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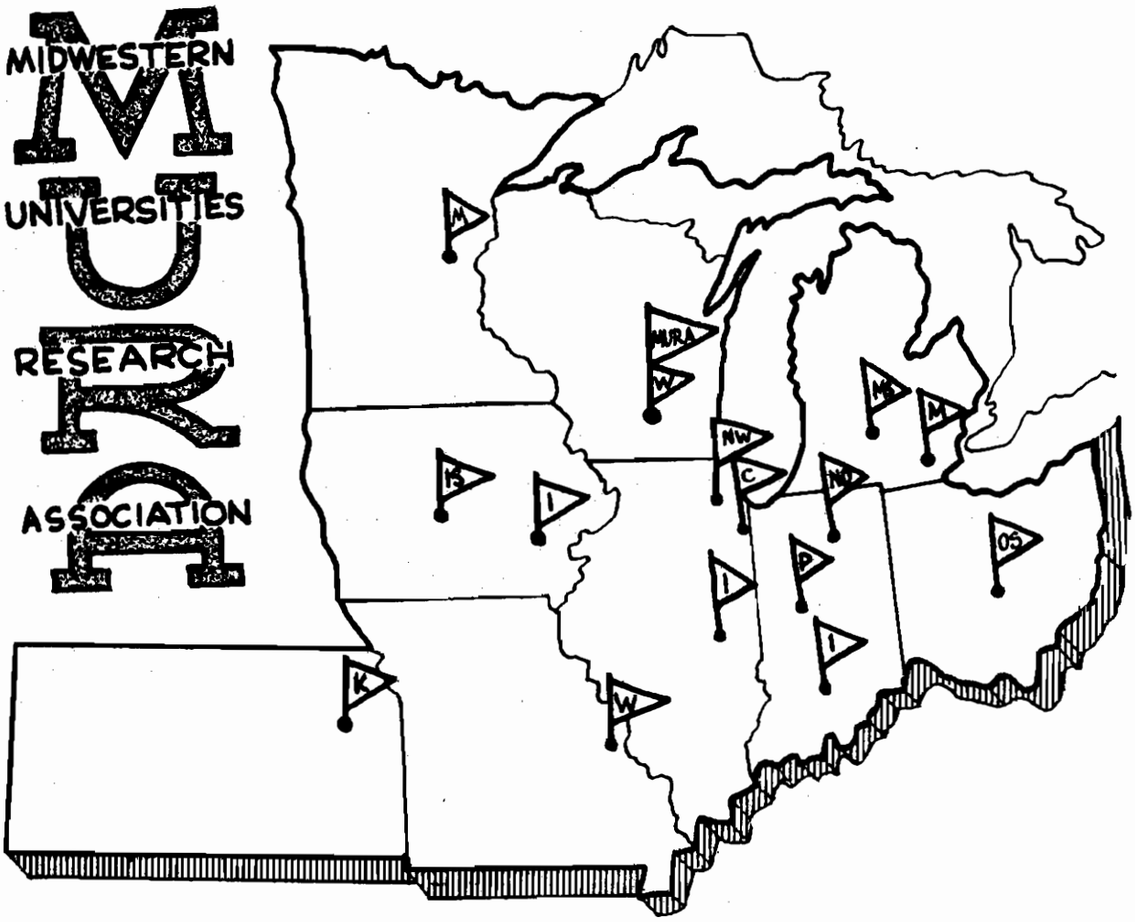

SCREENING WITH CURRENT SHEETS

J. Van Bladel

REPORT NUMBER 619

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I. INTRODUCTION

The following problem has been proposed by K. R. Symon, in connection with the design of air core magnets. Consider an annular volume V_1 (the vacuum chamber of an accelerator, for instance) which contains no currents and no magnetic material. It is desired to establish a given magnetic field $\bar{h}_0$ in V_1 by means of surface currents $\bar{K}_1$ flowing on the boundary S_1. It is furthermore desired to investigate the existence of a surrounding surface S_2, tangent to the magnetic field at each of its points. Notice that, if S_2 exists, a system of surface currents $\bar{K}_2 = -\bar{u}_{n2} \times \bar{h}$ (where $\bar{h}$ is the value just inside S_2) would cancel out the magnetic field outside the sausage-like volume V_2. The question is: does such a surface exist, and is the problem soundly formulated from a theoretical point of view? We advance the following answer to this question: "Given two arbitrary surfaces S_1 and S_2 of the type shown in Fig. 2, it is always possible to find surface currents $\bar{K}_1$ and $\bar{K}_2$ which cancel out the magnetic field outside. Furthermore, the solution is not unique, and an infinite number of combinations of $\bar{K}_1$ and $\bar{K}_2$ can be found."

