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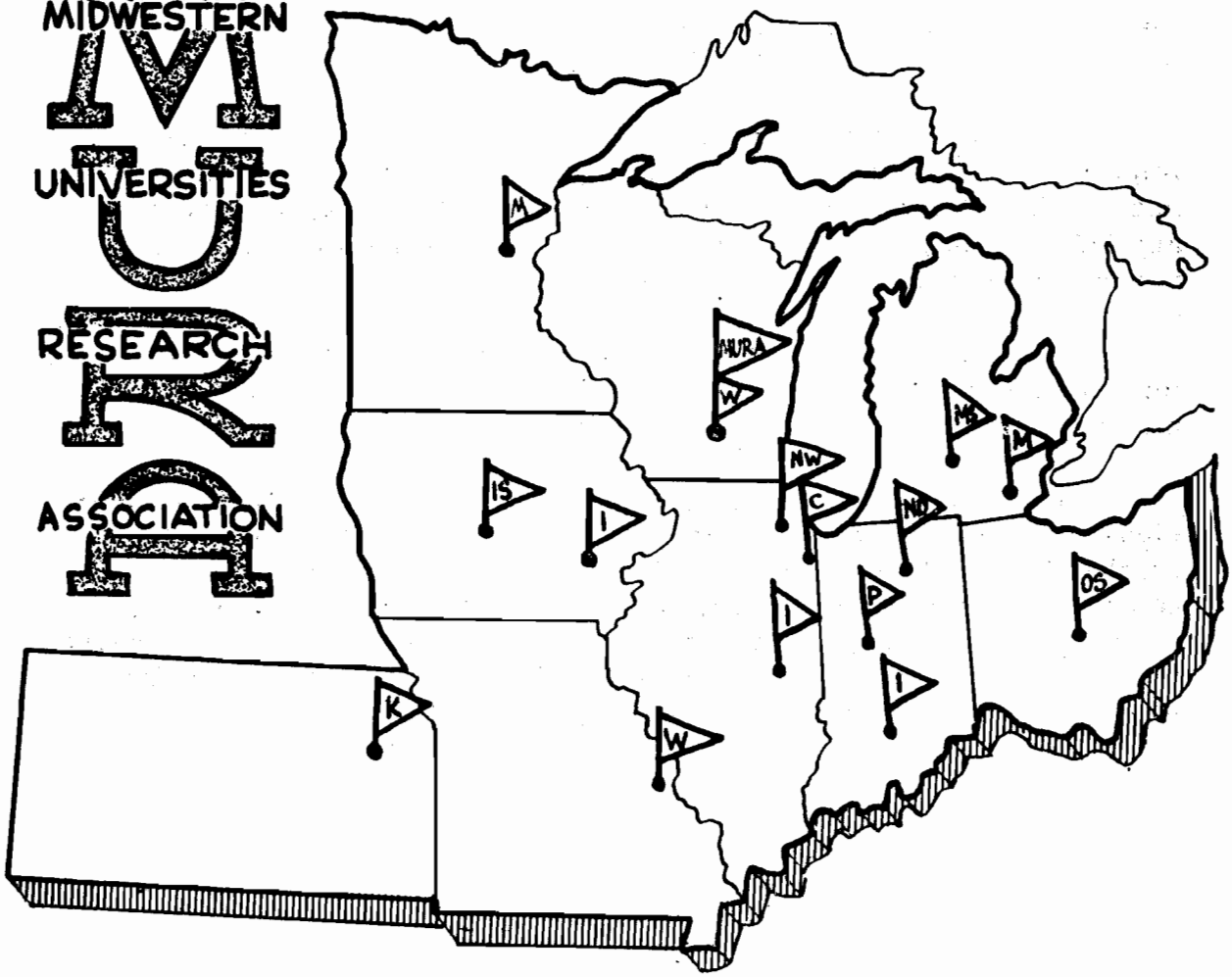

PERTURBATION THEORY FOR ACCELERATORS WITH A GENERAL MAGNETIC FIELD

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PERTURBATION THEORY FOR ACCELERATORS WITH A GENERAL MAGNETIC FIELD

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ABSTRACT

The problem of calculating the effect of field perturbation on the linear orbit properties of an accelerator with a general magnetic field is treated. The perturbing magnetic field in the median plane is assumed to have only a vertical component and to have the same periodicity as the unperturbed field. Results are found for the change in the equilibrium orbit, and in the betatron oscillation frequencies caused by the perturbation.

