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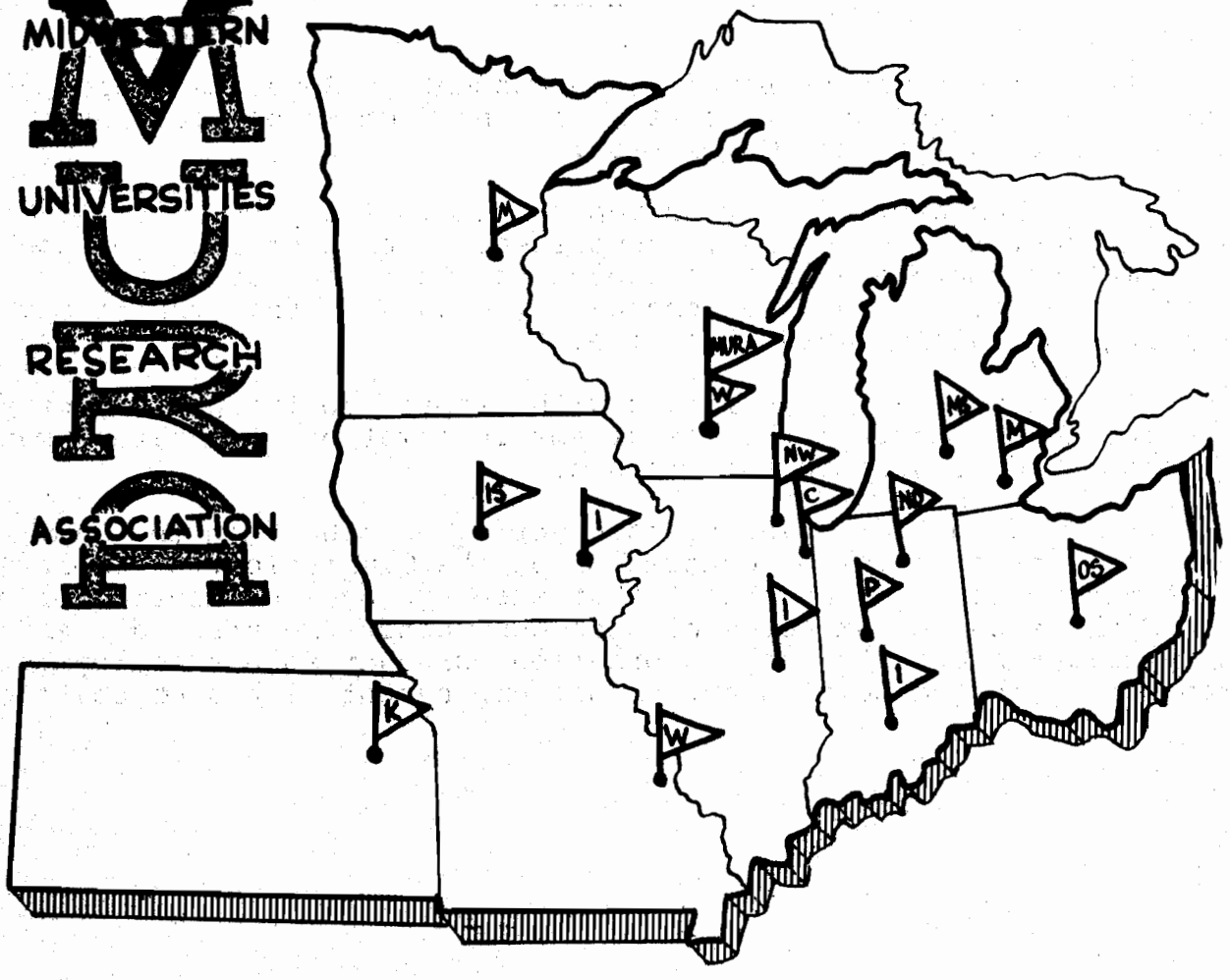

EFFECTS OF RADIAL STRAIGHT SECTIONS FROM A GENERAL POINT OF VIEW

W. N. Wong

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EFFECTS OF RADIAL STRAIGHT SECTIONS FROM A GENERAL POINT OF VIEW

February 27, 1961

W. N. Wong

ABSTRACT

General properties of equilibrium orbits and techniques in the amplitude modulation of the field are discussed in preparation for a formal proof to deny the existence of any radial straight section distribution scheme, be it purely of academical interest or of actual constructional practicability, that will preserve the scaling property of the equilibrium orbits of a certain class of field pattern.

