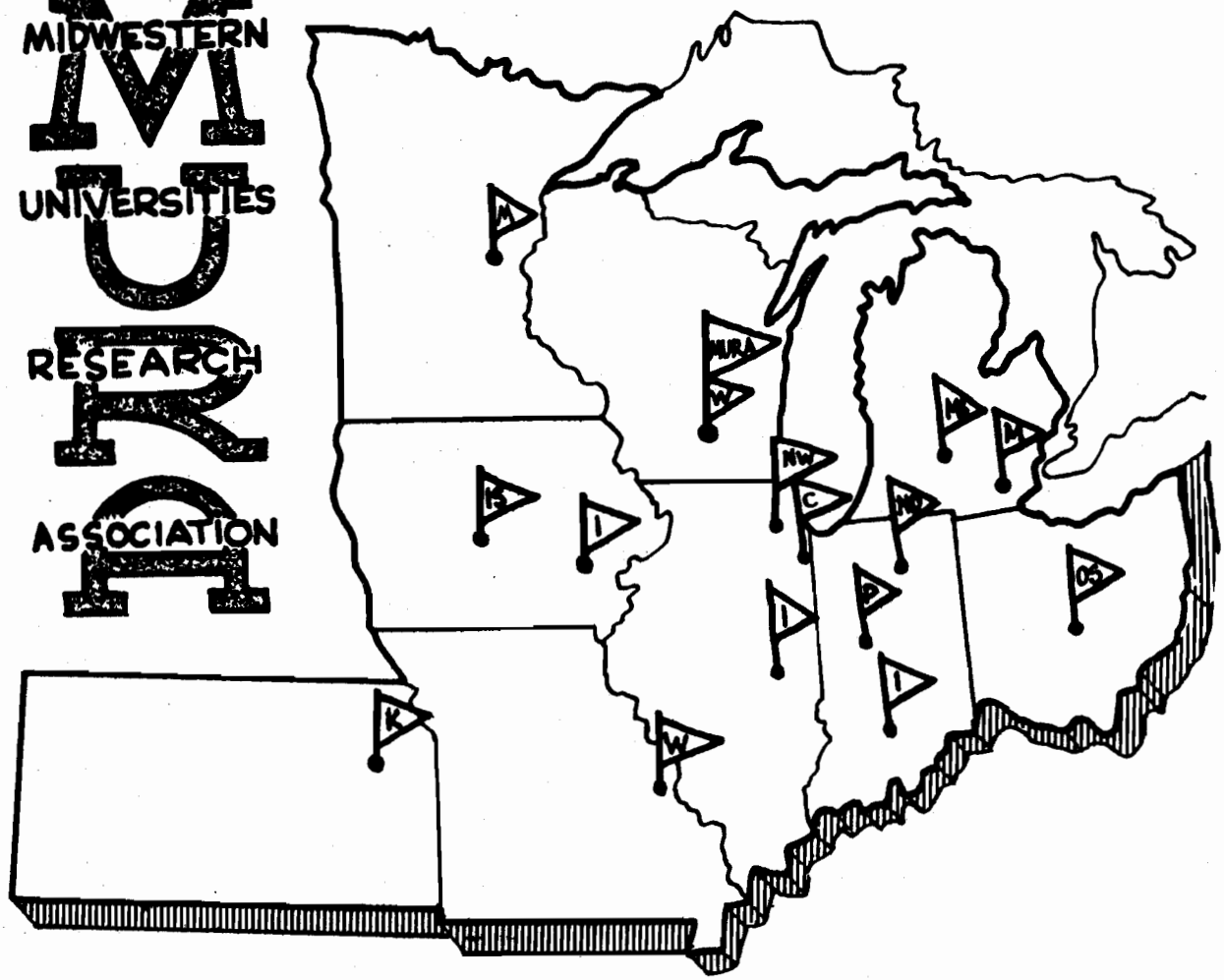
633

Fermilab Library
0 1160 0038897 9

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



SOME REMARKS ON FORCE-FREE COILS

J. Van Bladel

REPORT **NUMBER** 633

MURA-633
UC-28, Particle Accelerators and
High Voltage Machines
TID-4500 (16th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION

2203 University Avenue, Madison, Wisconsin

SOME REMARKS ON FORCE-FREE COILS

J. Van Bladel**

December 8, 1961

ABSTRACT

The possibility of establishing force-free current distributions in infinite cylindrical volumes is examined. It is argued that circular boundaries and fields with circular symmetry yield the only solution to the problem.

* AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission, Contract No. AT(11-1)-384.

**Electrical Engineering Department of the University of Wisconsin

I. Introduction

A force-free region is obtained when the magnetic field generated by a current distribution $\bar{j}$ is everywhere parallel to the current. Such a configuration is highly desirable for applications where high currents and magnetic fields are needed, and where the mechanical strength of the conducting material (a superconductor for example) presents a major problem. The critical magnetic field for superconductors is higher for fields parallel to the current than for transverse fields, and the magnetoresistive coefficient of an ordinary conductor is lower for parallel than for transverse magnetic fields. For these reasons also force-free current configurations would be of interest in the design of high-field magnets.

The basic equation to be satisfied is

$$\operatorname{curl} \bar{h} = \lambda \bar{h} \tag{1}$$

A particular solution of this equation has been known for some time[1]. For a circular cylindrical region, it is:

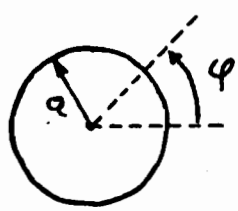


Fig. 1

$$\bar{h} = J_0(\lambda r)\bar{u}_z + J_1(\lambda r)\bar{u}_\varphi \tag{2}$$

A rather abundant literature has been devoted to the search for other suitable field configurations with more general boundaries (e.g. surfaces of revolution) both for constant λ[2,3] and for the more flexible situation where λ is

allowed to vary from point to point[4-7]. In this note we intend to look critically at the simplest version of the problem, obtained when the currents are contained in a cylindrical volume of arbitrary cross-section, and when λ is constant. More general structures will be analyzed in future reports.

