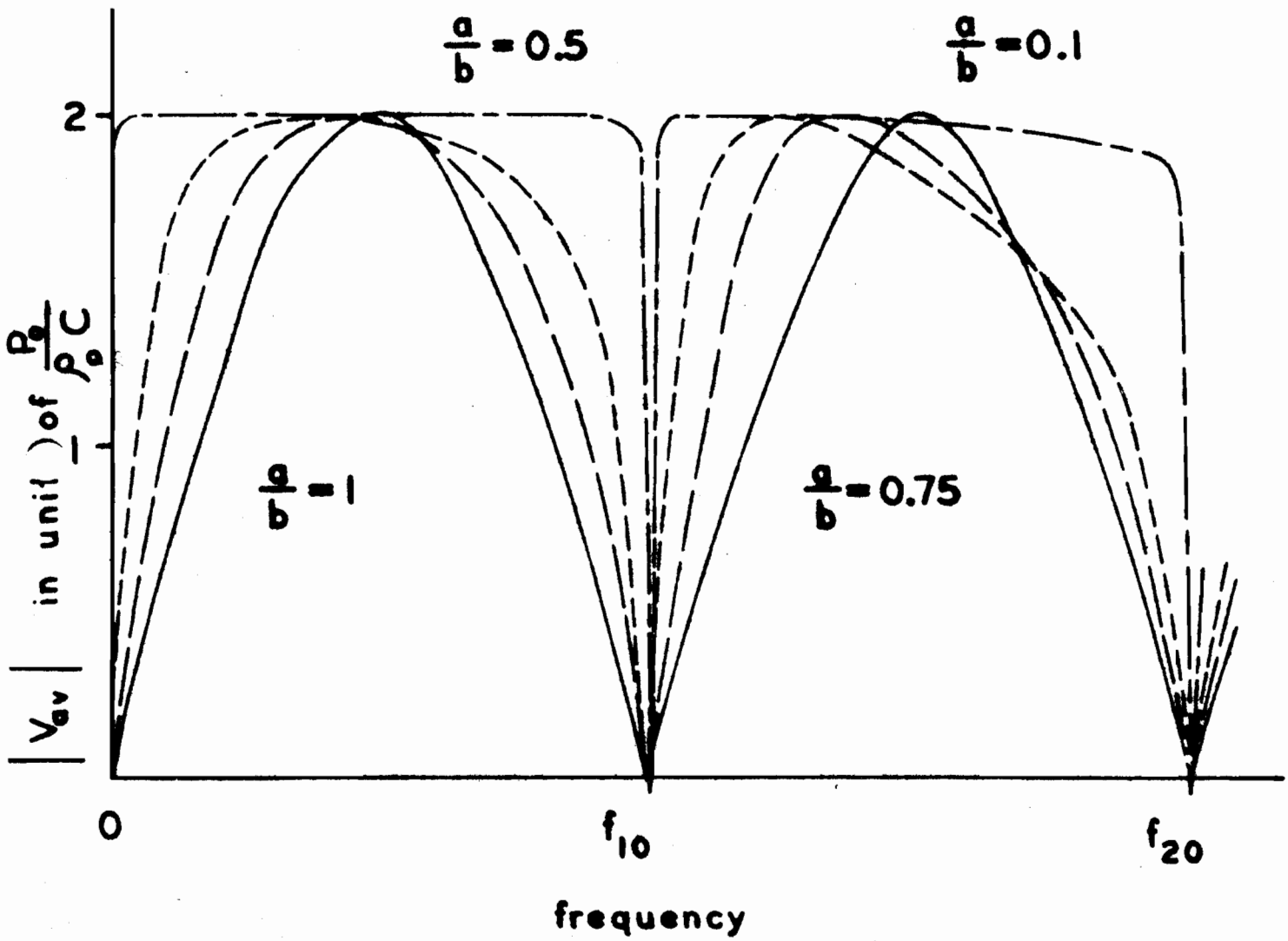



FIG 1

FIG 2





$\frac{a}{b} = 0.5$
$\frac{a}{b} = 0.1$
$\frac{a}{b} = 1$
$\frac{a}{b} = 0.75$
2
1
$|V_{av}|$ in unit of $\frac{P_0}{\rho_0 C}$
0
$f_{10}$
$f_{20}$
frequency