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I. Introduction

This paper treats the problem of calculating the effect of field perturbations on the linear orbit properties of an accelerator with a general magnetic field. We will consider the case where only $\Delta H_z \neq 0$ in the median plane.

We write the field in the median plane as

$$\overline{H}_z = H_z + \Delta H_z \tag{1.1a}$$

$$H_z = - \sum_n G_n (r)\, e^{in\theta} \tag{1.1b}$$

$$\Delta H_z = - \sum_n \Delta G_n (r)\, e^{in\theta} \tag{1.1c}$$

$$\overline{H}_z = - \sum_n \overline{G}_n (r)\, e^{in\theta} \tag{1.1d}$$

$$\overline{G}_n = G_n + \Delta G_n \,. \tag{1.1e}$$

The unperturbed field H_z has the periodicity $2\pi/N$, so that $G_n = 0$ unless $n = 0, \pm N, \pm 2N \cdots$.

The effect of the perturbing field ΔH_z may be broken into two parts, the effect due to the harmonics ΔG_n for which $n = 0, \pm N, \pm 2N \cdots$, and the effect due to the harmonics ΔG_n for which $n \neq 0, \pm N, \pm 2N \cdots$.

In this paper we will consider the effect of the harmonics for which $n = 0, \pm N, \pm 2N \cdots$, and the effect of the harmonics for which $n \neq 0, \pm N, \pm 2N \cdots$ will be treated in a later paper.

We will also make the same assumptions about the unperturbed field as were made in MURA-397.

II. Effect of the Harmonics $n = 0, \pm N, \pm 2N \cdots$

We treat the effect of the harmonics of the perturbing field corresponding to $n = 0, \pm N, \pm 2N \cdots$.

The Equilibrium Orbit

The r-equation of motion may be written (see MURA-397)

$$r'' - r = \frac{er^2}{pc} \overline{H}_z . \tag{2.1}$$

We put $r = r_s(\theta) + \rho$, where $r_s(\theta)$ is the equilibrium orbit of the unperturbed field corresponding to the same momentum p. $r_s(\theta)$ is given in MURA-397, 454.

Then ρ will satisfy

$$\rho'' - \rho = \frac{er^2}{pc} \overline{H}_z - \left(\frac{er^2}{pc} H_z\right)_{r_s} \tag{2.2}$$

where $(H_z)_{r_s}$ means $H_z(r, \theta)$ is evaluated at $r = r_s(\theta)$. We expand the right side of (2.2) in powers of ρ, keeping up to linear terms in ρ and find

$$\rho'' - \rho = \left[\frac{er^2}{pc} \Delta H_z\right]_{r_s} + \left[\frac{\partial}{\partial r}\left(\frac{er^2}{pc} H_z\right)\right]_{r_s} \rho , \tag{2.3a}$$

or

$$\overline{x}'' + \overline{n}(\theta)\overline{x} = \Delta f(\theta) , \tag{2.3b}$$

where

$$\overline{x} = \rho/R, \tag{2.4a}$$

$$\begin{aligned}
\overline{n}(\theta) &= - \frac{\partial}{\partial r}\left[\left(\frac{er^2}{pc} \overline{H}_z\right)\right]_{r_s} - 1, \\
&= - \frac{er_s}{pc}\left[r \overline{H}_{z,r} + 2\overline{H}_z\right]_{r_s} - 1 , \\
&= - \frac{eR}{pc}\Big[(r \overline{H}_{z,r} + 2\overline{H}_z)_R + x\,(R^2 H_{z,rr} \\
&\qquad + 4R\overline{H}_{z,r} + 2\overline{H}_z)_R\Big] - 1 ,
\end{aligned} \tag{2.4b}$$

$$\Delta f(\theta) = \frac{1}{R}\left[\frac{er^2}{pc}\ \Delta H_z\right]_{r_s} \tag{2.4c}$$
$$= \frac{eR}{pc}\left[\Delta H_z + x\,(r\,\Delta H_{z,r} + 2\,\Delta H_z)\right]_R .$$

R is the average radius of the unperturbed orbit, and we have put $r_s(\theta) = R(1 + x)$ and expanded in powers of x to obtain the last forms of $\bar{n}(\theta)$ and $\Delta f(\theta)$.