II. Z-independent Fields

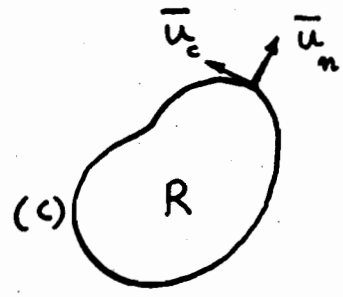


Fig. 2.

Let the currents be enclosed in a simply-connected region R, and depend on the transverse coordinates x,y only. The basic equation now yields

$$\frac{\partial h_z}{\partial y}\bar{u}_x - \frac{\partial h_z}{\partial x}\bar{u}_y + \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\bar{u}_z = \lambda h_x \bar{u}_x + \lambda h_y \bar{u}_y + \lambda h_z \bar{u}_z$$

This entails the following equation for h_z

$$\nabla^2 h_z + \lambda^2 h_z = 0 \tag{3}$$

while the transverse part of $\bar{h}$ is related to h_z by

$$\bar{h}_t = \frac{1}{\lambda}\,\mathrm{grad}\, h_z \times \bar{u}_z \tag{4}$$

At the boundary, $\bar{j}$ and $\bar{h}_t$ must be tangential. This implies the boundary condition

$$h_z = \text{constant on (c)}$$

The value of the constant is easy to find. Remember, indeed, that $\bar{h}$ is irrotational outside R. The expression for curl $\bar{h}$ shows that h_z must be independent of x and y. But this constant must be zero because of the required behavior at

infinity. We conclude that h_z vanishes on (c) too, or more explicitly, that the z component of a force-free field must be an eigenfunction of (3) with a Dirichlet boundary condition

$$h_z = 0 \quad \text{on } (c) \tag{5}$$

The proportionality constant λ is obviously the square root of the corresponding eigenvalue. An immediate application follows: the particular solution (2) cannot represent a self-consistent force-free field except if λ satisfies

$$J_0(\lambda a) = 0$$

But there is more. Equations (3) and (5) represent a necessary condition on h_z, but by no means a sufficient one. Let us rephrase this in a somewhat different fashion by considering the eigenfunctions

$$\begin{aligned} \nabla^2 A_m + \lambda_m^2 A_m &= 0 \\ A_m &= 0 \quad \text{on } (c) \end{aligned} \tag{6}$$

We now ask: does the current distribution

$$\bar{j} = \bar{j}_z + \bar{j}_t = \lambda_m A_m \bar{u}_z + \text{grad}\, A_m \times \bar{u}_z$$

really generate a magnetic field $\frac{1}{\lambda_m} \bar{j}$ inside R? In other words, is the solution of the complete problem

$$\begin{aligned} &\text{curl}\, \bar{h} = \lambda_m A_m \bar{u}_z + \text{grad}\, A_m \times \bar{u}_z \quad \text{inside } R \\ &\text{curl}\, \bar{h} = 0 \quad \text{outside } R \\ &\bar{h} \text{ tangent to } C \\ &\bar{h} \text{ continuous everywhere} \\ &\bar{h} \text{ regular at infinity} \end{aligned} \tag{7}$$

really $\bar{h} = \frac{\bar{j}}{\lambda_m}$? The analysis is clarified by examining the partial fields generated by $\bar{j}_z$ and $\bar{j}_t$ separately, and by writing the first and second equations as:

$$\begin{cases} \text{grad } h_z \times \bar{u}_z = \text{grad } A_m \times \bar{u}_z & \text{inside } R \\ -\text{div}(\bar{u}_z \times \bar{h}_t) = \bar{u}_z \cdot \text{curl } \bar{h}_t = \lambda_m A_m \end{cases} \tag{8}$$

$$\begin{cases} \text{grad } h_z = 0 & \text{outside } R \\ -\text{div}(\bar{u}_z \times \bar{h}_t) = 0 \end{cases}$$

We immediately see that h_z differs from A_m by an additive constant, but also that this constant is zero because both h_z and A_m vanish on the contour. The z components are consequently taken care of: h_z is indeed $\frac{1}{\lambda_m} j_z$. But the transverse components are more delicate to handle. We can look at the problem in two equivalent ways. First, we can write

$$\bar{h}_t = \frac{1}{\lambda_m} \bar{j}_t + \bar{\alpha}$$

and find out whether $\bar{\alpha}$ is zero. We notice that both $\bar{h}_t$ and $\bar{j}_t$ have zero divergence. This means that $\bar{\alpha}$ partakes of the same property, i.e. that there is a function β such that

$$\bar{\alpha} = \text{grad}\,\beta \times \bar{u}_z$$

On the other hand, (7) and (8) imply that curl $\bar{h}_t = \frac{1}{\lambda_m}$ curl $\bar{j}_t$. This entails curl $\bar{\alpha} = 0$ or, equivalently: $\nabla^2 \beta = 0$ inside R. If we now consider the region outside R, we can similarly write

$$\bar{h}_t = \text{grad}\,\psi \times \bar{u}_z$$
$$\nabla^2 \psi = 0$$
$$\lim_{r \to \infty} \psi = \frac{I}{2\pi} \ln \frac{1}{r}$$

where I is the total axial current flowing in the cylinder. At the boundary, β and ψ are related by the continuity condition on $\bar{h}_t$. This implies

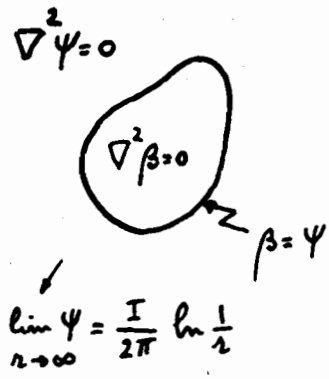


Fig. 3

$$\frac{\partial\psi}{\partial c}=\frac{\partial\beta}{\partial c}$$

$$\frac{\partial\psi}{\partial n}-\frac{\partial\beta}{\partial\eta}=\frac{1}{\lambda_m}\frac{\partial A_m}{\partial n}$$