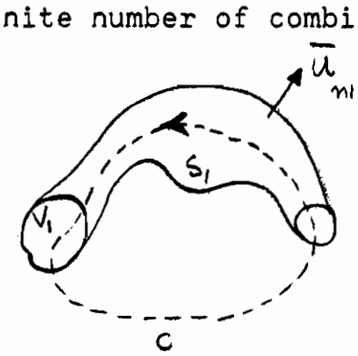

Fig. 1

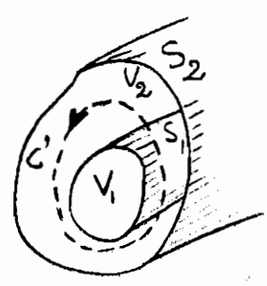

Fig. 2

II. THE GENERAL PROBLEM

Let us make a few preliminary remarks before giving a general proof of our statement. Consider the doubly-connected volume V_1 of Fig. 1. It is possible to show* that a non-zero vector $\bar{h}_1$ exists, which has zero divergence and zero curl inside the volume, and is tangent to the boundary surface. In mathematical terms:

$$\begin{aligned} \operatorname{div} \bar{h}_1 &= 0 \\ \operatorname{curl} \bar{h}_1 &= 0 \\ \bar{u}_n \cdot \bar{h}_1 &= 0 \end{aligned} \tag{1}$$

In a simply-connected volume there is no such vector. In V_1, however, there is an infinity of solutions to (1), all multiples of each other, and derivable from a multi-valued scalar potential. We make the solution unique by normalizing $\bar{h}_1$ to have unit circulation around a curve C surrounding the hole of the "donut"

$$\int_C \bar{h}_1 \cdot d\bar{l} = 1$$

When considering volume V_2, it immediately appears that <u>two</u> independent solutions to (1) exist: first, a solution $\bar{h}_1$ which has unit circulation around a curve C surrounding the hole of the donut; second, a solution $\bar{h}_2$ which has unit circulation around the "inner tube"

$$\int_{C'} \bar{h}_2 \cdot d\bar{l} = 1$$

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* H. Weyl. "Uber die Randwertaufgabe der Strahlungstheorie und Asymptotische Spektralgesetze" J. Reine Angew. Math. <u>143</u>. pp 177-202, 1913

and zero circulation around C (similarly, $\bar{h}_1$ has zero circulation around C'). Any linear combination of $\bar{h}_1$ and $\bar{h}_2$ is a solution of (1), and conversely.

We are now ready to proceed to the general problem. The magnetic field in V_2 must satisfy the following requirements

$$\begin{aligned} &\operatorname{div} \bar{h} = 0 \\ &\operatorname{curl} \bar{h} = 0 \\ &\bar{u}_{n1} \cdot \bar{h} = \bar{u}_{n1} \cdot \bar{h}_0 \text{ on } S_1 \\ &\bar{u}_{n2} \cdot \bar{h} = 0 \text{ on } S_2 \end{aligned} \tag{2}$$

This is not sufficient to determine $\bar{h}$ uniquely. Our previous considerations show, indeed, that $\bar{h}$ is determined with the exception of a linear combination of $\bar{h}_1$ and $\bar{h}_2$. We make the problem unique by specifying the circulation of $\bar{h}$ around C and C'. Let us consider, in particular, the solution $\bar{h}'$ which has <u>zero</u> circulation around C and C'

$$\int_C \bar{h}' \cdot d\bar{l} = \int_{C'} \bar{h}' \cdot d\bar{l} = 0 \tag{3}$$

This solution is unique. It also <u>exists</u>. With conditions (3), indeed, $\bar{h}'$ derives from a scalar potential which is single-valued throughout V_2. Writing $\bar{h}' = \operatorname{grad} \psi'$, we obtain the differential system

$$\begin{aligned} &\nabla^2 \psi' = 0 \\ &\frac{\partial \psi'}{\partial n} = 0 \text{ on } S_2 \\ &\frac{\partial \psi'}{\partial n} = \bar{u}_{n1} \cdot \bar{h}_0 \text{ on } S_1 \end{aligned} \tag{4}$$