To solve equation (2.3), we write it as

$$\bar{x}'' + (\bar{E}_s - \bar{g}(\theta)\,\bar{x} = \Delta f(\theta)\,, \tag{2.5}$$

where $\bar{n}(\theta) = \bar{E}_s - \bar{g}(\theta)$, $\bar{E}_s$ is the average value of $\bar{n}(\theta)$, and we expand x in the set of functions $\bar{u}_n(\theta)$ which are the solutions of the equation

$$\frac{d^2\,\bar{u}_n}{d\theta^2} + (\bar{E}_n - \bar{g}(\theta))\,\bar{u}_n = 0. \tag{2.6}$$

The E_n are chosen so that the Floquet functions $\bar{u}_n(\theta)$ have the tune $\nu = n$ where $n = 0, \pm N, \pm 2N \cdots$. We write

$$\bar{x}(\theta) = \sum a_n\ \bar{u}_n\,, \tag{2.7}$$

where the a_n are given by

$$a_n = \frac{1}{\bar{E}_s - \bar{E}_n}\,\frac{N}{2\pi}\int_0^{2\pi/N} d\theta\ \bar{u}_n^{\,*}(\theta)\ \Delta f(\theta). \tag{2.8}$$

The $\bar{E}_n$ depend on the harmonics of $\bar{g}(\theta)$, $\bar{g}_n$. A relatively simple expression for $\bar{E}_n$ is

$$\bar{E}_n = n^2 - \sum_{m>o}^{N} \frac{|\bar{g}_m|^2}{m^2 - 4n^2}\,, \tag{2.9}$$

where $\sum_{m>o}^{N}$ means $m = N, 2N \cdots$.

We find for $\bar{E}_s$ from (2.4b)

$$\bar{E}_s = E_o + \Delta E_s\,, \tag{2.10a}$$

$$E_o = \frac{eR}{pc} (R\, G_o' + 2\, G_o) - 1 \tag{2.10b}$$

$$+ \left(\frac{eR}{pc}\right) \sum_{m \neq o}^{N} x_m \left(R^2\, G_m'' + 4\, R\, G_m' + 2\, G_m\right),$$

$$E_s = \frac{eR}{pc} (\Delta G_o' + 2\, \Delta G_o) + \tag{2.10c}$$

$$+ \left(\frac{eR}{pc}\right) \sum_{m}^{N} x_{-m} \left(R^2\, \Delta G_m'' + 4\, R\, \Delta G_m' + 2\, \Delta G_m\right)$$

where the x_m are the harmonics of the unperturbed orbit. We find for $\bar{g}_m$,

$$\bar{g}_m = g_m + \Delta g_m, \tag{2.11a}$$

$$g_m = -\frac{eR}{pc} (R\, G_m' + 2\, G_m), \tag{2.11b}$$

$$\Delta g_m = -\frac{eR}{pc} (R\, \Delta G_m' + 2\, \Delta G_m). \tag{2.11c}$$

We may note that E_o is related to the unperturbed radial tune ν_r (see MURA-397)

$$E_o = \nu_r^2 - \sum_{m > o}^{N} \frac{|g_m|^2}{m^2 - 4\, \nu_r^2} \quad . \tag{2.12}$$

If, in calculating $\bar{E}_s - \bar{E}_n$, we neglect the terms of order ΔH_z which would give an error which is second order in ΔH_z in the a_n, then we can put

$$\bar{E}_s - \bar{E}_n = E_o - E_n\,, \tag{2.13a}$$

$$\simeq \nu_r^2 - n^2\,. \tag{2.13b}$$

The $\bar{u}_n(\theta)$ are given by (see MURA-397)

$$\bar{u}_n(\theta) = e^{in\theta} \left\{ 1 - \sum_{m \neq o, -2n}^{N} \frac{\bar{g}_m}{(m+n)^2 - \bar{E}_n}\, e^{im\theta} \right\} . \tag{2.14}$$