The problem is summarized in Fig. 3. Clearly, ψ and β are respectively the exterior and interior potentials which appear when the cylindrical boundary (c) is covered with a charge density $\eta_0=-\varepsilon_0\frac{1}{\lambda_m}\frac{\partial A_m}{\partial n}$. Condition $\bar{d}=0$ is, consequently, <u>not</u> automatically satisfied. As a matter of fact, it will be satisfied if, and only if, η_0 is proportional to the charge density η which appears when the cylinder is metallized and charged. (This charge density has the property of creating an inside field equal to zero). Our criterion can be formulated as follows:

$$\frac{\partial A_m}{\partial n} \text{ proportional to } \eta \text{ on } (C) \tag{9}$$

The second way to look at the problem is perhaps more illuminating. The transverse magnetic field outside has zero divergence and curl. It is tangent to C, and vanishes at least like $1/r$ at large distances, where its lines of force become circular. There is only one vector $\bar{h}_0$, (determined with the exception of a multiplicative constant) which satisfies all these requirements. This vector is, furthermore, $\bar{u}_z\times\bar{e}_0$, where $\bar{e}_0$ is the electrostatic field outside the charged metallized cylinder. If $\frac{1}{\lambda_m}\,\text{grad}\,A_m\times\bar{u}_z$ is to be the magnetic field inside C, it must join

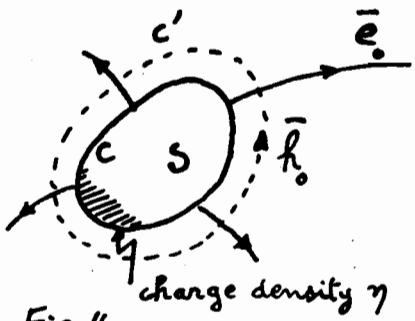


Fig.4

smoothly to $B\bar{h}_o$, where B is a suitable constant. This means that $\frac{\partial A_m}{\partial n}$ must be proportional $\bar{u}_n \cdot \bar{e}_o$, i.e to η, and we again obtain condition (9).

III. Realizability of z-independent Force-free Fields.

The results obtained up to this point can be summarized as follows:

(a) The realizability of a force-free field reduces to the search for a contour C possessing an eigenfunction A_m whose normal gradient is proportional to the charge density η.

(b) The magnetic field outside the force-free region must necessarily be a multiply of $\bar{h}_o$, where $\bar{h}_o$ is obtained by rotating by 90° the electrostatic field outside the charged metallized cylinder. The multiplicative constant is determined from Ampere's law

$$\int_{c'} \bar{h} \cdot d\bar{c} = I = \iint_S j_z \, dS = \lambda_m \iint_S A_m \, dS$$

The two conditions appearing in equations (6) and (9) involve an exterior and an interior problem, and are normally incompatible. The problem has a solution, however, when the configuration has the kind of symmetry which automatically ensures satisfaction of (9). We are thinking here of the circular boundary, for which η is simply a constant, and where (9) is automatically satisfied for all φ-independent eigenfunctions. General statements in other cases are hampered by the lack[8] of known properties of the charge distribution η. The possibility of force-free fields

obviously does not exist for a rectangular cross-section, where

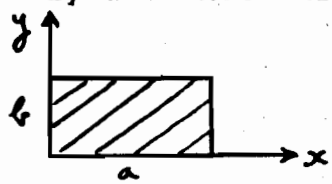


Fig. 5

$$A_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

while η is positive all along the contour and approaches infinity at the corners. Qualitatively, we feel that any excursion from the circular boundary will destroy the possibility of force-free fields of the type we are investigating, because η will be large, and $\frac{\partial A}{\partial n}$ small, in regions of strong outward curvature such as the corner of a rectangle. These views are borne out by Appendices I and II, where we calculate η and $\frac{\partial A}{\partial n}$ for a small deformation of the circle. Finally, we can also check our statement in the case of the elliptic cross-section. Using elliptic coordinates:

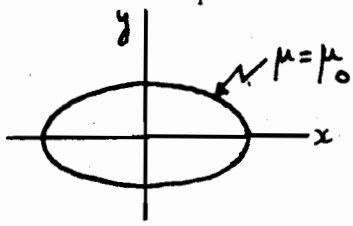


Fig. 6

$$x = \frac{g}{2}\cosh\mu\cos\theta$$
$$y = \frac{g}{2}\sinh\mu\sin\theta \qquad (10)$$

we easily find that the charge density is

$$\eta = \frac{}{(\cosh^2\mu_0 - \cos^2\theta)^{1/2}}\cdot\frac{Q}{\pi g} \qquad (11)$$

where Q is the total charge in Coul m^{-1}. On the other hand, the eigenfunctions are[9]

$$A_{mn} = R_{\binom{e}{o}m}(\mu, \alpha_{mn})\cdot S_{\binom{e}{o}m}(\theta, \alpha_{mn}) \quad \text{with} \quad R_{\binom{e}{o}m}(\mu_0, \alpha_{mn}) = 0$$

where e and o refer to even and odd functions respectively, and R and S are m^{th} order radial and angular Mathieu functions. The

normal derivative of A_{mm} for $\mu = \mu_0$ is proportional to

$$\frac{1}{(\cosh^2\mu_0 - \cos^2\theta)^{1/2}} \cdot S_{\binom{e}{o}m}(\theta, \alpha_{mn})$$

The equality of η and $\frac{\partial A}{\partial n}$ would require that $S_{\binom{e}{o}m}(\theta, \alpha_{mn})$ be constant for all θ, which is obviously not the case.[10]

IV. Doubly-connected regions

Fig. 7

Consider a current distribution with a hollow central region such as pictured in Fig. 7. In this central region, $\bar{h}$ is irrotational, which implies that h_z is constant. The transverse component $\bar{h}_t$ has zero divergence, and can consequently be written as

$$\bar{h}_t = \text{grad}\,\gamma \times \bar{u}_z$$

with $\nabla^2\gamma = 0$ because curl $\bar{h} = 0$. On the other hand, $\bar{h}_t$ must be tangent to C_1, which requires that γ be constant along C_1. This entails, from well-known results of potential theory, that γ be constant throughout the central region, i.e. that $\bar{h}_t$ vanish. Here again we find out that a force-free region does not leave any flexibility as far as the external magnetic fields are concerned. Inside they are bound to be homogeneous and parallel to the z-axis, outside they must be transverse and a multiple of $\bar{h}_0$. Furthermore, the very possibility of these force-free regions is open to question. The basic differential problem is now, for the region between C_1 and C_2,

$$\begin{aligned} &\nabla^2 A + \lambda^2 A = 0 \\ &A = 0 \text{ and } \frac{\partial A}{\partial n} \text{ proportional to } \eta \text{ on } C_2 \\ &A = \text{constant and } \frac{\partial A}{\partial n} = 0 \text{ on } C_1 \end{aligned} \qquad (12)$$

