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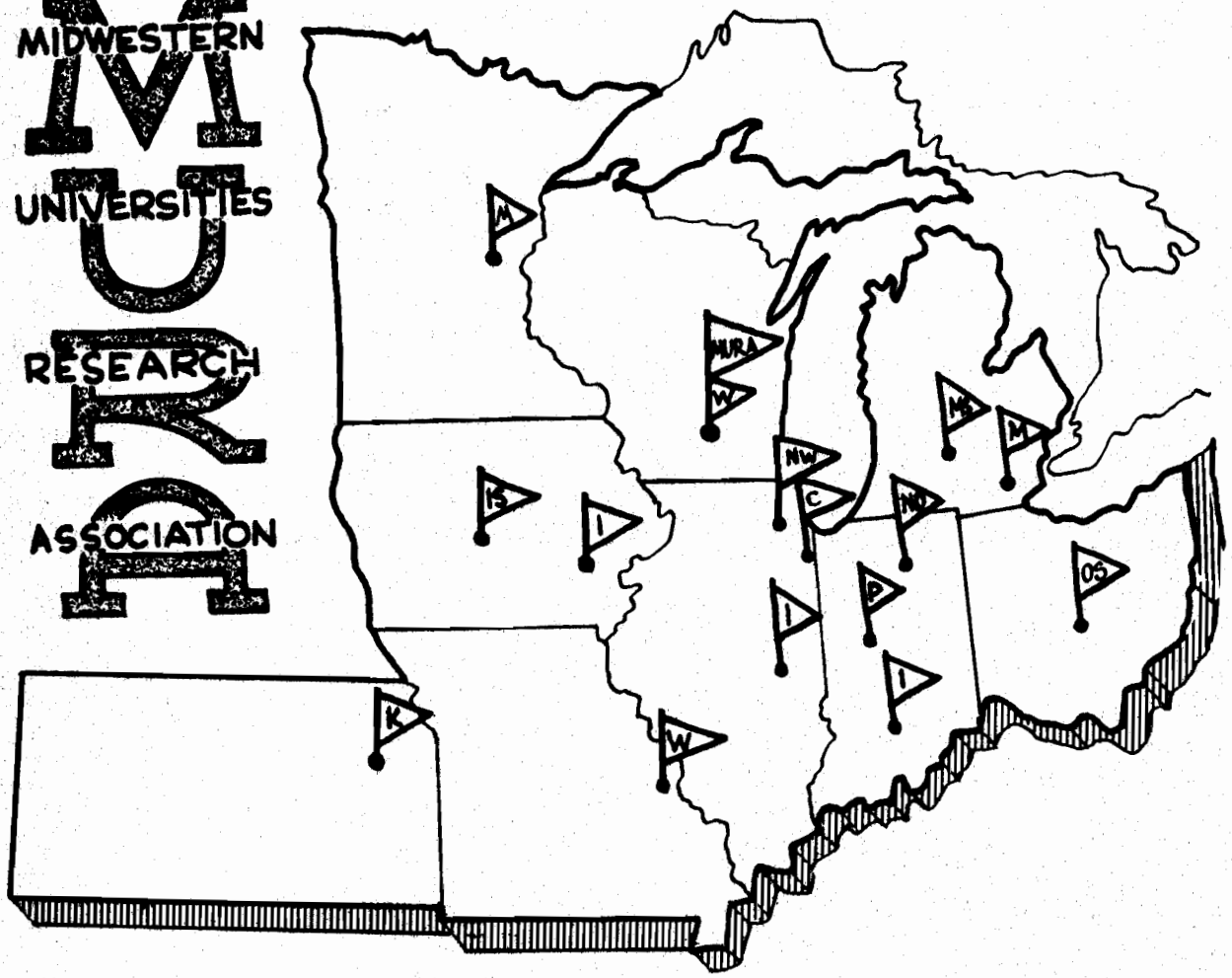

TIME MODES OF A COASTING BEAM IN A SIMPLE PERTURBATION APPROACH

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REPORT NUMBER 611

MURA-611
UC-28, Particle Accelerators and High Voltage Machines
TID-4500 (16th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

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February 27, 1961

ABSTRACT

A simple linear perturbation treatment is attempted in the solution of the coupled Maxwell equations and Boltzmann's equation of the distribution function pertaining to a charged coasting beam of weak intensity confined to a donut-like space of metallic boundaries. These coupled equations are cast into a single integral equation and treated as an eigenvalue problem to determine the various time modes. Difficulties encountered at various stages of approximation are appreciated.