We find then

$$\frac{N}{2\pi}\int_{o}^{2\pi/N} d\theta\ \bar{u}_n^*\ \Delta f(\theta) = \Delta f_n - \sum_{m\neq o,-2n}^{N} \frac{\bar{g}_{-m}}{(m+n)^2 - \bar{E}_n}\ \Delta f_{n+m}\,, \tag{2.15}$$

the Δf_n are the harmonics of $\Delta f(\theta)$ and are given by

$$\Delta f_n = -\frac{eR}{pc}\,\Delta G_n - \frac{eR}{pc}\sum_{m}^{N} x_{-m}\,(R\,\Delta G'_{n+m} + 2\,\Delta G_{n+m})\,. \tag{2.16}$$

Thus we have for the a_n,

$$a_n = \frac{1}{\nu_r^2 - n^2}\left\{\Delta f_n - \sum_{m\neq o,-2n}^{N} \frac{\bar{g}_{-m}}{(m+n)^2 - \bar{E}_n}\ \Delta f_{n+m}\right\} \tag{2.17a}$$

$$a_n = \frac{1}{n^2 - \nu_r^2}\left\{\frac{eR}{pc}\,\Delta G_n + \frac{eR}{pc}\sum_{m}^{N} x_{-m}\,(R\,\Delta G'_{n+m} + 2\,\Delta G_{n+m}) + \sum_{m\neq o,-2n}^{N}\left(\frac{eR}{pc}\right)^2 \frac{(R\,G'_{-m} + 2\,G_{-m})}{(m+n)^2 - E_n}\,\Delta G_{m+n}\right\}. \tag{2.17b}$$

We find then that

$$a_o = -\frac{1}{\nu_r^2}\left\{\frac{eR}{pc}\,\Delta G_o + \left(\frac{eR}{pc}\right)^2 \times \sum_{m\neq o}^{N}\left[\frac{G_{-m}}{m^2}\,(R\,\Delta G'_m + 2\,\Delta G_m) + \frac{\Delta G_{-m}}{m^2 - E_o}\,(R\,G'_m + 2\,G_m)\right]\right\} \tag{2.18a}$$

$$a_n = \frac{1}{n^2 - \nu_r^2}\,\frac{eR}{pc}\,\Delta G_n\,, \quad n \neq o\,. \tag{2.18b}$$

Since the a_n for $n \neq o$ are smaller than a_o by about $(\nu/N)^2$, we keep only the first term in (2.18b) when $n \neq o$.

The harmonics of $\bar{x}(\theta)$ may be found using (2.19) and (2.7)

$$\bar{x}(\theta) = \sum_n a_n e^{in\theta} \left\{ 1 - \sum_{m \neq o, -2n}^{N} \frac{\bar{g}_m}{(m+n)^2 - E_n} e^{im\theta} \right\} \tag{2.19a}$$

$$\bar{x}_n = a_n - \sum_{m \neq o}^{N} a_m \frac{g_{n-m}}{n^2 - E_m} . \tag{2.19b}$$

We find then, keeping only dominant terms,

$$\bar{x}_o = a_o \tag{2.20a}$$

$$\bar{x}_n = a_n - a_o \frac{g_n}{N^2} , \quad n \neq o. \tag{2.20b}$$

Since $a_o >> a_n$, we keep the second term in x_n, $n \neq o$, but not in x_o.

The Tune Shift

We now find the shift in the tune ν_r, ν_z due to the harmonics $n = o, \pm N, \pm 2N \cdots$ of the perturbing field.