The boundary conditions on C_1 are obtained by expressing that grad A $\times \bar{u}_z$ is tangent to C_1. Clearly, the differential system (12) is again overdetermined, except for a configuration where C_1 and C_2 are concentric circles of radii b and a. In that case, the φ -independent modes are determined by

$$A = J_0(\lambda r) + \alpha N_0(\lambda r)$$
$$J_0(\lambda a) + \alpha N_0(\lambda a) = 0$$
$$J_1(\lambda b) + \alpha N_1(\lambda b) = 0$$

which quantizes λ and α .

V. z-dependent Fields

We now turn our attention to a current distribution which is periodic in z with period L. Such a configuration is obtained, for example, by "developing" a toroidal region. The magnetic field can be Fourier expanded as

$$\bar{h} = \sum_m \left(A_m \sin\frac{m\pi z}{L}\bar{u}_z + \bar{F}_m \cos\frac{m\pi z}{L}\right) + \left(B_m \cos\frac{m\pi z}{L}\bar{u}_z + \bar{G}_m \sin\frac{m\pi z}{L}\right)$$

where A_m, B_m, $\bar{F}_m$ and $\bar{G}_m$ depend on x and y only, and $\bar{F}_m$ and $\bar{G}_m$ are transverse vectors. The curl and divergence of $\bar{h}$ are then

$$\operatorname{curl}\bar{h} = \sum_m \left[-\operatorname{div}(\bar{u}_z \times \bar{F}_m)\cos\frac{m\pi z}{L}\bar{u}_z - \sin\frac{m\pi z}{L}\cdot\frac{m\pi}{L}\cdot\bar{u}_z \times \bar{F}_m - \sin\frac{m\pi z}{L}\bar{u}_z \times \operatorname{grad} A_m\right]$$
$$+ \left[-\operatorname{div}(\bar{u}_z \times \bar{G}_m)\sin\frac{m\pi z}{L}\bar{u}_z + \cos\frac{m\pi z}{L}\cdot\frac{m\pi}{L}\cdot\bar{u}_z \times \bar{G}_m - \cos\frac{m\pi z}{L}\bar{u}_z \times \operatorname{grad} B_m\right]$$

$$\operatorname{div}\bar{h} = \sum_m \left[\frac{m\pi}{L}A_m + \operatorname{div}\bar{F}_m\right]\cos\frac{m\pi z}{L} + \left[-\frac{m\pi}{L}B_m + \operatorname{div}\bar{G}_m\right]\sin\frac{m\pi z}{L}$$

The expansions we have written are for the inside region R. Similar expansions, with primed coeffcients, can be used for the outside region, where curl $\bar{h} = \operatorname{div}\bar{h} = 0$. This entails, for $m \neq 0$

$$\bar{F}'_m = -\frac{L}{m\pi}\operatorname{grad} A'_m$$
$$\operatorname{div}\bar{F}'_m = -\frac{m\pi}{L}A'_m$$

or, by elimination,

$$\nabla^2 A'_m - \left(\frac{m\pi}{L}\right)^2 A'_m = 0$$

Let us examine the behavior at the boundaries. On C, $\bar{h}$ must be tangential, i.e. $\bar{u}_n \cdot \bar{F}'_m = 0$ or $\frac{\partial A'_m}{\partial n} = 0$. At large distances, $\bar{h}$ vanishes at least like $1/r$, i.e. A'_m and $\bar{u}_r \cdot \bar{F}'_m$ (or $\frac{\partial A'_m}{\partial r}$) vanish at least like $1/r$. We are now in a position to apply Green's theorem to the region outside the currents:

$$\iint \left[A'_m \nabla^2 A'_m + (\text{grad}\, A'_m)^2 \right] dS = \iint \left[\left(\frac{m\pi}{L}\right)^2 (A'_m)^2 + (\text{grad}\, A'_m)^2 \right] dS$$

$$= \int_C A'_m \frac{\partial A'_m}{\partial n}\, dc + \int_\infty A'_m \frac{\partial A'_m}{\partial r}\, r\, d\varphi$$

Both contour integrals vanish, which allows us to conclude that A'_m and $\bar{F}'_m$ vanish everywhere. A similar conclusion can be reached for B'_m and $\bar{G}'_m$. Turning now to the inside fields, we notice that the fundamental equation (1) requires that

$$\frac{m\pi}{L} \bar{F}_m + \text{grad}\, A_m = \lambda (\bar{u}_z \times \bar{G}_m)$$

$$\text{div}\, \bar{F}_m = -\frac{m\pi}{L} A_m$$

$$\text{div}\, (\bar{u}_z \times \bar{G}_m) = -\lambda A_m$$

By taking the divergence of the first equation, we find that

$$\nabla^2 A_m + \left[\lambda^2 - \left(\frac{m\pi}{L}\right)^2 \right] A_m = 0 \tag{13}$$

Furthermore, $\bar{h}$ must vanish at the boundary because it vanishes outside. In other words,

$$A_m = 0$$

$$\frac{\partial A_m}{\partial n} = 0 \qquad \text{on } (C) \tag{14}$$

The only solution of (13) (14) is zero. We conclude that there is no self-consistent force-free current distribution for $m \neq 0$ or, by letting L approach infinity and replacing Fourier series by integrals, that there are no z -dependent force-free fields for periodic or infinite cylindrical current distribution.

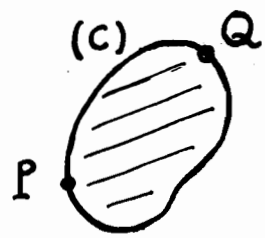


Fig. 8.

APPENDIX I: Perturbation in the Charge Density.