*AEC Research and Development Report. Research supported by the U.S. Atomic Energy Commission, Contract No. AT(11-1)-384.

INTRODUCTION

The problem of the stability of a coasting beam in a donut-like tube is one of immense mathematical complexity. Even by accepting the conjecture that if the initial beam is not so intense as to prohibit the collective effects from being treated as a perturbation in the lowest order, there still are difficulties which cannot be solved without drastically over-simplifying the physical situation.

Realistic approximations aimed at applications to accelerator physics have been done by a group of authors at Berkeley[8] by invcking the negligibility of certain contributions in the azimuthal electric field in order to simplify the algebra in the dispersion relation obtained from the Boltzmann equation (with the collision term deleted). In this study the less complicated case of an ideal beam (without the presence of rf cavities to compensate the radiation energy loss) is examined in some detail in order to appreciate the various difficulties one has to deal with when the effect of the azimuthal electric field is considered in full. In addition to the solution of the unstable time modes, eigenfunctions associated with these modes are also discussed in relation to the question: what initial distribution will minimize the occurrence of those most undesirable unstable modes.

The problem is first manipulated into a one-dimensional case and an integral equation obtained by coupling the Maxwell equations and the Boltzmann-Liouville equation together is then

treated as an eigenvalue problem. The technique employed in handling such an eigenvalue problem is based, in spirit, upon the work by K.M. Case in his "Plasma Oscillation."[1] The various time modes into which the perturbation increment of some given initial distribution is resolved then supply an idea about the stability conditions of the beam. Because of the limitation of perturbation theory, those time modes which for short duration indicate possible instability may not actually cause instability after time long enough to invalidate any predictions based on a linear theory.

The study is purely of exploratory nature in search for an approach and is not intended as a downright attack to this complex problem

II FORMULATION OF THE PROBLEM

We formulate the problem by first considering a swarm of neutral particles constrained to move on circular orbits by some non-energy-contributing agent. The law of constraint will be specified later. The system will be referred to the curvilinear coordinates based on a circle of radius R with $\hat{e}_x x$ radially directed toward the center of R, $\hat{e}_y y$ vertically perpendicular to the plane of R and $\hat{e}_\theta \theta$ counterclockwise; so that $\hat{e}_\theta = \hat{e}_x \times \hat{e}_y$ form a right-handed triad.[2] The single particle Lagrangian appropriate for the motion here may be put as

$$L = p\dot{s} - E \tag{1}$$

where s is differentially defined by ds = (R-x)dθ and p is the conjugate variable to s. E is a constant and is seen to be the particle energy. To put (1) in terms of familiar MURA variables w, θ we may carry out a canonical transformation

$$\iint dp\, ds = \iint dp\,(R-x)\, d\theta \equiv \iint dw\, d\theta \tag{2}$$

so that $dw = (R-x)dp = (R-x)\frac{E}{pc^2}\, dE = \frac{dE}{\dot{\theta}}$ is conjugate to θ and the Lagrangian may be put as

$$L = w\dot{\theta} - E \tag{1a}$$

which will be referred to as the "unperturbed problem." The variables p_x & p_y respectively conjugate to x and y, do not enter in this unperturbed problem.

The distribution Ψ of a swarm of particles is determined by initial conditions and satisfies the Boltzmann-Liouville

equation

$$\frac{\partial \Psi}{\partial t} + \dot{w}\frac{\partial \Psi}{\partial w} + \dot{\theta}\frac{\partial \Psi}{\partial \theta} + \dot{p}_x\frac{\partial \Psi}{\partial p_x} + \dot{p}_y\frac{\partial \Psi}{\partial p_y} + \dot{x}\frac{\partial \Psi}{\partial x} + \dot{y}\frac{\partial \Psi}{\partial y} = 0 \qquad (3)$$

in this particular coordinate system.