To find the radial tune, we expand the r-equation of motion around the shifted equilibrium orbit found above. The r-equation may be written as

$$r'' - r = \frac{er^2}{pc} \bar{H}_z . \tag{2.21}$$

We put $r = \bar{r}_s(\theta) + \rho$, where $\bar{r}_s(\theta)$ is the perturbed equilibrium orbit, and get

$$\rho'' - \rho = \left[\frac{\partial}{\partial r} \left(\frac{er^2}{pc} \bar{H}_z \right) \right]_{\bar{r}_s} \rho , \tag{2.22}$$

or

$$\rho'' + \bar{n}_u(\theta) \rho = 0 . \tag{2.23}$$

$$\bar{n}_u(\theta) = -\left[\frac{\partial}{\partial r} \left(\frac{e}{pc} r^2 \bar{H}_z \right) \right]_{\bar{r}_s} - 1 , \tag{2.24a}$$

$$\bar{n}_u(\theta) = -\frac{e}{pc}\left[r^2 \bar{H}_{z,r} + 2\, r\, \bar{H}_z\right]_{\bar{r}_s} - 1\,, \tag{2.24b}$$

$$\bar{n}_u(\theta) = -\frac{e}{pc}\left\{(r^2 \bar{H}_{z,r} + 2\, r\, \bar{H}_z)_{r_s} + \frac{\partial}{\partial r}(r^2 \bar{H}_{z,r} + 2\, r\, \bar{H}_z)_{r_s} \cdot R\,\bar{x} + \cdots\right\} - 1\,, \tag{2.24c}$$

where in (2.24c) we expanded $\bar{n}_u$ in powers of the orbit shift $\bar{x}$.

We can also write for $\bar{n}_u$, by expanding in powers of x, where $r_s = R\,(1 + x)$ is the unperturbed orbit,

$$\begin{aligned}\bar{n}_u = -\frac{eR}{pc}\Big\{ & (R\,\bar{H}_{z,r} + 2\,\bar{H}_z)_R \\ & + x\,(R^2 \bar{H}_{z,rr} + 4\,R\,\bar{H}_{z,r} + 2\,\bar{H}_z)_R + \cdots \\ & + \bar{x}\Big[(R^2 \bar{H}_{z,rr} + 4\,R\,H_{z,r} + 2\,H_z)_R \\ & \quad + x\,(R^3 H_{z,rrr} + 6\,R^2 H_{z,rr} + 6\,R\,H_{z,r})_R + \cdots\Big] \\ & + \cdots\Big\} - 1\,.\end{aligned} \tag{2.25}$$

If we now write $\bar{n}_u$ as

$$\bar{n}_u(\theta) = \bar{E}_o - \bar{g}(\theta)\,, \tag{2.26}$$

where $\bar{E}_o$ is the average value of $\bar{n}_u(\theta)$, and

$$\bar{E}_o = E_o + \Delta E_o\,,$$

$$\bar{g}_n = g_n + \Delta g_n, \tag{2.27b}$$

$$\begin{aligned}\Delta E_o = \frac{eR}{pc}\Big\{ & (R\,\Delta G_o' + 2\,\Delta G_o)_R \\ & + \sum_m x_{-m}\,(R^2\,\Delta G_m'' + 4\,R\,\Delta G_m' + 2\,\Delta G_m)_R \\ & + \bar{x}_o\Big[(R^2 G_o'' + 4\,R\,G_o' + 2\,G_o)_R \\ & \quad + \sum_m x_{-m}\,(R^3 G_m''' + 6\,R^2 G_m'' + 6\,R\,G_m')_R\Big] \\ & + \sum_{m\neq o} \bar{x}_m\,(R^2 G_m'' + 4\,R\,G_m' + 2\,G_m)_R\Big\}\,,\end{aligned} \tag{2.27c}$$

$$\Delta g_n = -\frac{eR}{pc}\left\{ (R\,\Delta G_n' + 2\,\Delta G_n)_R + \bar{x}_o\,(R^2 G_n'' + 4\,R\,G_n' + 2\,G_n)_R \cdots \right\}. \tag{2.27d}$$

Given ΔE_o and Δg_n, the change in the r-tune may be found according to (see Appendix A)

$$\nu_r^2 - \nu_r^2 = \Delta E_o + 2\sum_{m>o}^{N} \frac{g_m\,\Delta g_{-m} + g_{-m}\,\Delta g_m}{m^2 - 4\,\nu_r^2}. \tag{2.28}$$

Equation (2.23) is correct to first order in the perturbation. More accurate expressions are given in Appendix A.