Consider a metallic cylinder carrying a charge Λ Coul m^{-1}. This charge distributes itself with a density η, which is related to the potential V of the cylinder by the integral relation:

$$V(P) = \frac{1}{2\pi\varepsilon_0} \int_C \eta(Q) \ln \frac{1}{|\bar{r}_p - \bar{r}_Q|} \, dc \qquad (15) .$$

The integral is an improper integral, and Q is the integration point. The potential must be independent of P, which means that the correct charge distribution must make the right-hand member equal to a constant. If we consider a circular cylinder of unit radius covered with unit density $\eta = 1$, then the constant is (with $|\bar{r}_p - \bar{r}_Q| = 2 \sin \frac{|\varphi_p - \varphi_Q|}{2}$) equal to

$$\frac{1}{2\pi\varepsilon_0} \int_0^{2\pi} \ln \frac{1}{2 \sin |\varphi - \varphi_p|/2} \, d\varphi = 0$$

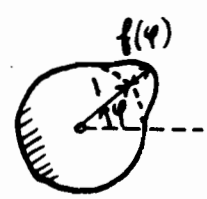


Fig. 9.

For a more general shape, (15) becomes an integral equation for η. Let us consider, in particular, a cylinder obtained by slightly distorting the

circle. That is, let the equation of the contour be

$$r = 1 + \varepsilon f(\varphi)$$

More particularly, let $f(\varphi)$ be Fourier-expanded as:

$$f(\varphi) = f_0 + \sum_1^\infty A_m \cos m\varphi + \sum_1^\infty B_m \sin m\varphi \tag{16}$$

In the limit of very small ε, we write, to higher orders in ε,

$$|\bar{r}_p - \bar{r}| = 2 \sin \frac{|\varphi_p - \varphi|}{2} \cdot \left[1 + \frac{\varepsilon}{2} f(\varphi) + \frac{\varepsilon}{2} f(\varphi_p)\right]$$

$$dc = \left[1 + \varepsilon f(\varphi)\right] d\varphi$$

If we maintain a total charge equal to the original one (i.e. 2π), the new density becomes

$$\eta = 1 + \varepsilon \left[\sum_1^\infty \alpha_m \cos m\varphi + \sum_1^\infty \beta_m \sin m\varphi\right] = 1 + \varepsilon N(\varphi) \tag{17}$$

If we insert these power series in (15), we find that the coefficient of the term in ε is

$$C = \frac{1}{2\pi\varepsilon_0}\left\{\int_0^{2\pi} \left[f(\varphi) + N(\varphi)\right] \ln \frac{1}{2 \sin |\varphi - \varphi_p|/2} d\varphi - \frac{1}{2}\int_0^{2\pi} \left[f(\varphi) + f(\varphi_p)\right] d\varphi\right\}$$

This coefficient must be independent of φ_p. We now insert the Fourier expansions (16) and (17) in the expression for C and obtain, with $\varphi' = \varphi - \varphi_p$

$$2\pi\varepsilon_0 C = \sum_1^\infty \left[(\alpha_m + A_m)\cos m\varphi_p + (\beta_m + B_m)\sin m\varphi_p\right] \int_0^{2\pi} \cos m\varphi' \ln \frac{1}{2 \sin \varphi'/2} d\varphi'$$

$$+ \left[(\beta_m + B_m)\cos m\varphi_p - (\alpha_m + A_m)\sin m\varphi_p\right] \int_0^{2\pi} \sin m\varphi' \ln \frac{1}{2 \sin \varphi'/2} d\varphi'$$

$$- \pi f_0 - \pi f(\varphi_p)$$

The values of the integrals are easy to calculate if we remark that they are proportional to the potential at $\varphi' = 0$ due to charge densities cos mφ' and $\sin m\varphi'$ respectively. The first integral turns out to be π/m, and the second one is

simply zero. This gives

$$2\pi\varepsilon_0 C = -2\pi f_0 + \sum_1^\infty \cos m\varphi_p \left[\frac{\pi\alpha_m}{m} + \frac{\pi A_m}{m} - \pi A_m\right] + \sum_1^\infty \sin m\varphi_p \left[\frac{\pi\beta_m}{m} + \frac{\pi B_m}{m} - \pi B_m\right]$$

The requirement that C be independent of φ_p immediately yields, by setting the brackets equal to zero, the value of α_m and β_m. We consequently find the new charge distribution to be

$$\eta(\varphi) = 1 + \varepsilon \sum_1^\infty (m-1) A_m \cos m\varphi + (m-1) B_m \sin m\varphi \tag{18}$$

It is interesting to compare this charge distribution to the curvature, and check whether high (or low) curvatures imply high (or low) charge densities. The curvature $\frac{1}{R}$ is given by

$$\frac{1}{R} = \frac{r^2 + 2\left(\frac{dr}{d\varphi}\right)^2 - r\frac{d^2r}{d\varphi^2}}{\left[r^2 + \left(\frac{dr}{d\varphi}\right)^2\right]^{\frac{3}{2}}} = 1 - \varepsilon\left(f + \frac{d^2f}{d\varphi^2}\right) + \cdots = 1 + \varepsilon\rho(\varphi) + \cdots$$

$$= 1 + \varepsilon \sum_1^\infty (m^2-1) A_m \cos m\varphi + (m^2-1) A_m \sin m\varphi \tag{19}$$

Clearly, the two Fourier expansions are different. Let us investigate the correlation on the particular deformation shown in Fig. 10, which evidences an inward and an outward bump. The formulas are

$$f(\varphi) = \frac{16}{\pi} \sum_1^\infty \left(1 - \cos\frac{m\pi}{2}\right) \frac{1}{m(m^2-16)} \sin m\varphi$$

$$N(\varphi) = \frac{16}{\pi} \sum_1^\infty \left(1 - \cos\frac{m\pi}{2}\right) \frac{m-1}{m(m^2-16)} \sin m\varphi$$

$$\rho(\varphi) = -\left(\frac{1}{2} + \frac{15}{2}\cos 4\varphi\right) \quad \text{from } 0 \text{ to } \frac{\pi}{2}$$

$$= 0 \quad \text{from } \frac{\pi}{2} \text{ to } \pi$$

Numerical results are shown in Fig. 10, and are self explanatory.