FIG. 3

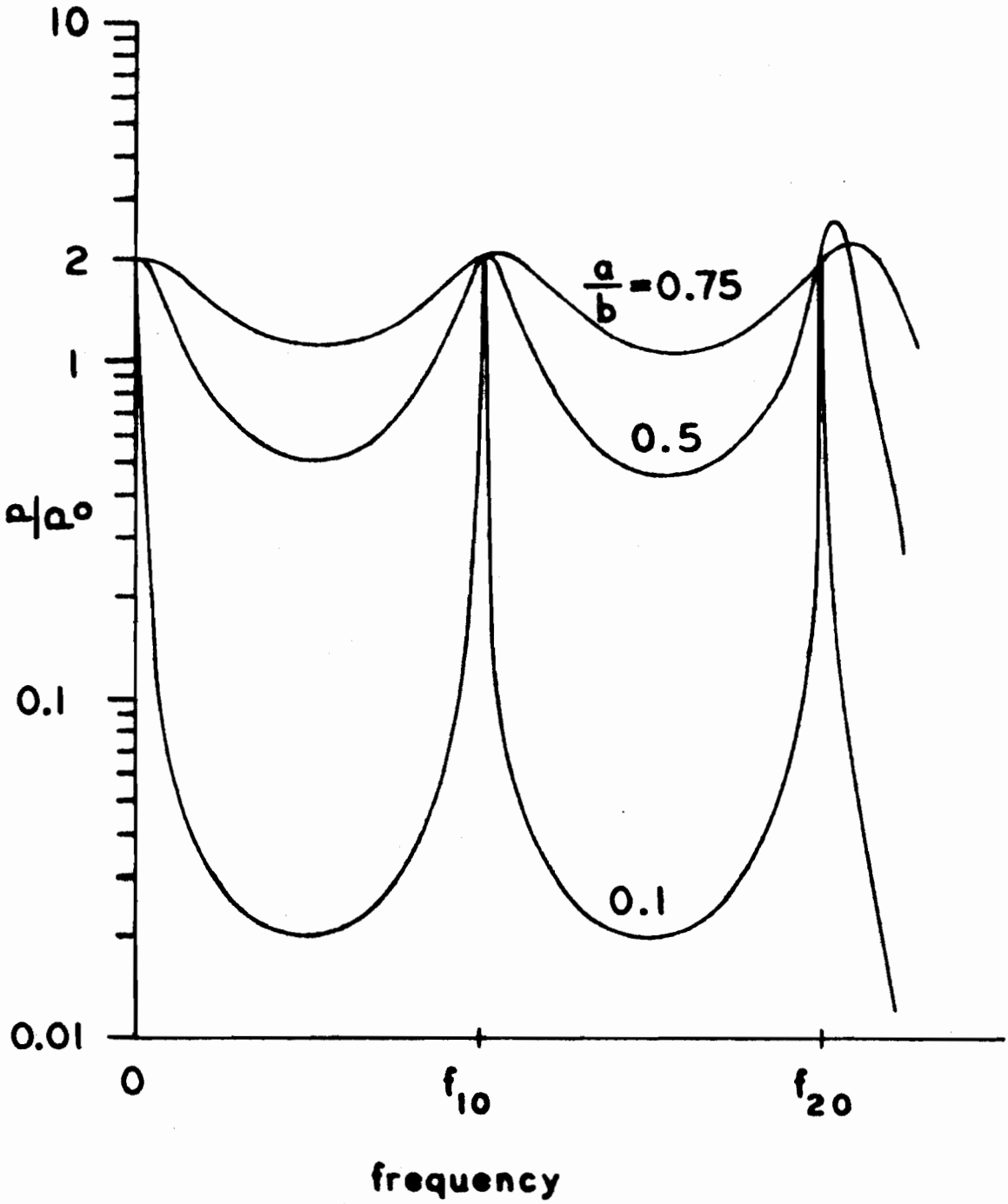