We now treat the shift in the z-tune, ν_z. The z-equation of motion may be written

$$z'' = \frac{er}{pc}\left[r'\,H_\theta - r\,H_r \right] \tag{2.29a}$$

$$= \frac{er}{pc}\left[\frac{r'}{r}\,H_{z,\theta} - r\,H_{z,r} \right] \cdot z\,. \tag{2.29b}$$

We can write the z-equation as

$$z'' + \bar{n}_z(\theta)\,z = 0, \tag{2.30a}$$

$$\bar{n}_z(\theta) = -\frac{e}{pc}\left[r'\,\bar{H}_{z,\theta} - r^2\,\bar{H}_{z,r} \right] \bar{r}_s\,, \tag{2.30b}$$

$$\bar{n}_z(\theta) = -\frac{e}{pc}\left\{ (r'\,\bar{H}_{z,\theta} - r^2\,H_{z,r})_{r_s} + \frac{\partial}{\partial r}(r'\,H_{z,\theta} - r^2\,H_{z,r})_{r_s}\,R\,\bar{x} + \frac{\partial}{\partial r'}(r'\,H_{z,\theta} - r^2\,H_{z,r})_{r_s}\,R\,\bar{x}' + \cdots \right\} \tag{2.30c}$$

where in Eq. (2.30) we have expanded $\bar{n}_z$ in powers of the orbit shift $\bar{x}$.

We expand now the coefficients of this expansion of $\bar{n}_z$ in powers of x, where $r_s = R\,(1 + x)$ is the unperturbed orbit,

$$\bar{n}_z = -\frac{eR}{pc}\Big\{ (x' \bar{H}_{z,\theta} - R\,\bar{H}_{z,r})_R \tag{2.31}$$
$$+ x\,(-R^2 H_{z,rr} - 2R\,H_{z,r})_R$$
$$+ \bar{x}\Big[(-R^2 H_{z,rr} - 2R\,H_{z,r})_R$$
$$+ x\,(-R^3 H_{z,rrr} - 4R^2 H_{z,rr} - 2R\,H_{z,r})_R$$
$$+ x'\,(R\,H_{z,\theta r})_R + \cdots\Big]$$
$$+ \bar{x}'\Big[(H_{z,\theta})_R$$
$$+ x'\,(R\,H_{z,\theta r})_R + \cdots\Big]$$
$$+ \cdots \Big\}.$$

If we now write $\bar{n}_z$ as

$$\bar{n}_z(\theta) = \bar{E}_o' - \bar{f}(\theta), \tag{2.32}$$

where $\bar{E}_o'$ is the average value of $\bar{n}_z(\theta)$ and

$$\bar{E}_o' = E_o' + \Delta E_o' \tag{2.33a}$$

$$\bar{f}_n = f_n + \Delta f_n \tag{2.33b}$$

$$\Delta E_o' = \frac{eR}{pc}\Big\{ R\,(\Delta G_o')_R \tag{2.33c}$$
$$+ \sum_m x_{-m}\,(-\omega_m^2\,\Delta G_m + R^2\,\Delta G_m'' + 2R\,\Delta G_m')_R$$
$$+ \bar{x}_o\Big[(R^2 G_o'' + 2R\,G_o')_R$$
$$+ \sum_m x_{-m}\,(R^3 G_m''' + 4R^2 G_m'' + 2R\,G_m' - \omega_m^2\,G_m')_R\Big]$$
$$+ \sum_{m\neq o} \bar{x}_{-m}\,(R^2 G_m'' + 2R\,G_m' - \omega_m^2\,G_m)_R\Big\}$$

$$\Delta f_n = \frac{eR}{pc}\Big\{ R\,(\Delta G_n')_R \tag{2.33d}$$
$$+ \bar{x}_o\,(R^2 G''_n + 2R\,G'_n)_R + \cdots \Big\}.$$

The change in the z-tune may then be calculated from

$$\bar{\nu}_z^2 - \nu_z^2 = \Delta E_o' + 2\sum_{m>o}^{N} \frac{f_m\,\Delta f_{-m} + f_{-m}\,\Delta f_m}{m^2 - 4\,\nu_z^2}\,. \tag{2.34}$$