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I. General Properties of Equilibrium Orbits

The vertical component of the field on the median plane will be denoted by H and the particle scalar momentum by p. Then

$$\kappa = \frac{1}{p} H \tag{1}$$

is the curvature of the orbit (e=c=1 as unit). If the equilibrium orbit (abbreviated E.O. in the following) is $r(\theta)$, θ being the polar angle, κ is given by

$$\kappa = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} \qquad \text{(where prime denotes } \frac{d}{d\theta}\text{)} \tag{2}$$

A small quanitity, $r = R(1+x)$, is defined such that $\bar{r} = R$ and $\bar{x} = o$, where a bar denotes the operation of averaging over θ, i.e.,

$$\overline{f(\theta)} = \frac{1}{T}\int_0^T d\theta\, f(\theta) \qquad (T = 2\pi \text{ usually}).$$

Thus, x is essentially the ratio of the AC component to the DC component in r, so in the Fourier resolution of x,

$$x = \sum_{\nu \neq 0} x_\nu(R)\, e^{i\nu\theta}, \tag{3}$$

there is no $\nu = o$ term. As the notation indicates, the complex amplitudes $x_\nu(R)$ are in general functions of R. In particular such R-dependence may appear just in the form of an overall multiplicative factor $\sigma(R)$ and a phase $\Omega(R)$:

$$x = \sigma(R) \sum_{\nu \neq 0} x_\nu e^{i\nu(\theta + \Omega(R))} \tag{3a}$$

in which x_ν are now complex numbers depending on ν but not on R. In still more particular case, σ(R) = 1, then the orbits are said to be "Scaling".

<u>Theorem 1</u> A necessary and sufficient condition for E.O. to scale is that the R-dependence in the ratio,

$$\frac{H(R(1+x), \theta)}{\overline{H}(R(1+x), \theta)},$$

can be completely transformed away by a rotation.

<u>Proof</u> The following expressions are easily obtained from (1) and (2)

$$R\kappa = \frac{2x'^2 - (1+x)x'' + (1+x)^2}{\left[x'^2 + (1+x)^2\right]^{3/2}}, \quad \overline{\kappa} = \frac{1}{\rho}\overline{H}, \quad \frac{R\kappa}{R\overline{\kappa}} = \frac{H}{\overline{H}}.$$

Since $R\kappa$ depends on R only through x, so should be the ratio $\frac{H}{\overline{H}}$. If a solution of (1) has the form

$$x = \sum_{\nu \neq 0} x_\nu e^{i\nu(\theta + \Omega(R))},$$

then $\frac{H}{\overline{H}}$, whatever it is, can only assume a similar functional structure

$$\frac{H}{\overline{H}} = \sum_{\mu} f_\mu e^{i\mu(\theta + \Omega(R))}.$$

Consequently it is necessary that

$$\frac{H}{\bar{H}} = \mathcal{F}(x, \theta + \Omega(R)).$$

The rotation $\phi \equiv \theta + \Omega(R)$ eliminates R-dependence completely. For sufficiency, we look at the equation

$$\frac{Rx}{R\bar{x}} = \mathcal{F}(x, \phi), \tag{1a}$$

which is equivalent to (1). Since there is no R in this equation, periodicity requirement restricts the solution to the form.

$$x = \sum_{\nu \neq 0} x_\nu e^{i\nu\phi} = \sum_{\nu \neq 0} x_\nu e^{i\nu(\theta + \Omega(R))}.$$

Q.E.D.

If the phase $\Omega(R)$ in E.O. is differentiable and $\Omega'(R)$ is continuous within the range of R interested, we may call this E.O. to be "continuously scaling" in the sense that when R changes by dR, it rotates by an angle $\Omega'(R)dR$. In particular $\Omega'(R)$ may be zero, in which case, the orbits for different R are all similarly orientated with respect to the phase of the field.

<u>Theorem 2</u> If $H(r, \theta)$ within a finite angular span $\theta_1 \leq \theta \leq \theta_2$ (or spans) <u>regardless of r</u> possesses a certain property not shared by the field outside this span, and correspondingly, that portion of the E.O. within this angular span inherits from this property of H a unique property of its own not enjoyed by any other portion of the E.O. outside this span, then continuous scaling is impossible unless $\Omega'(R) = 0$.

The proof is trivial. For, if $\Omega'(R) \neq 0$, then for <u>any</u> finite change ΔR of R, the E.O. rotates a finite angle $\Omega'(R)\Delta R \neq 0$. Hence that portion of the E.O. initially inside the span ($\theta_1 \leq \theta \leq \theta_2$) now goes out of it. This contradicts the conditions specified in the hypothesis.