This is now a typical Neumann problem of the kind encountered in potential theory. It always admits a solution, provided

$$\iint_S \frac{\partial \Psi}{\partial n}\, dS = 0$$

on the boundary surface. But this is automatically ensured by the divergence-less character of $\bar{h}_0$. The solution for Ψ' is, of course, determined with the exception of an additive constant, but this is without effect on the value of $\bar{h}'$, which is unique.

To ensure continuity of the normal components of the magnetic field, then, we discover that the magnetic field in V_2 must be of the form

$$\bar{h} = \bar{h}' + A\bar{h}_1 + B\bar{h}_2 \tag{5}$$

The condition on the tangential components of $\bar{h}$ implies that the surface currents must be

$$\begin{aligned} \bar{K}_1 &= \bar{u}_{n1} \times \bar{h}' - \bar{u}_{n1} \times \bar{h}_0 + A\,\bar{u}_{n1} \times \bar{h}_1 + B\,\bar{u}_{n1} \times \bar{h}_2 \\ \bar{K}_2 &= -\bar{u}_{n2} \times \bar{h}' - A\,\bar{u}_{n2} \times \bar{h}_1 - B\,\bar{u}_{n2} \times \bar{h}_2 \end{aligned} \tag{6}$$

Conversely, and this is important, the system of surface currents (6) will create the desired magnetic field, i.e: zero outside S_2, $\bar{h}$ in V_2 and $\bar{h}_0$ in V_1. This is evident on intuitive grounds, but might be worthwhile checking mathematically. The magnetic field created by $\bar{K}_1$ and $\bar{K}_2$ satisfies the conditions

$$\operatorname{div} \bar{h} = 0$$
$$\operatorname{curl} \bar{h} = 0$$

$\bar{h}$ regular at infinity

$\bar{u}_n \times \bar{K}$ jump in tangential component of $\bar{h}$ on S_1 and S_2

$\bar{u}_n \cdot \bar{h}$ continuous on S_1 and S_2 (7)

The solution to this problem is unique. Assume, indeed, that two different solutions exist. Their difference $\bar{h}_d$ would satisfy

$$\operatorname{div} \bar{h}_d = 0$$
$$\operatorname{curl} \bar{h}_d = 0$$

$\bar{h}_d$ regular at infinity

and be continuous everywhere. The circulation of $\bar{h}_d$ would be zero around any closed curve of space, allowing us to write $\bar{h}_d = \operatorname{grad} \Psi_d$ with

$$\nabla^2 \Psi_d = 0$$

Ψ_d and first derivatives continuous

Ψ_d regular at infinity

Ψ_d single valued

The only solution to this problem is $\Psi_d = 0$. We conclude that the solution of (7) is unique, and that the desired magnetic field, which satisfies (7), is indeed the field generated by $\bar{K}_1$ and $\bar{K}_2$.

The significance of the constants A and B can be clarified by the following comments. The total current flowing <u>longitudinally</u> along the inner tube S_1 is given by

$$i_1 = \int_{C_1} \bar{K}_1 \cdot (\bar{u}_{n1} \times \bar{u}_{c1})\, dc_1 = B$$

Similarly, the longitudinal current along the outer tube S_2 is

$$i_2 = -B$$

The total longitudinal current is zero, as expected from the fact that the magnetic field vanishes outside S_2. The current <u>encircling</u> the inner tube is

$$I_1 = \int_c \bar{K}_1 \cdot (\bar{u}_c \times \bar{u}_{n1})\, dc$$
$$= \int_c \bar{h}_0 \cdot d\bar{c} - A = I_0 - A$$

the positive senses being indicated on Fig. 4. The current encircling the outer tube, in the same sense, is + A. The net encircling current for a curve of type C is I_0, as expected.

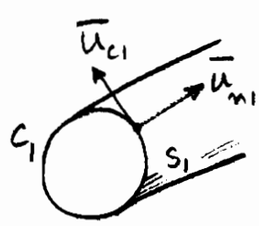


Fig. 3.