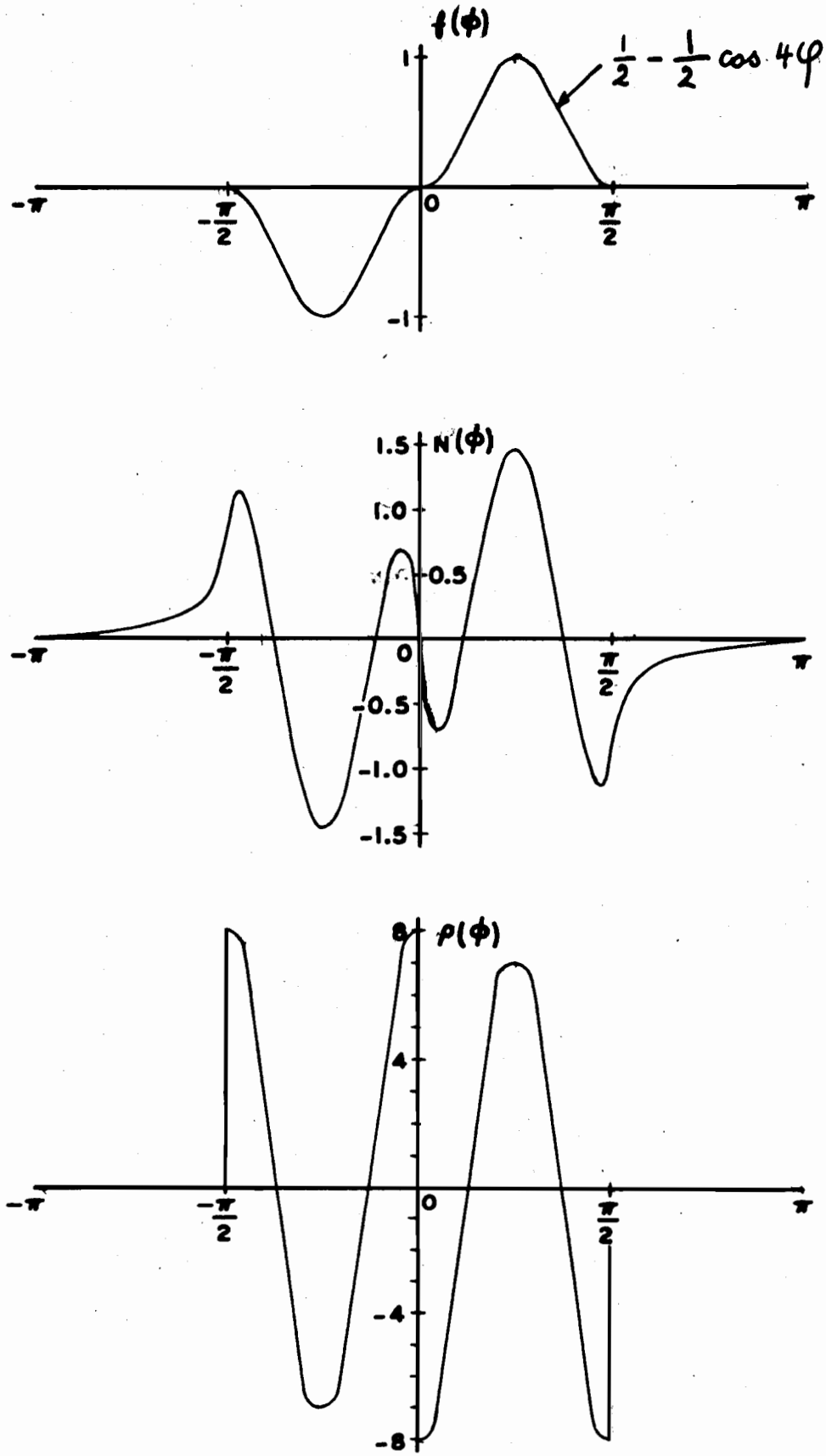

f(φ)
1
-1
$\frac{1}{2} - \frac{1}{2}\cos 4\varphi$
-π
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
N(φ)
1.5
1.0
0.5
-0.5
-1.0
-1.5
ρ(φ)
8
4
-4
-8

Fig. 10

APPENDIX II: Perturbation in $\frac{\partial A}{\partial n}$.

For the circular boundary sketched in Fig. 9, the eigenfunctions leading to force-free fields are $\alpha J_0(\lambda r)$, where α is a normalization constant defined by

$$2\pi\alpha^2\int_0^1 [J_0(\lambda r)]^2 r\,dr = \pi\alpha^2[J_1(\lambda)]^2 = 1$$

and λ is a root of $J_0(\lambda)=0$. If the boundary is deformed as indicated in Fig. 9, the eigenfunction will be perturbed, and becomes:

$$\psi = \alpha\left[J_0(\lambda r) + \varepsilon j(r,\varphi) + \cdots\right] \tag{20}$$

Notice that, to maintain normalization,

$$\iint_{C+\Sigma} \psi^2 r\,dr\,d\varphi = \iint_{C+\Sigma} \alpha^2[J_0(\lambda r)]^2 r\,dr\,d\varphi + 2\varepsilon\alpha^2\iint_{C+\Sigma} J_0(\lambda r)\,j(r,\varphi)\,r\,dr\,d\varphi + \cdots = 1$$

where C is the area of the circle, and Σ the supplementary area, which is of order ε. Remembering that $J_0(\lambda r)$ vanishes at $r=1$, so that its value in Σ is of order ε, we find that normalization is kept if

$$\iint_C J_0(\lambda r)\,j(r,\varphi)\,r\,dr\,d\varphi = 0 \tag{21}$$

or, in other words, if $j(r,\varphi)$ is orthogonal to $J_0(\lambda r)$. To proceed with our study of $j(r,\varphi)$, let us examine the behavior at the boundary. The Dirichlet boundary condition entails that

$$J_0(\lambda + \lambda\varepsilon f) + \varepsilon\, j[1+\varepsilon f(\varphi), \varphi] + \cdots = 0$$