Beat Factors

The Floquet function giving the linear r-motion is given by

$$u_\nu = e^{i\nu\theta} \left\{ 1 - \sum_{n \neq o}^{N} u_{\nu,n} e^{in\theta} \right\} \tag{2.35a}$$

$$u_{\nu,n} = \frac{g_n}{(n+\nu)^2 - E_o} \quad . \tag{2.35b}$$

The change in the $u_{\nu,n}$ is then

$$\Delta u_{\nu,n} = \frac{1}{(n+\nu)^2 - E_o} \left\{ \Delta g_n + u_{\nu,n} \left[2(n+\nu)\Delta\nu - \Delta E_o \right] \right\} . \tag{2.36}$$

The radial beat factor is given by

$$B_r = \frac{1 + \left| u_{\nu,N} + u_{\nu,-N} \right|}{1 - \left| u_{\nu,N} + u_{\nu,-N} \right|} \quad .$$

The change in the beat factor is then

$$\Delta B_r \doteq \frac{\Delta \left| u_{\nu,N} + u_{\nu,-N} \right|}{1 - \left| u_{\nu,N} + u_{\nu,-N} \right|} (1 + B_r) .$$

The results for the z-motion are similar to the above.

Appendix A

We treat here the problem of finding the tune change when the change in the linear equation of motion is known. We write the linear equation of motion as

$$\frac{d^2}{d\theta^2} u + (\overline{E}_o - \overline{g}(\theta) u = 0, \tag{A.1a}$$

$$\overline{E}_o = E_o + \Delta E_o , \tag{A.1b}$$

$$\overline{g}(\theta) = g(\theta) + \Delta g(\theta) . \tag{A.1c}$$

where $g(\theta)$ has the period $2\pi/N$ and $\Delta g(\theta)$ has the period 2π .

We consider the following eigenvalue problem. The unperturbed linear equation has the eigenfunction $u_n(\theta)$ with E-values E_n which are solutions of the equation,

$$\left[\frac{d^2}{d\theta^2} + (E_n - g(\theta))\right] u_n = 0 \tag{A. 2}$$

and which have the tune $\nu = n$.

The perturbation $\Delta g(\theta)$ will shift the E-value E_s, which corresponds to the tune $\nu = s$ to the E-value $\bar{E}_s$, and $\bar{E}_s$ is given by

$$\bar{E}_s = E_s + (\Delta g)_{s,s} + \sum_{\ell \neq s} \frac{|(\Delta g)_{\ell,s}|^2}{\bar{E}_s - E_\ell} , \tag{A. 3a}$$

$$(\Delta g)_{\ell,s} = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, u_\ell^*(\theta)\, \Delta g(\theta)\, u_s(\theta) , \tag{A. 3b}$$

$$\ell = s + n, \quad n = 0, \pm 1, \pm 2 \cdots . \tag{A. 3c}$$

Equation (A. 3) is the usual quantum mechanical perturbation theory result for the shift in the energy levels.

We now evaluate $(\Delta g)_{s,s}$ and $(\Delta g)_{\ell,s}$. $u_s(\theta)$ is given by

$$u_s = e^{is\theta}\left(1 - \sum_{m \neq o}^{N} \frac{g_m}{(m+s)^2 - E_s} e^{im\theta}\right) . \tag{A. 4}$$

Thus we find

$$(\Delta g)_{s,s} = \frac{2\pi}{N} \int_0^{2\pi/N} d\theta \left[1 - \sum_{m \neq o}^{N} \frac{e^{-im\theta}}{(m+s)^2 - E_s} g_m^*\right] \tag{A. 5a}$$

$$\times\, \Delta g \left[1 - \sum_{m \neq o}^{N} e^{im\theta} \frac{g_m}{(m+s)^2 - E_s}\right] ,$$

$$(\Delta g)_{s,s} = - \sum_{m \neq o}^{N} \frac{g_{-m}\, \Delta g_m + g_m\, \Delta g_{-m}}{(m+s)^2 - E_s} . \tag{A. 5b}$$