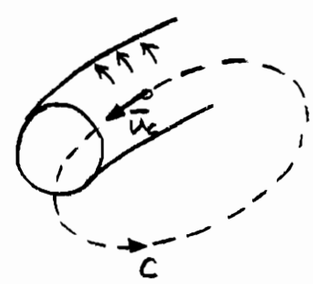


Fig. 4.

Formulas (6) clearly show that many possible surface current distributions are capable of screening the inside magnetic field. In fact, one disposes of two degrees of freedom which will allow one to optimize the current distribution with respect to some criterion, such as maximum allowable current density, or minimum heat dissipation in the conductors.

III. VOLUMES OF REVOLUTION

The simplification here is that $\bar{h}_1$ is known explicitly

$$\bar{h}_1 = \frac{\bar{u}_\varphi}{2\pi r}$$

while $\bar{h}_2$ is purely meridian and satisfies

$$\begin{aligned} \operatorname{curl} \bar{h}_2 &= \frac{\partial h_{2r}}{\partial z} - \frac{\partial h_{2z}}{\partial r} = 0 \\ \operatorname{div} \bar{h}_2 &= \frac{\partial h_{2r}}{\partial r} + \frac{1}{r} h_{2r} + \frac{\partial h_{2z}}{\partial z} = 0 \\ \bar{u}_n \cdot \bar{h}_2 &= 0 \\ \int_{C'} \bar{h}_2 \cdot d\bar{c} &= 1 \end{aligned} \tag{8}$$

In other words, $\bar{h}_2$ is given by

$$\begin{aligned} \bar{h}_2 &= \operatorname{grad} \psi_2 \\ \frac{\partial^2 \psi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z^2} &= 0 \\ \frac{\partial \psi_2}{\partial n} &= 0 \quad \text{on } C_1 \text{ and } C_2 \\ \psi_2 &\text{ multi-valued with period one around } C_1 \end{aligned} \tag{9}$$

This problem can be solved explicitly for certain simple geometries (e.g., in toroidal coordinates).

The Joule losses in the current sheets are proportional to

$$\iint_{S_1} |\bar{K}_1|^2 dS_1 + \iint_{S_2} |\bar{K}_2|^2 dS_2 = \iint_{S_1} |\bar{h}' - \bar{h}_0|^2 dS + A^2 \iint_{S_1+S_2} |\bar{h}_1|^2 dS + B^2 \iint_{S_1+S_2} |\bar{h}_2|^2 dS$$

$$+ \iint_{S_2} |\bar{h}'|^2 dS + 2A \iint_{S_1} \bar{h}_1 \cdot (\bar{h}' - \bar{h}_0) dS_1 + 2B \iint_{S_1} \bar{h}_2 \cdot (\bar{h}' - \bar{h}_0) dS_1 \qquad (10)$$

$$+ 2A \iint_{S_2} \bar{h}' \cdot \bar{h}_1 \, dS_2 + 2B \iint_{S_2} \bar{h}' \cdot \bar{h}_2 \, dS_2$$

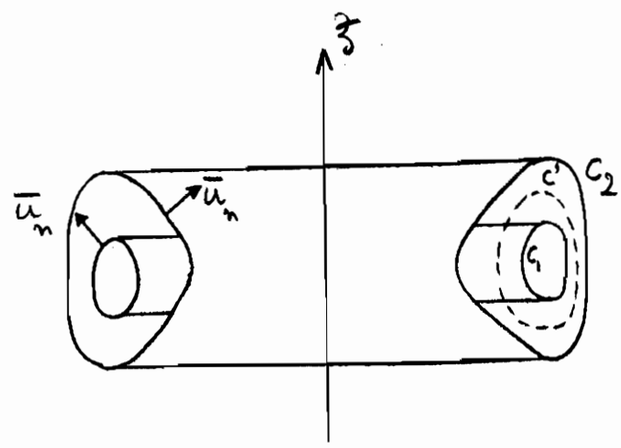


Fig. 5

The value of A which minimizes this quantity is

$$A = - \frac{\iint_{S_1} \bar{h}_1 \cdot (\bar{h}' - \bar{h}_0) dS_1 + \iint_{S_2} \bar{h}' \cdot \bar{h}_1 \, dS_2}{\iint_{S_1+S_2} |\bar{h}_1|^2 dS} = + \frac{\frac{1}{2\pi} \left[\int_{C_1} \frac{dc}{r} \int_0^{2\pi} \bar{h}_0 \cdot \bar{u}_\varphi \, r \, d\varphi \right]}{\frac{1}{2\pi} \int_{C_1+C_2} \frac{dc}{r}}$$