A temporally stationary and azimuthally uniform distribution of the form $\Psi = Q(x,y)\Psi^{o}(w)$ is certainly a solution of this equation for the unperturbed problem and shall be considered as given The factor $Q(x,y)$ specifies the lateral dimension of the unperturbed beam (i.e. the cross-sectional extension of the beam) and usually can be considered as Gaussian in shape for both the x- and y- extensions. In the case of a beam of weak intensity, these Gaussian curves are very flat and $\frac{\partial Q}{\partial x}$ & $\frac{\partial Q}{\partial y}$ are small quantities almost constant. This latter approximation is essential to the simplification of the problem in what follows.

The actual distribution of a beam of charged particles coasting in a donut tube of metallic walls will be looked upon as a perturbed problem with $Q(x,y)\Psi^{o}(w)$ as the zeroth-order function The perturbing sources consist of the interparticle long-range electromagnetic interaction and boundary image forces which are collectively referred to as the "collective effect." They are described by the interaction Lagrangian ($e = c = 1$ as unit)

$$\Delta L = -\phi + \vec{A}\cdot\vec{v}$$
$$= -\phi + A_\theta (R-x)\dot{\theta} + A_x\dot{x} + A_y\dot{y}$$

in which ($\vec{A}, \phi$) is the electromagnetic 4-potential created by the beam itself and compatible with the boundary conditions of the donut space under consideration. This four-potential is of course a complicated functional of the distribution function we are looking for.

Let us denote the distribution function of the perturbed problem by Ψ^*, which in general should depend on all three pairs of conjugate variables as well as explicitly on the time t. In the following, all variables that pertain to the perturbed problem will be superlabelled with a star (*). Then

$$\Psi^*(\theta^*, w^*, x^*, y^*, p_x^*, p_y^*, t) = Q(x^*, y^*)\Psi^\circ(w^*) + \Delta\Psi(\theta^*, w^*, x^*, y^*, p_x^*, p_y^*, t) ,$$

and any of these starred variables can be expressed in terms of the unstarred variables (i.e. variables pertaining to the unperturbed problem) through the relation

$$q^* = q + \Delta q$$

where q stands for any of the six variables while Δq, the perturbation increment, is to be expressed as a function of q's and will be (as we hope it may be) treated linearly in a first-order theory. Then the linearized version of equation (3) for the perturbed problem expressed in terms of unperturbed variables and perturbation increment, has the general form

$$\frac{\partial\Delta\Psi}{\partial t} + \dot{w}\frac{\partial\Delta\Psi}{\partial w} + \dot{\theta}\frac{\partial\Delta\Psi}{\partial\theta} + \dot{p}_x\frac{\partial\Delta\Psi}{\partial p_x} + \dot{p}_y\frac{\partial\Delta\Psi}{\partial p_y} + \dot{x}\frac{\partial\Delta\Psi}{\partial x} + \dot{y}\frac{\partial\Delta\Psi}{\partial y}$$

$$= -\Delta\dot{w}\frac{\partial\Psi}{\partial w} - \Delta\dot{p}_x\frac{\partial\Psi}{\partial p_x} - \Delta\dot{p}_y\frac{\partial\Psi}{\partial p_y} - \Delta\dot{\theta}\frac{\partial\Psi}{\partial\theta} - \Delta\dot{x}\frac{\partial\Psi}{\partial x} - \Delta\dot{y}\frac{\partial\Psi}{\partial y}$$

In view of the special structure of $\Psi = Q(x,y)\Psi^{\circ}(w)$, assumed for the unperturbed problem and the absence of the lateral motion in the unperturbed Lagrangian (1a), a number of terms in the above equation vanish, the surviving ones being:

$$\frac{\partial \Delta\Psi}{\partial t} + \dot{\theta}\frac{\partial \Delta\Psi}{\partial \theta} = -\Delta\dot{w}\, Q(x,y)\frac{\partial \Psi^{\circ}}{\partial w} - \Delta\dot{x}\,\Psi^{\circ}(w)\frac{\partial Q}{\partial x} - \Delta\dot{y}\,\Psi^{\circ}(w)\frac{\partial Q}{\partial y} \tag{5}$$