As an example, we look at the field very generally represented (in Parzen's notation) by

$$H(r,\theta) = \sum_n G_n(r)\, e^{-i\beta_n(r)} e^{inN\theta} \qquad N \sim \text{integers}$$

If $G_n(r)$ is factorable and $\beta_n(r)$ separable respectively like:

$$G_n(R(1+x)) = \rho(R)\, g_n(x) \tag{4a}$$

$$\beta_n(R(1+x)) = \psi_n(x) - nN\Omega(R), \tag{4b}$$

then
$$\frac{H}{\bar{H}} = \frac{\cancel{\rho(R)} \sum_n g_n(x)\, e^{-i\psi_n(x)}\, e^{inN(\theta+\Omega(R))}}{\cancel{\rho(R)}\, \frac{1}{2\pi}\int_0^{2\pi} d\theta \sum_n g_n(x)\, e^{-i\psi_n(x)}\, e^{inN(\theta+\Omega(R))}} .$$

The rotation $\phi = \theta + \Omega(R)$ eliminates all R, so such a field yields scaling E.O. The spiral FFAG is one of this variety and is continuously scaling. The two conditions listed are very stringent. They are sufficient to ensure $\frac{H}{\bar{H}} = \mathcal{F}(x,\phi)$ but nothing can be said about necessity which we shall not investigate.

II. Techniques in Amplitude Modulation

The simplest representation of the most general square wave function for amplitude modulation in the straight section problem has the form

$$S_+(\mu(\theta)) \equiv \frac{1}{2}\left(1 + \frac{\mu(\theta)}{|\mu(\theta)|}\right), \tag{5}$$

where $\mu(\theta)$ is a periodic function of θ of a certain period (say 2π) with a number of simple zeros on the reference circle R. These zeros should coincide with the "entrance" and "exit" points of the desired straight sections. $\mu(\theta)$ should be continuous, single valued in the neighbourhood of any zero (a finite number of jump discontinuities not in the neighborhood of the zeros is permitted) and is otherwise completely arbitrary. The precise meaning of the foregoing statement can be easily grasped by looking at Fig. 1.

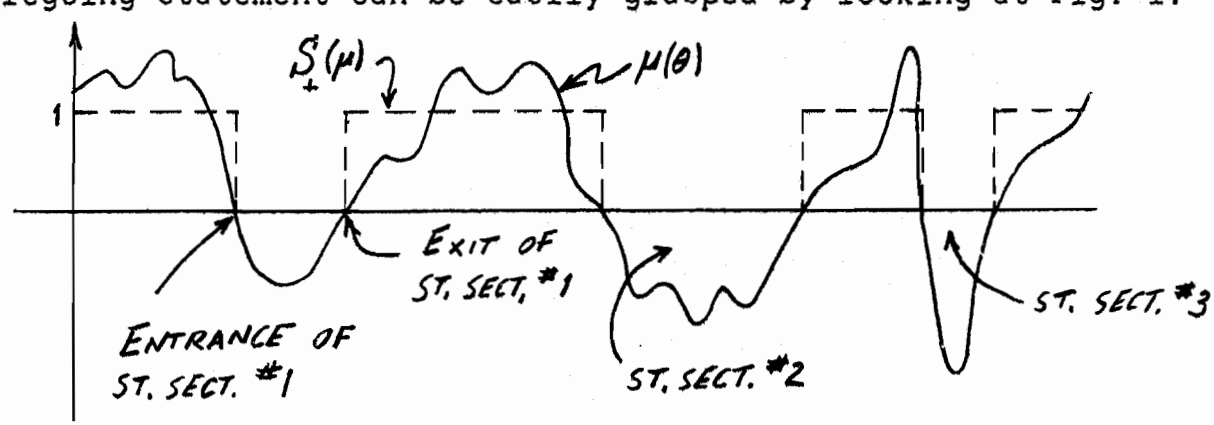

FIG. 1

S_+ can also be conveniently put in the following forms:

$$S_+(\mu) = \frac{1}{2\pi i}\int_{\Omega} dt\,\frac{e^{-i\mu t}}{t} = \lim_{\eta\to 0}\frac{-1}{2\pi i}\int_{-\infty}^{\infty} dt\,\frac{e^{-i\mu t}}{t+i\eta} \tag{5a}$$

where the path Ω implies detouring into UHP of the complex t-plane when by-passing the pole at the origin. The value of this integral (1 or 0) is determined by the sign of $\mu(\theta)$ and when $\mu\to 0$, whether it approaches zero from right or from left determines the value of this integral.