if we take into account that $\int_0^{2\pi} (\bar{h}' \cdot \bar{u}_\varphi) r \, d\varphi = 0$ from (3). On the

other hand, $\int_0^{2\pi} \bar{h}_0 \cdot \bar{u}_\varphi \, r \, d\varphi = I_0$, so that the optimum value of A turns out to be

$$A = I_0 \frac{\int_{c_1} \frac{dc}{r}}{\int_{c_1} \frac{dc}{r} + \int_{c_2} \frac{dc}{r}} \tag{11}$$

Let us assume, as often happens in practice, that the desired field is perpendicular to the equatorial plane, that surfaces S_1 and S_2 are symmetric with respect to that plane, and that the lines of force have the symmetry evidenced in Fig. 6 (a). It is immediately apparent that the optimum value of A is zero in these conditions, because $I_0 = 0$. In other words, we don't want any part of the field $\bar{h}_1 = \frac{\bar{u}_\varphi}{2\pi r}$ in our magnetic field if it is desired to minimize current losses. Our particular geometry also entails, as can be seen from (4), that ψ' will be anti-symmetric with respect to the equatorial plane, so that $\bar{h}'$ possess the same symmetry as $\bar{h}_0$.

(a) (b)

Fig. 6

IV. STRAIGHT TUBES

It is often permissible to neglect the effect of curvature in the configuration of Fig. 5, and to develop the donut into a cylindrical structure with periodic boundary conditions at $z=0$ and $z=L$. Much of the analysis presented above can easily be adapted to the new configuration. The basic equations for $\bar{h}$ are precisely as in (2), with the added requirement that $\bar{h}$ take identical values at homologous points in the $z=0$ and $z=L$ planes. Equations (2) again determine $\bar{h}$ with the exception of a linear combination of $\bar{h}_1$ and $\bar{h}_2$, where $\bar{h}_1$ is $\frac{\bar{u}_z}{L}$, and $\bar{h}_2$ is grad ψ_2 where

$$\nabla^2_{xy}\,\psi_2 = 0$$

$$\frac{\partial \psi_2}{\partial n} = 0 \quad \text{on } C_1 \text{ and } C_2$$

$$\psi_2 \text{ is multi-valued with period one around } C_1 \tag{12}$$

We make the problem unique, and obtain a field $\bar{h}'$, by requiring that $\int_{C_1} \bar{h}'\cdot d\bar{l} = \int_{C_2} \bar{h}'\cdot d\bar{l} = 0$ and $\int_0^L h'_z\, dz = 0$. This ensures that $\bar{h}'$ derives from a scalar potential ψ' which is single-valued and periodic, i.e., takes identical values at homologous points in the $z=0$ and $z=L$ planes. This being settled, equations (4) to (6) are still valid.

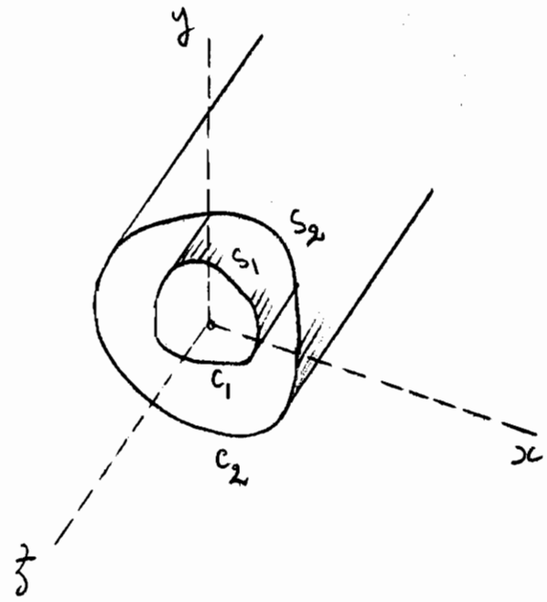


Fig. 7

The simplicity of the cylindrical structure allows us to tackle an elementary numerical example. Let S_1 and S_2 be coaxial circular cylindrical surfaces, and let $\bar{h}_0$ be a homogeneous field