FIG. 4

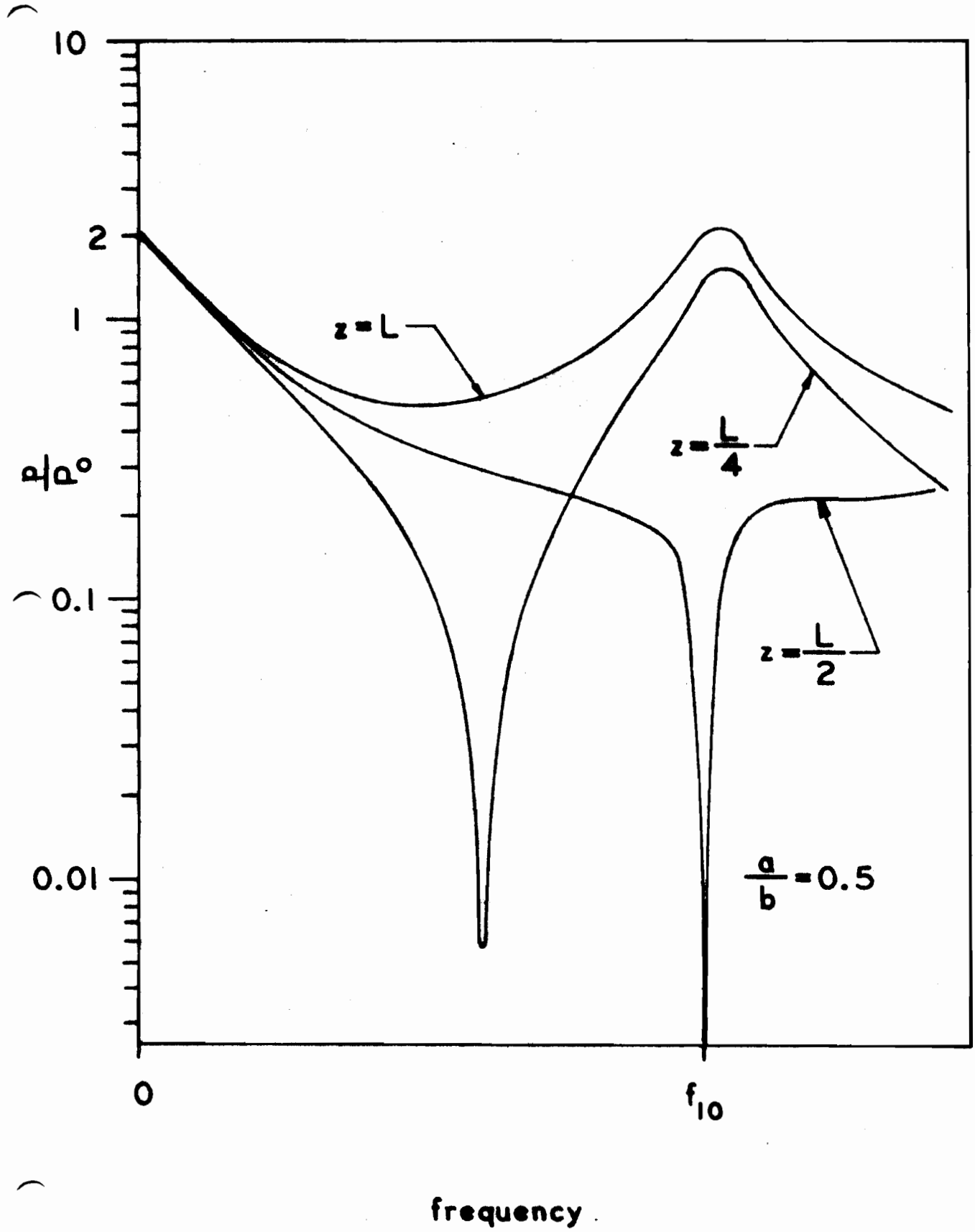