We find for $(\Delta g)_{\ell, s}$

$$(\Delta g)_{\ell, s} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[1 - \sum_{m \neq o}^{N} \frac{e^{-im\theta} g_m^*}{(m+\ell)^2 - E_\ell} \right] e^{-i\ell\theta} \quad \text{(A.6a)}$$

$$\times \; \Delta g \left[1 - \sum_{m \neq o}^{N} \frac{e^{im\theta} g_m}{(m+s)^2 - E_s} \right] e^{is\theta} ,$$

$$(\Delta g)_{\ell, s} = \Delta g_n - \sum_{m \neq o}^{N} \left[\frac{g_{-m} \Delta g_{n+m}}{(m+\ell)^2 - E_\ell} + \frac{g_m \Delta g_{-m+n}}{(m+s)^2 - E_s} \right] \quad \text{(A.6b)}$$

where $\ell = s + n$, $n = \pm 1, \pm 2, \cdots$.

In many instances one can put simple

$$(\Delta g)_{\ell, s} = \Delta g_n \quad . \quad \text{(A.7)}$$

One may also note that if $\Delta g_n \neq o$ only when $n = \pm N, \pm 2N$, then the contribution due to $(\Delta g)_{\ell, s}$ may be neglected if $(\Delta g)_{s, s}$ does not vanish as $(\Delta g)_{\ell, s}$ gives a second-order shift in tune, whereas $(\Delta g)_{s, s}$ gives a first-order shift.

If $\Delta g_n \neq 0$ only when $n \neq \pm N, \pm 2N$, then $(\Delta g)_{s, s} = 0$, and only $(\Delta g)_{\ell, s}$ contributes. To find the tune change from (A.3), we observe that in our problem $s = \bar{\nu}$ and $\bar{E}_s$ is known being $\bar{E}_o = E_o + \Delta E_o$ where $\bar{\nu}$ is the perturbed tune. We also note that if ν is the unperturbed tune, $E_o = E_\nu$, $E_s = E_{\bar{\nu}}$. Thus we get

$$E_\nu + \Delta E_o = E_{\bar{\nu}} + (\Delta g)_{\bar{\nu}, \bar{\nu}} + \sum_\ell \frac{|(\Delta g)_{\ell, \bar{\nu}}|^2}{E_o + \Delta E_o - E_\ell} \quad \text{(A.8)}$$

or

$$E_{\bar{\nu}} - E_\nu = \Delta E_o - (\Delta g)_{\bar{\nu}, \bar{\nu}} + \sum_\ell \frac{|(\Delta g)_{\ell, \bar{\nu}}|^2}{E_\ell - (E_o + \Delta E_o)} \quad \text{(A.9)}$$

$$\ell = \bar{\nu} + n, \quad n = \pm 1, \pm 2, \cdots .$$

$$(\Delta g)_{\bar{\nu},\bar{\nu}} = - \sum_{m \neq o}^{N} \frac{g_{-m}\, \Delta g_m + g_m\, \Delta g_{-m}}{(m + \bar{\nu})^2 - E_{\bar{\nu}}} \tag{A.10a}$$

$$(\Delta g)_{\ell,\bar{\nu}} = \Delta g_n - \sum_{m \neq o}^{N} \left[\frac{g_{-m}\, \Delta g_{n+m}}{(m + \ell)^2 - E_\ell} + \frac{\Delta g_m\, g_{-m+n}}{(m + \bar{\nu}) - E_{\bar{\nu}}} \right]. \tag{A.10b}$$

One can often put

$$E_{\bar{\nu}} - E_\nu = \bar{\nu}^2 - \nu^2 , \tag{A.11a}$$

$$E_\ell - E_o = (\bar{\nu} + n)^2 - \nu^2 . \tag{A.11b}$$

Using (A.10), one can write

$$\bar{\nu}^2 - \nu^2 = \Delta E_o + \sum_{m \neq o}^{N} \frac{\Delta g_{-m}\, g_m + \Delta g_m\, g_{-m}}{(m + \nu)^2 - E_o} \tag{A.12}$$

$$+ 2 \sum_{n > o} \frac{|\Delta g_n|^2}{n^2 - 4\nu^2} .$$