The connection between Δw and ΔL is given by the Euler-Lagrangian equation of motion[3]:

$$\begin{aligned}\frac{\partial \Delta w}{\partial t} &= \frac{\partial \Delta L}{\partial \theta} - \frac{d}{dt}\frac{\partial \Delta L}{\partial \dot{\theta}} \\ &= (R-x)\left\{-\hat{e}_{\theta}\cdot\nabla\phi - \frac{\partial A_{\theta}}{\partial t}\right\} + \dot{x}\,(R-x)\left\{\frac{1}{R-x}\frac{\partial A_x}{\partial \theta} - \frac{\partial A_{\theta}}{\partial x} + \frac{A_{\theta}}{R-x}\right\} \\ &\quad + \dot{y}\,(R-x)\left\{\frac{1}{R-x}\frac{\partial A_y}{\partial \theta} - \frac{\partial A_{\theta}}{\partial y}\right\}\end{aligned} \tag{6}$$

It can be easily shown[4] that in the particular coordinate system used here, the quantities in the three braces in (6) are respectively $\mathcal{E}_{\theta}$ (the azimuthal component of the electric field intensity), $\mathcal{H}_y$ and $\mathcal{H}_x$ (the lateral components of the magnetic field intensity). The latter two terms, because of the factors $\dot{x}$ and $\dot{y}$ being zero in the unperturbed problem, are to be dropped in keeping with the policy of approximation.

By definition, $\Delta x = x^{*} - x$ so

$$\Delta\dot{x} = \dot{x}^{*} \sim A_x \sim \int dV \frac{j_x}{r}$$

Since j_x and j_y are zero in the unperturbed problem and in view

of the smallness of $\frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial y}$ assumed, the last two terms in (5) are quantities of second-order effect. Consequently, in a lowest order estimate, we essentially have

$$\frac{\partial \Delta\Psi}{\partial t} + \dot{\theta}\,\frac{\partial \Delta\Psi}{\partial \theta} \simeq (R-x)\,Q(x,y)\,\frac{\partial \Psi^{\circ}}{\partial W}\left\{\hat{e}_{\theta}\cdot\nabla\phi + \frac{\partial\,\hat{e}_{\theta}\cdot\vec{A}}{\partial t}\right\} \tag{7}$$

The form of this equation is almost apparent even without going through the details[5]. We went through the details just to examine that if the collective effects may indeed be adequately treated as a perturbation at its lowest order, then the integro-differential equation we have to face is essentially one-dimensional in structure, at least in the case of a weak beam.

III. THE INTEGRAL EQUATION

We are now to express the potentials $\hat{e}_\theta \cdot \vec{A}$ and ϕ in terms of the current source density $\hat{e}_\theta \cdot \vec{j}$ (only the azimuthal component), which can be obtained by taking the first moment of the distribution function itself. We shall consider a donut tube of the same geometry as the one studied in MURA-555. Briefly, the donut is to have a rectangular cross section of height 2Y and width R(a+b) where a and b are small parameters fixing the location of the beam relative to the radial extension of the donut. The eigenfunctions associated with free-space oscillation in such an empty donut space can be used as the basis for expansion of those functions within our interest in the following scheme:

(1) Any function of θ which is periodic of period 2π can be resolved in terms of the eigenfunctions $e^{in\theta}$, $n = 0, \pm$ integers.

(2) Any function of y which vanishes on $y = \pm Y$ boundaries can be resolved in terms of the eigenfunctions $\sin \frac{m\pi}{2Y}(y-Y)$, m = integers

(3) Any function of x which vanishes on x = Ra, -Rb boundaries can be resolved in terms of the eigenfunctions.

$$\mathcal{D}_{n\sigma}(x) \equiv J_n(k_\sigma(R-x)) + C\, N_n(k_\sigma(R-x))$$

in which J_n & N_n are respectively the Bessel functions of the first and second kind of order n and C and k_σ are determined by the requisite boundary conditions (Dirichlet).

In addition, we shall assume with no loss of generality, the time dependence in all functions to be of the form $e^{-i\omega t}$. Essentially, a function $F(\theta, x, y, t)$ satisfying the above conditions can be put as

$$F(\theta, x, y, t) = F_\omega(\theta, x, y)\, e^{-i\omega t} = e^{-i\omega t} \sum_{nm\sigma} F_{\omega nm\sigma} \langle \theta, x, y | nm\sigma \rangle$$

$$\text{with} \quad \langle \theta, x, y | nm\sigma \rangle \equiv e^{in\theta} \sin \frac{m\pi}{2Y}(y-Y)\, \mathcal{D}_{n\sigma}(x)$$