FIG. 5

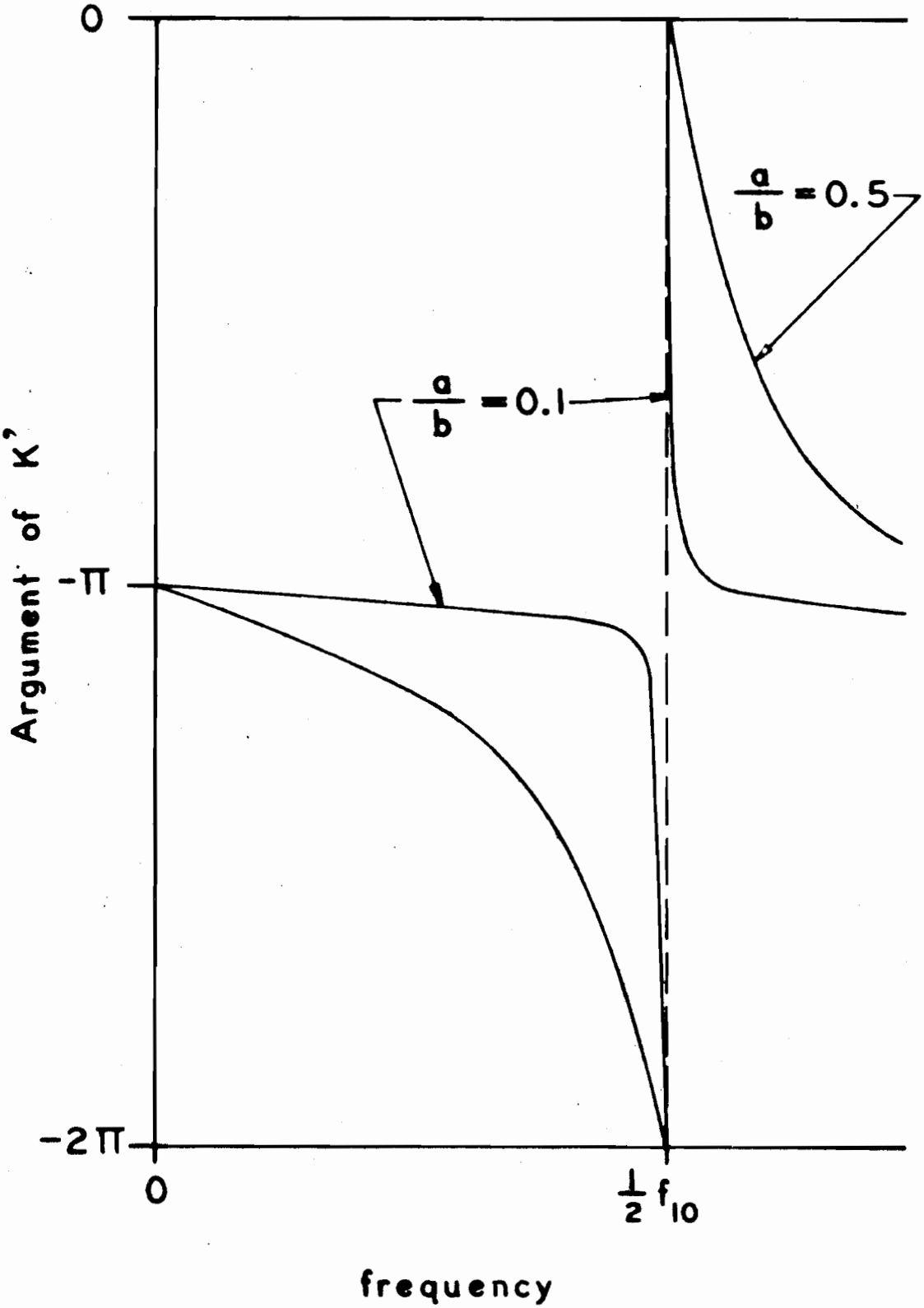



FIG 6