Using a Taylor expansion, we immediately obtain

$$j(1,\varphi) = \lambda J_1(\lambda) f(\varphi) \tag{22}$$

The differential equation satisfied by $j(r,\varphi)$ is:

$$\nabla^2\left[J_0(\lambda r)+\varepsilon j(r,\varphi)\right]+(\lambda^2+\varepsilon B^2)\left[J_0(\lambda r)+\varepsilon j(r,\varphi)\right]=0$$

Keeping terms in ε only, we find

$$\nabla^2 j(r,\varphi)+\lambda^2 j(r,\varphi)=-B^2 J_0(\lambda r) \tag{23}$$

At this point it is useful to Fourier-expand $j(r,\varphi)$. We write:

$$j(r,\varphi)=P_0(r)+\sum_1^\infty P_m(r)\cos m\varphi+\sum_1^\infty Q_m(r)\sin m\varphi$$

Inserting in (22) and (23), we obtain the differential system

$$\frac{d^2P_0}{dr^2}+\frac{1}{r}\cdot\frac{dP_0}{dr}+\lambda^2 P_0=-B^2 J_0(\lambda r) \text{ and } P_0(1)=\lambda J_1(\lambda)f_0 \tag{24}$$

$$\frac{d^2P_m}{dr^2}+\frac{1}{r}\cdot\frac{dP_m}{dr}+\left(\lambda^2-\frac{m^2}{r^2}\right)P_m=0 \quad \text{and } P_m(1)=\lambda J_1(\lambda)A_m \tag{25}$$

$$\frac{d^2Q_m}{dr^2}+\frac{1}{r}\cdot\frac{dQ_m}{dr}+\left(\lambda^2-\frac{m^2}{r^2}\right)Q_m=0 \quad \text{and } Q_m(1)=\lambda J_1(\lambda)B_m \tag{26}$$

Equation (24) is particularly interesting, because it will allow us to find εB^2, i.e. the shift in eigenvalue produced by the boundary perturbation. This is seen by writing the differential equation satisfied by $J_0(\lambda r)$:

$$\frac{d^2J_0}{dr^2}+\frac{1}{r}\cdot\frac{dJ_0}{dr}+\lambda^2 J_0=0 \qquad J_0(\lambda)=0 \tag{27}$$

Multiplying (24) by J_0, (27) by P_0, subtracting and integrating yields

$$\int_0^1\left[rJ_0\frac{d^2P_0}{dr^2}+J_0\frac{dP_0}{dr}-rP_0\frac{d^2J_0}{dr^2}-P_0\frac{dJ_0}{dr}\right]dr=-B^2\int_0^1 J_0^2\, r\, dr$$

But the integrand is a perfect differential, and one obtains, for the left member

$$\left[rJ_0\frac{dP_0}{dr}-rP_0\frac{dJ_0}{dr}\right]_0^1=-P_0(1)\left.\frac{dJ_0(\lambda r)}{dr}\right|_{r=1}=P_0(1)\,\lambda J_1(\lambda)$$

from which follows that

$$B^2 = -\frac{\lambda^2 [J_1(\lambda)]^2}{\int_0^1 J_0^2\, r\, dr} f_0 = -2\lambda^2 f_0$$

The new eigenvalue is consequently

$$\lambda^2 [1 - 2\varepsilon f_0] \tag{28}$$

Turning now to the value of P_0, we notice that $r J_1(\lambda r)$ is a particular solution of (24). In fact:

$$\frac{d^2}{dr^2}[r J_1(\lambda r)] + \frac{1}{r}\frac{d}{dr}[r J_1(\lambda r)] + \lambda^2 [r J_1(\lambda r)] = 2\lambda J_0(\lambda r)$$

The solution of (24) must then be of the form

$$P_0 = D J_0(\lambda r) + \lambda f_0 r J_1(\lambda r)$$

Here, D is a still-undetermined constant, which can be found from the normalization condition (21):

$$\int_0^1 [\lambda f_0 r J_1(\lambda r) + D J_0(\lambda r)] J_0(\lambda r)\, r\, dr = 0$$

which yields $D = -f_0$.

The solution of (25) and (26) is immediate, because $J_m(\lambda r)$ is the only bounded solution of these equations. We are now ready to write the final form of j:

$$j(r,\varphi) = f_0[\lambda r J_1(\lambda r) - J_0(\lambda r)] + \sum_1^\infty \cos m\varphi \frac{\lambda J_1(\lambda) A_m}{J_m(\lambda)} J_m(\lambda r) + \sum_1^\infty \sin m\varphi \frac{\lambda J_1(\lambda) B_m}{J_m(\lambda)} J_m(\lambda r) \tag{29}$$

Our ultimate aim is to calculate the new value of $\frac{\partial \psi}{\partial n}$. For the small deformation of Fig. 9,

$$\frac{\partial \psi}{\partial n} = \bar{u}_n \cdot \operatorname{grad} \psi = \alpha\left(\bar{u}_r - \varepsilon \frac{df}{d\varphi}\bar{u}_\varphi + \cdots\right) \cdot \left[\operatorname{grad} J_0(\lambda r) + \varepsilon \operatorname{grad} j(r,\varphi) + \cdots\right]_{at\, r = 1+\varepsilon f}$$

or

$$\frac{1}{\alpha}\frac{\partial \psi}{\partial n} = -\lambda J_1(\lambda) - \varepsilon \lambda^2 f(\varphi) J_1'(\lambda) + \varepsilon \left.\frac{\partial j}{\partial r}\right|_{r=1}$$

$$= -\lambda J_1(\lambda)\left[1 - 2\varepsilon f_0 - \varepsilon \sum_1^\infty (A_m \cos m\varphi + B_m \sin m\varphi)\left(1 + \frac{\lambda J_m'(\lambda)}{J_m(\lambda)}\right)\right].$$

Normalizing the φ-independent term to be one, we find that $\frac{\partial\psi}{\partial n}$ is proportional to

$$\frac{\partial\psi}{\partial n} \doteq 1 + \varepsilon \sum_1^\infty (A_m \cos m\varphi + B_m \sin m\varphi)\left(-1 - \frac{\lambda J_m'(\lambda)}{J_m(\lambda)}\right)$$

$$\doteq 1 + \varepsilon \sum_1^\infty (A_m \cos m\varphi + B_m \sin m\varphi)\left(m - 1 - \lambda \frac{J_{m-1}(\lambda)}{J_m(\lambda)}\right) \qquad (30)$$

This must be compared to (18), the expansion for the charge density. The two are obviously unequal except when the term $m=1$ is the only one to be present in the expansion. The corresponding deformation is trivial: it conserves constant curvature, i.e. reduces to a simple translation of the initial circle.