In exactly the same manner, another function

$$S_- \equiv \frac{1}{2\pi i}\int_{\cup} dt\,\frac{e^{-i\mu t}}{t} = \lim_{\eta\to 0}\int_{-\infty}^{\infty} dt\,\frac{e^{-i\mu t}}{t-i\eta}\,\frac{1}{2\pi i} \tag{5b}$$

may be used to achieve the modulation purpose, only now the straight sections are generated whenever μ goes to positive instead of negative as in S_+ .

Out of S_+ and S_- and with the aid of the relation

$$\frac{1}{t\pm i\eta} = \mathcal{P}\frac{1}{t} \mp \pi i\,\delta(t) \tag{6}$$

where $\mathcal{P}$ implies "principal value of" and $\delta(t)$ is the usual δ -function, we may yet construct another function which will be used in § III . This function is

$$S_0(\mu) \equiv \mathcal{P}\int_{-\infty}^{\infty} dt\,\frac{\sin\mu t}{t} \tag{5c}$$

so that $\frac{1}{\pi^2}S_0^2 = 0$ for $\mu = 0$ and is otherwise always unity. The modulation effect of S_0^2 is to create a sharp notch at the origin.

One can find an infinite number of $\mu(\theta)$ that will, through $S_{\pm}$, generate the same straight section pattern. But, with no loss of generality, we can always choose $\mu(\theta)$ the smoothest function possible, viz, the harmonic functions. The recipe for constructing a $\mu(\theta)$ with parameters associated with a particular straight section pattern we desire, follows as:

Let $x_i \equiv \cos\theta_i$ with θ_i's coincident with the entrance and exit points of the straight sections. Then the polynomial $(\pm)\prod_i (x - x_i)$ adjusted appropriately as to its overall sign (according to S_+ or S_- is used) is a qualified $\mu(\theta)$ for the purpose. This polynomial may be transformed into other convenient forms if one likes, such as

$$\prod_i (\cos\theta - \cos\theta_i) = \sum_\ell A_\ell P_\ell(\cos\theta) = \sum_n B_n \cos n\theta$$

and the coefficients A_ℓ or B_n may be used as straight section parameters.

As an example, a set of equally spaced identical straight sections can be generated by

$$\mu(\theta) = \sin M(\theta - \theta_0) + \sin\gamma \qquad 0 < \gamma < \pi/2$$

where M is the number of straight sections, θ_0 affixes the θ-location of the straight sections in relation to the phase of the field being modified, and $\sin\gamma$ is a bias constant that adjusts the length of the straight sections.

The spiral FFAG field to be modified has the form

$$\mathcal{H}(r,\theta) \equiv \sum_n h_n r^{k_n} e^{inN\theta}, \quad k_n \equiv k - inK \tag{7}$$

so the modified field is

$$[\mathcal{H}]_\mu \equiv \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt}{t+i\eta} e^{-it[\sin M(\theta-\theta_0)+\sin\gamma]} \mathcal{H}(r,\theta). \tag{8}$$

After carrying out the following:

(1) Expanding $e^{-it\sin M(\theta-\theta_0)} = \sum_{-\infty}^{\infty} J_\nu(t) e^{-i\nu M(\theta-\theta_0)}$

where $J_\nu(t)$ is the Bessel function of the first kind,

(2) Truncating the series at some optimum term: $\nu = \nu_{op}$ and $n = n_{op}$ so that interchange of summation and integration is rendered legitimate without further justification,

(3) Evaluating the integral

$$\Lambda_\nu(\gamma) \equiv \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt}{t+i\eta} e^{-it\sin\gamma} J_\nu(t)$$

with the aid of (6) and general properties of $J_\nu(t)$,

the whole thing looks like

$$[\mathcal{H}]_\mu \sim \sum_\nu \sum_n |h_n \Lambda_\nu(\gamma)| r^{k_n} e^{i\nu M(\theta-\theta_0)+i\alpha_n+i\delta_{\nu,ODD}\frac{\pi}{2}} e^{inN\theta-i\nu M\theta} \tag{8a}$$

where α_n is defined by $h_n = |h_n| e^{i\alpha_n}$

and $\delta_{\nu,ODD}$ is zero for ν even and unity for ν odd.