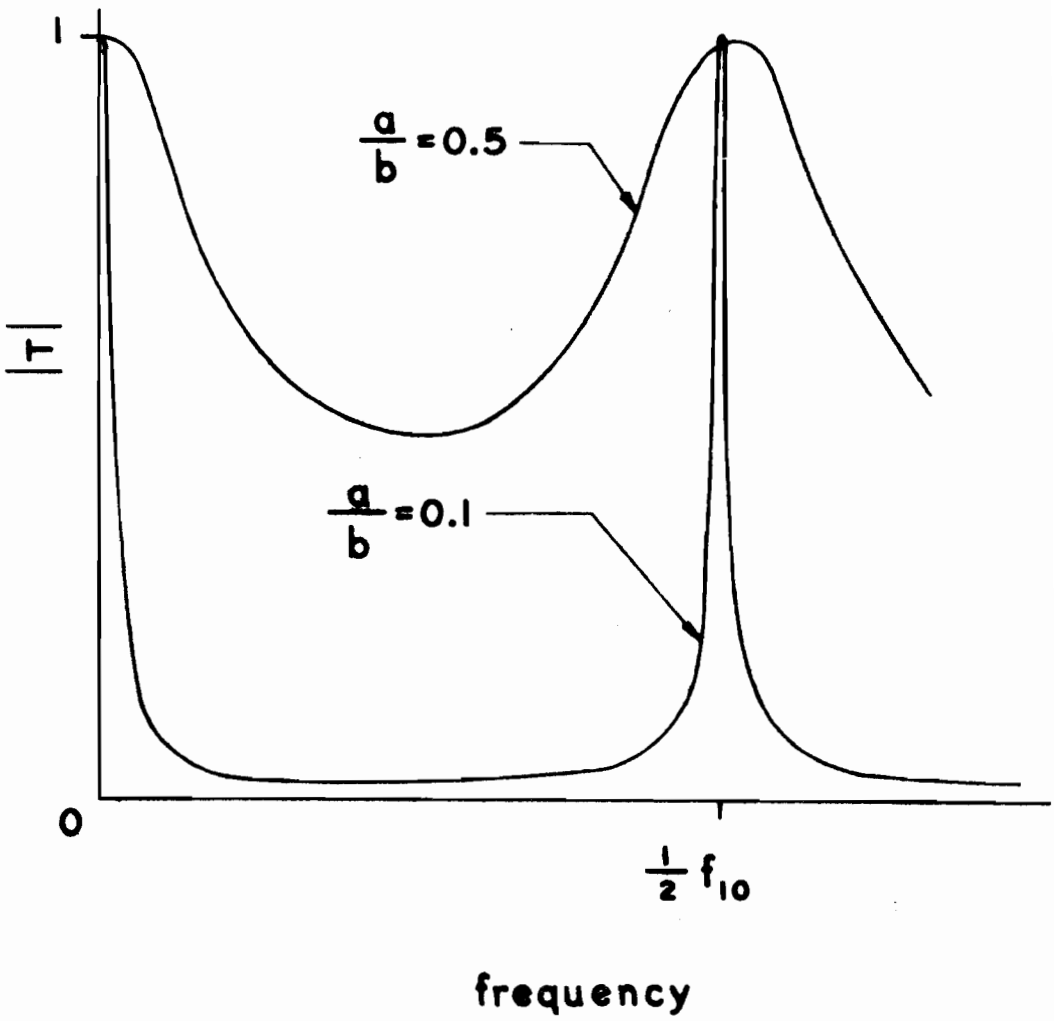



FIG. 7