$$\bar{h}_0 = \bar{u}_y = \bar{u}_r \sin\varphi + \bar{u}_\varphi \cos\varphi$$

The potential ψ' will obviously be independent of z, and will satisfy

$$\nabla^2 \psi' = \frac{\partial^2 \psi'}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \psi'}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 \psi'}{\partial \varphi^2} = 0$$

$$\frac{\partial \psi'}{\partial r} = 0 \quad \text{at } r = a$$

$$\frac{\partial \psi'}{\partial r} = \bar{u}_r \cdot \bar{h}_0 = \sin\varphi \quad \text{at } r = b$$

The single-valued solution is

$$\psi' = -\frac{ab^2}{a^2-b^2}\sin\varphi\left[\frac{a}{r}+\frac{r}{a}\right]$$

giving a field

$$\bar{h}' = \text{grad}\,\psi' = \frac{b^2}{r^2}\frac{a^2-r^2}{a^2-b^2}\sin\varphi\,\bar{u}_r - \frac{b^2}{r^2}\frac{a^2+r^2}{a^2-b^2}\cos\varphi\,\bar{u}_\varphi \tag{13}$$

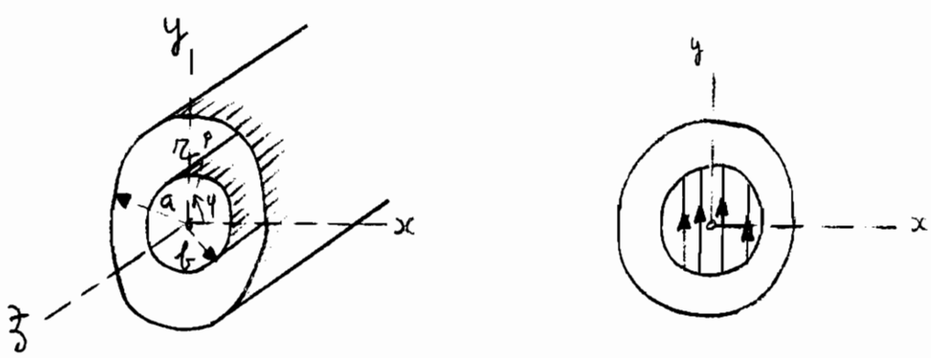
and other possible fields

$$\bar{h} = \bar{h}' + \frac{B}{2\pi r}\bar{u}_\varphi + \frac{A}{L}\bar{u}_z$$

which entail surface currents

$$\bar{K}_1 = \bar{u}_r \times (\bar{h}-\bar{h}_0) = -\frac{A}{L}\bar{u}_\varphi + \left[\frac{B}{2\pi b} - \frac{2a^2}{a^2-b^2}\cos\varphi\right]\bar{u}_z$$

$$\bar{K}_2 = -\bar{u}_r \times \bar{h} = \frac{A}{L}\bar{u}_\varphi - \left[\frac{B}{2\pi a} - \frac{2b^2}{a^2-b^2}\cos\varphi\right]\bar{u}_z \tag{14}$$



(a) (b)

Fig. 8

Figure 9 shows a typical plot of $\bar{h}'$, and indicates the current distributions which generate this field, and which are obtained by setting $A = B = 0$ in (14). It is a simple matter to check that this choice of coefficients minimizes heat losses.

Let us extend our analysis to further examples. Consider again the structure of Fig. 8, but let the magnetic field $\bar{h}_0$ be non-uniform, while keeping the kind of symmetry indicated in Fig. 6. The potential from which $\bar{h}_0$ derives will be odd in φ. It is a solution of Laplace's equation, and can be expressed as

$$\psi_0 = \sum_{m=1}^{\infty} A_m r^m \sin m\varphi$$

Coefficients A_m can be determined from a knowledge of $|\bar{h}_0|$ along the x axis. Indeed

$$\bar{h}_0 = \sum_{m=1}^{\infty} m A_m r^{m-1} \sin m\varphi \, \bar{u}_r + m A_m r^{m-1} \cos m\varphi \, \bar{u}_\varphi$$

or, along the x axis

$$|\bar{h}_0| = \sum_{m=1}^{\infty} m A_m x^{m-1}$$

A power series expansion of $|\bar{h}_0|$ will yield coefficients A_m, and allow plotting of $\bar{h}_0$ everywhere. Instead of carrying this analysis further in full generality, let us limit ourselves to an example where two terms are present.