By invoking the sufficiency conditions cited at the end of $\S I$ to ensure $[\mathcal{H}]$ satisfies

$$\frac{[\mathcal{H}]}{\overline{[\mathcal{H}]}} = \mathcal{F}(x, \theta + \Omega(R))$$

the best we can do turns out to be the approximate situation stated in Cole & Morton's theorem. Since these sufficiency conditions are very stringent and we cannot prove that they are also necessary, we are not in a position to conclude that there is no equally spaced identical radial straight section pattern that will preserve the scaling property of the equilibrium orbits of the spiral FFAG field.

III. The Self-Consistent Approach

The preparatory work done in §I & §II will be applied in this section.

We note that within a straight section and only within a straight section, H=O. Correspondingly, from the E.O. equation, we can see that this property of H endows every point on the segment of E.O. within the straight section the unique property of K = o, or equivalently $2x'^2-(1+x)x''+(1+x)^2=0$. This statement is at least true up to the exclusion of a finite number of isolated points scattered elsewhere for which $2x'^2-(1+x)x''+(1+x)^2=0$ might be satisfied. However, they may be considered as points within straight sections of zero length. Thus Theorem II leads us to conclude: Any field pattern in which a finite number of radial straight sections are distributed will yield continuously scaling E.O. only of the form $x=\sum_{\nu\neq 0} x_\nu e^{i\nu\theta}$, i.e., the phase function $\Omega(R)$ = constant.

Next, let us suppose that there exists a certain radial straight section distribution corresponding to the modulating function $S_+(\mu_1(\theta))$ with a suitably chosen $\mu_1(\theta)$ such that

$$S_+(\mu_1(\theta))\,\mathcal{H}(R(1+x),\theta)$$

preserves the scaling property of the spiral FFAG field $\mathcal{H}$. Then the E.O. $x=\sum_{\nu\neq 0} x_\nu e^{i\nu\theta}$ is given by the equation

$$K=\frac{1}{p}S_+(\mu_1(\theta))\,\mathcal{H}.$$

We now claim that, as a necessary consistancy requirement, this solution must also satisfy the following equation

$$\kappa = \frac{1}{p}\frac{1}{\pi^2} S_0'^2(\mathcal{X})\mathcal{H} \qquad \mathcal{X} \equiv 2x'^2 - (1+x)x'' + (1+x)^2 \tag{9}$$

Or, written out explicitly and eliminating p by using $\bar{\kappa} = \frac{1}{p}\bar{H}$

$$\frac{2x'^2 - (1+x)x'' + (1+x)^2}{\left[x'^2 + (1+x)^2\right]^{3/2}} = \frac{R\bar{\kappa}}{\pi^2}\frac{S_0'^2(2x'^2 - (1+x)x'' + (1+x)^2)\mathcal{H}(R(1+x),\theta)}{\overline{S_0'^2\mathcal{H}}} \tag{9a}$$

The reason is obvious, for $S_0'^2(\mathcal{X})$ should have the same modulating effect as $S_+'(\mu_1(\theta))$ provided x is the desired solution of the E.O. equation.

In this consistency equation $R\kappa$ depends on R only through x while in

$$\frac{S_0'^2(\mathcal{X})\mathcal{H}}{\overline{S_0'^2\mathcal{H}}} \sim \frac{R^k S_0'^2(\mathcal{X})\sum_n h_n (1+x)^{k-inK} e^{-inN(\theta - \frac{K}{N}\ln R)}}{\frac{R^k}{2\pi}\int_0^{2\pi} d\theta\, S_0'^2(\mathcal{X})\sum_n h_n (1+x)^{k-inK} e^{-inN(\theta - \frac{K}{N}\ln R)}}$$

the remaining R-dependence can be completely transformed away by $\phi = \theta - \frac{K}{N}\ln R$. So this consistency equation is essentially of the structure

$$F(x, x', x'', \phi) = 0$$

and periodicity requirement limits any of its solutions to the form $x = \sum_\mu x_\mu e^{i\mu\phi} = \sum_\mu x_\mu e^{i\mu(\theta - \frac{K}{N}\ln R)}$ in which $\Omega'(R) \neq 0$ unless K = 0, but $K \neq 0$ lest the spiral field reduces to a conventional circular field. Consequently, none of the solutions has the form $\sum_{\nu \neq 0} x_\nu e^{i\nu\theta}$ and we may conclude, either $\mu_1(\theta)$ does not exist or else its effect is just to generate straight sections of zero length and shall be of no interest to us.

The same argument also applies to the whole class of field specified by (4a) & (4b) in which the spiral FFAG is a member.