The potentials ϕ and $\hat{e}_\theta \cdot \vec{A}$ both satisfy the requisite boundary conditions and can be accordingly resolved. The azimuthal current density being the first moment of the distribution function also meets the requirement. The solution of Maxwell equations (adopting a Coulomb gauge $\nabla \cdot \vec{A} = 0$) gives the following relations linking the expansion amplitudes of these three functions[6]:

$$\phi_{\omega nm\sigma} = \frac{n}{\omega} \gamma_{nm\sigma}\, j_{\theta\omega nm\sigma} \qquad (\omega \neq 0) \tag{8a}$$

$$A_{\theta\omega nm\sigma} = \frac{n^2}{k_{nm\sigma}^2 - \omega^2} \lambda_{nm\sigma}\, j_{\theta\omega nm\sigma} \qquad (c = 1 \text{ as unit}) \tag{8b}$$

in which $k_{nm\sigma}^2 = k_\sigma^2 + \left(\frac{m\pi}{2Y}\right)^2$ and $\gamma_{nm\sigma}$ & $\lambda_{nm\sigma}$ are quantities

depending on the various quantum numbers, geometrical parameters and certain normalization convention. Their detailed structure shall not concern us here.

Likewise, we expand the perturbed distribution

$$\Psi^* = e^{-i\omega t} \sum_{nm\sigma} \psi_{\omega nm\sigma}(w) \langle \theta, x, y | nm\sigma \rangle \tag{8c}$$

and by definition of the current density

$$\hat{e}_\theta \cdot \vec{j} = \int dw\, \dot{\theta}(w)\, \Psi^* \quad ,$$

we have the amplitude relation $(\omega \neq 0)$

$$j_{\theta\omega nm\sigma} = \int dw\, \dot{\theta}(w)\, \psi_{\omega nm\sigma}(w) \tag{8d}$$

where $\dot{\theta}(w)$ is specified by the law of constraint in the unperturbed problem and will be examined later.

The function $Q(x,y)$ presents some difficulty for a simple treatment of equation (7). In the following, the situation is analysed in detail in order to establish some justification for the simplification compatible with the assumptions already made in § II.

Let $Q(x,y)$ be a general function of x, y staisfying the requisite boundary conditions. We make the following resolutions:

$$Q(x,y) = \sum_{m'\sigma'} q^{(0)}_{nm'\sigma'} \langle x, y | nm'\sigma' \rangle \tag{8e}$$

$$\langle x, y | nm\sigma \rangle \equiv \sin \frac{m\pi}{2Y}(y-Y)\, \delta_{n\sigma}(x)$$

$$(R-x)\, Q(x,y) = \sum_{m'\sigma'} q^{(1)}_{nm'\sigma'} \langle x, y | nm'\sigma' \rangle \tag{8f}$$

Then, by adopting the abbreviating definitions

$$\frac{\omega}{n} \equiv \Omega \quad , \quad \frac{k_{nm\sigma}}{n} \equiv \kappa_{nm\sigma} \tag{9a}$$

equation (7) leads to the expression:

$$\begin{aligned}&(\dot{\theta}-\Omega)\sum_{m\sigma}\psi_{\Omega nm\sigma}\langle x,y|nm\sigma\rangle \\ &= \frac{\partial\Psi^0}{\partial W}\sum_{m\sigma}\sum_{m'\sigma'}\left\{\frac{\gamma_{nm\sigma}}{\Omega}q^{(0)}_{nm'\sigma'} - \frac{\Omega\lambda_{nm\sigma}}{\kappa^2_{nm\sigma}-\Omega^2}q^{(1)}_{nm'\sigma'}\right\}j_{\theta\Omega nm\sigma}\langle x,y|nm\sigma\rangle\langle x,y|nm'\sigma'\rangle\end{aligned} \tag{7a}$$

It is understood here, in the bra & ket notation, that for any F(x,y):

$$\langle nm_1\sigma_1|F(x,y)|nm_2\sigma_2\rangle \equiv \iint (R-x)\,dx\,dy\; F(x,y)\sin\frac{m_1\pi}{2Y}(y-Y)\,\mathcal{D}^*_{n\sigma_1}(x)\sin\frac{m_2\pi}{2Y}(y-Y)\,\mathcal{D}_{n\sigma_2}(x)$$