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SCREENING WITH CURRENT SHEETS

J. Van Bladel**

May 18, 1961

ABSTRACT

It is desired to establish a given magnetic field $\bar{h}_o$ inside the vacuum chamber of an accelerator, and to screen the surrounding space from this magnetic field. The present report investigates the possibility of achieving this purpose with suitably disposed electric current sheets.

* AEC Research and Development Report. Research supported by the U.S.

**EE, Dept. of University of Wisconsin

$$\psi_0 = \frac{r}{b}\sin\varphi + \frac{1}{4}\frac{r^2}{b^2}\sin 2\varphi$$

$$\bar{h}_0 = \left[\frac{\sin\varphi}{b} + \frac{r}{2b^2}\sin 2\varphi\right]\bar{u}_r + \left[\frac{\cos\varphi}{b} + \frac{r}{2b^2}\cos 2\varphi\right]\bar{u}_\varphi$$

We find

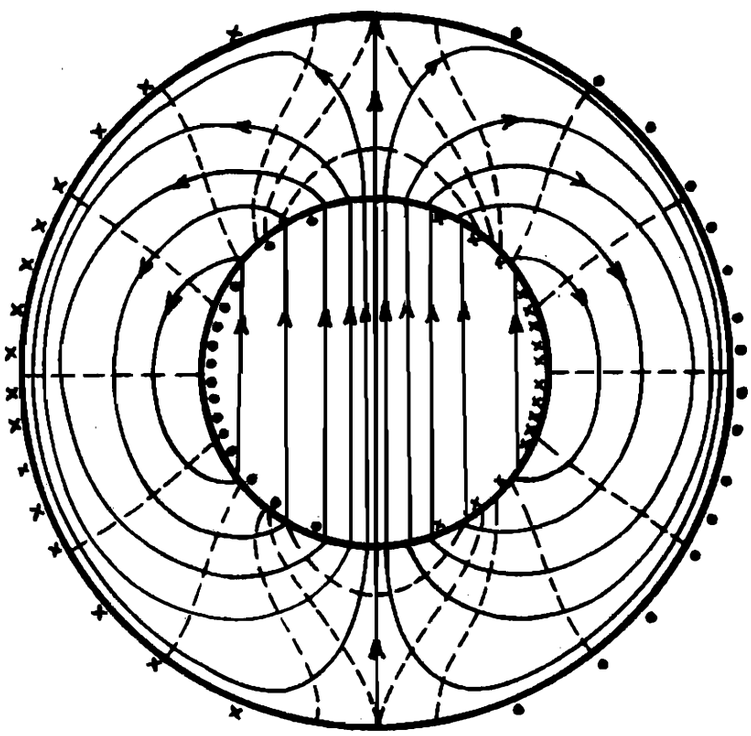
$$\psi' = -\frac{ab}{a^2-b^2}\sin\varphi\left[\frac{a}{r}+\frac{r}{a}\right] - \sin 2\varphi\,\frac{a^2b^2}{4(a^4-b^4)}\left[\frac{a^2}{r^2}+\frac{r^2}{a^2}\right]$$

$$\bar{h}' = -\frac{ab}{a^2-b^2}\left\{\left[\sin\varphi\left(\frac{1}{a}-\frac{a}{r^2}\right)+\frac{ab}{2(a^2+b^2)}\sin 2\varphi\left(\frac{r}{a^2}-\frac{a^2}{r^3}\right)\right]\bar{u}_r\right.$$
$$\left.+\left[\cos\varphi\left(\frac{1}{a}+\frac{a}{r^2}\right)+\frac{ab}{2(a^2+b^2)}\cos 2\varphi\left(\frac{r}{a^2}+\frac{a^2}{r^3}\right)\right]\bar{u}_\varphi\right\}$$

$$\bar{K}_1' = -\left[\frac{2a^2}{b(a^2-b^2)}\cos\varphi + \frac{a^4}{b(a^4-b^4)}\cos 2\varphi\right]\bar{u}_z$$

$$\bar{K}_2' = \left[\frac{2b}{a^2-b^2}\cos\varphi + \frac{ab^2}{a^4-b^4}\cos 2\varphi\right]\bar{u}_z$$