It is worth while checking our calculations on a small elliptical perturbation. Let ρ be the eccentricity $(1 - a^2/b^2)^{\frac{1}{2}}$

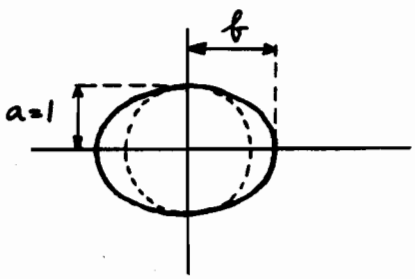


Fig. 11

For small ρ, and using elliptic coordinates, we find the equation of the ellipse to be

$$r = 1 + \frac{\rho^2}{2}\cos^2\varphi + \ldots = 1 + \frac{\rho^2}{4} + \frac{\rho^2}{4}\cos 2\varphi + \cdots$$

The small parameter ε is ρ^2 in our case, and $f_0 = \frac{1}{4}$, $A_2 = \frac{1}{4}$.

All other coefficents vanish. The eigenfunctions of the elliptic cylinder have been given above, and their limiting form can easily be found for small ρ and h (Reference 9 p. 1419):

$$S_{eo}(h, \cos\theta) = 1 + \frac{h^2}{8} - \frac{h^2}{8}\cos 2\theta + \cdots$$

$$R_{eo}(h, \cos\theta) = \sqrt{\frac{\pi}{2}}\left\{\left(1 + \frac{1}{8}h^2\right) J_0(h\cosh\mu) + \frac{1}{8}h^2 J_2(h\cosh\mu) + \cdots\right\}$$

The values of h are obtained from the roots of R_{eo}. Using the power series for the Bessel functions and relevant recursion

formulas, we find:

$$h = \lambda\rho + \frac{1}{8}\lambda^2\rho^3 \frac{J_2(\lambda)}{J_1(\lambda)} + \cdots = \lambda\rho\left[1 + \frac{\rho^2}{4} + \cdots\right]$$

where the λ's are, again, the roots of J_0 . Furthermore, we can go over from the μ, θ coordinates to polar coordinates r, φ by using the easily established relations

$$\cos\varphi = \cos\theta\left[1 + \frac{1}{2}\frac{\rho^2}{r^2}\sin^2\varphi + \cdots\right]$$

$$h\cosh\mu = \lambda r\left[1 - \frac{\rho^2}{4} + \frac{\rho^2}{2r^2}\sin^2\varphi + \cdots\right]$$

This yields the limiting form of the relevant eigenfunctions:

$$\psi = J_0(\lambda r) + \rho^2\left\{\frac{\lambda^2}{8}J_0(\lambda r) + \frac{1}{4}\lambda r J_1(\lambda r)\right\} + \frac{1}{8}\rho^2\lambda^2 J_2(\lambda r)\cos 2\varphi + \cdots$$

The new eigenvalue is easily obtained by calculating the Laplacian of ψ , and turns out to be $\lambda^2\left[1 - \frac{\rho^2}{2}\right]$, in harmony with (28). The eigenfunctions given above are not normalized, but can be normalized by dividing by $\left[\int \psi^2 dS\right]^{\frac{1}{2}}$ to yield:

$$\frac{1}{\sqrt{\pi}\, J_1(\lambda)}\left[J_0(\lambda r) - \frac{\rho^2}{4}J_0(\lambda r) + \frac{\rho^2}{4}\lambda r J_1(\lambda r) + \frac{\rho^2\lambda^2}{8}J_2(\lambda r)\cos 2\varphi + \cdots\right]$$

which is in agreement with (29) (to check the agreement for the $\cos 2\varphi$ term, use the property $\frac{2}{\lambda}J_1(\lambda) = J_2(\lambda)$).

LIST OF CAPTIONS

Figure 1: Circular cylinder.

Figure 2: Cylinder with arbitrary cross-section.

Figure 3: A potential problem.

Figure 4: The exterior magnetic field for a force-free region.

Figure 5: Rectangular cross-section.

Figure 6: Elliptical cross-section

Figure 7: Current distribution with central hole.

Figure 8: Metallic cylinder.

Figure 9: Slightly deformed circular boundary.

Figure 10: Perturbation in charge density and curvature resulting from a contour distortion.

Figure 11: Circular contour distorted into an ellipse.

1. S. Lundquist "Magneto-Hydrostatic Fields" Arkiv Fysik, 2, pp 361-365, 1951.

2. S. Chandrasekhar and P.C. Kendall "On Force-free Magnetic Fields" Astrophysical Jour. 126, pp 457-460, 1957.

3. H.P. Furth, M.A. Levine and R.W. Waniek "Production and Use of High Transient Magnetic Fields" II. "Rev.Sci. Instr. 28, pp 949-958, 1957.

4. R. Lüst and A. Schlüter "Kraft freie Magnet-felder" Zeits. Astrophys, 34, pp 263-282, 1954.

5. S. Chandrasekhar "On Force-free Magnetic Fields" Proc. Nat. Acad. Sci. 42, pp 1-5, 1956.

6. S. Chandrasekhar and L. Woltjer "On Force-free Magnetic Fields" Proc. Nat. Acad. Sci. 44, pp 285-289, 1958.

7. E. Richter "Zur Theorie kraftfreier Magnetfelder" Zeit. Phys, 159, pp 194-211, 1960.

8. R. Cade "The Charge Density Near a Sharp Point on a Conductor" Proc. Cambr. Phil. Soc. 53, pp 870-877, 1957.

9. P.M. Morse and H. Feshbach "Methods of Theoretical Physics" Ch. 11, McGraw-Hill Book Co. New York, 1953.

10. See e.g. the tables and curves in reference 9 p. 1934, or E. Jahnke and F. Emde "Tables of Functions", XI, Dover Publications, New York, 1945.