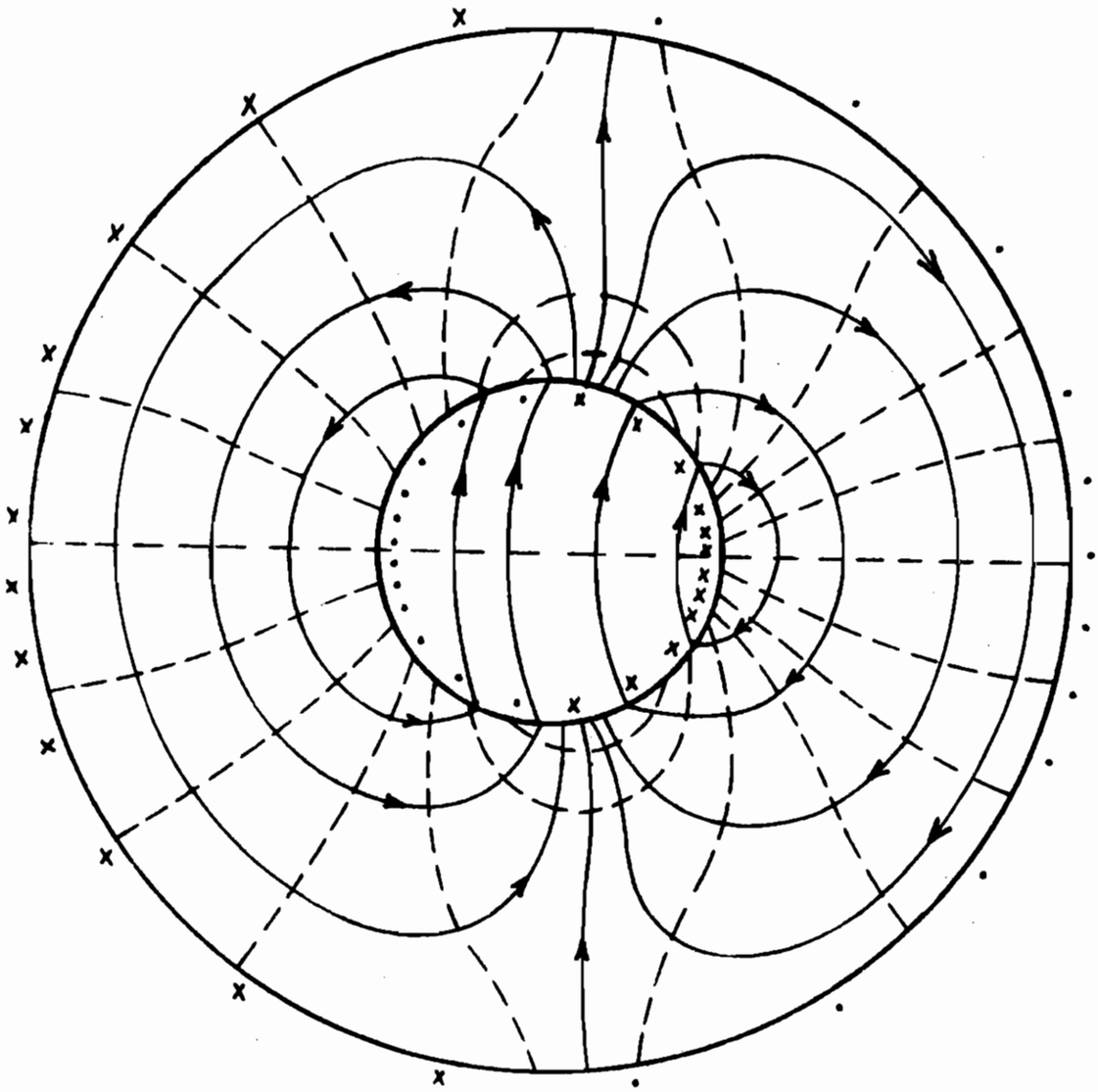
A typical sketch of these quantities appears in Figure 10.



--- Equipotentials ψ'

——— Lines of $\bar{h}'$

Fig. 9



---- Equipotentials ψ'

——— Lines of $\bar{h}'$

Fig. 10