(note the weight factor (R-x) in the integration element). Equation (7a) can then be manipulated to the form:

$$\begin{aligned}&(\dot{\theta}-\Omega)\psi_{\Omega nm''\sigma''}\langle nm''\sigma''|nm''\sigma''\rangle \\ &= \frac{\partial\Psi^0}{\partial W}\sum_{m\sigma}\sum_{m'\sigma'}\left\{\frac{\gamma_{nm\sigma}}{\Omega}q^{(0)}_{nm'\sigma'} - \frac{\Omega\lambda_{nm\sigma}}{\kappa^2_{nm\sigma}-\Omega^2}q^{(1)}_{nm'\sigma'}\right\}j_{\theta\Omega nm\sigma}\langle nm''\sigma''|\sin\frac{m'\pi}{2Y}(y-Y)\,\mathcal{D}_{n\sigma'}(x)|nm\sigma\rangle\end{aligned} \tag{7b}$$

Proceeding in a formal manner, without actual evaluation, we may have:

$$\langle x,y|nm'\sigma'\rangle\langle x,y|nm\sigma\rangle = \sum_{m''\sigma''} C_{nm''\sigma''}\langle x,y|nm''\sigma''\rangle \tag{10a}$$

and (7b) assumes the neat form:

$$(\dot{\theta}-\Omega)\psi_{\Omega nm''\sigma''} = \frac{\partial\Psi^0}{\partial W}\sum_{m\sigma}\sum_{m'\sigma'} C_{nm''\sigma''}\left\{\frac{\gamma_{nm\sigma}}{\Omega}q^{(0)}_{nm'\sigma'} - \frac{\Omega\lambda_{nm\sigma}}{\kappa^2_{nm\sigma}-\Omega^2}\right\}j_{\theta\Omega nm\sigma} \tag{7c}$$

Subject to (10a)

Thus there is an infinite number of terms on the r.h.s. for each $nm''\sigma''$-mode We are confronted with the problem of a set of coupled equations infinite in number, an obvious impasse.

For the sake of argument, let us actually evaluate the coefficients $G_{nm''\sigma''}$ with respect to the y-variable. The situation for the x-variable follows similarly (although an explicit evaluation will be very arduous) since $\sqrt{R-x}\,\mathcal{D}_{n\sigma}(x)$ in the limit $\frac{x}{R}\to 0$ is essentially also the sine function. Then

$$G_{nm''\sigma''} \sim \int dy \, \sin\frac{m''\pi}{2Y}(y-Y)\,\sin\frac{m'\pi}{2Y}(y-Y)\,\sin\frac{m\pi}{2Y}(y-Y)$$
$$\sim \left\{ \frac{1}{m-m'-m''} - \frac{1}{m+m'-m''} + \frac{1}{m+m'+m''} - \frac{1}{m-m'+m''} \right\} \tag{10b}$$

all denominators assume odd integers only.

Now if Q(x,y) is flat and smooth with respect to the y-variable, the expansions in (8) should depend on the m'-quantum number only very weakly, in which case (10b) approaches a small quantity unless $m \simeq \pm m''$

Let $\tilde{Q}$ be the DC Component in (8e) and define

$$\tilde{R}_{n\sigma} \equiv \frac{\int_{-R\sigma}^{Ra} dx\,(R-x)^2\,\mathcal{D}^*_{n\sigma}\,\mathcal{D}_{n\sigma}}{\int_{-R\sigma}^{Ra} dx\,(R-x)\,\mathcal{D}^*_{n\sigma}\,\mathcal{D}_{n\sigma}} \simeq R \quad \text{as} \quad \frac{x}{R}\to 0$$

then the equations (7c) are decoupled:

$$(\dot\theta-\Omega)\,\psi_{\Omega nm\sigma}(w) = \tilde{Q}\,\frac{\partial\Phi^0}{\partial w}\left\{\frac{\gamma_{nm\sigma}}{\Omega} - \frac{\Omega\lambda_{nm\sigma}}{K^2_{nm\sigma}-\Omega^2}\,\tilde{R}_{n\sigma}\right\}\int_I dw_1\,\dot\theta(w_1)\,\psi_{\Omega nm\sigma}(w_1) \tag{11}$$

Dropping all labelling quantum numbers and introducing the abbreviation

$$\tilde{Q}\frac{\partial \Psi^{\circ}}{\partial w} \equiv g(w) \quad , \qquad \frac{\gamma}{\Omega} - \tilde{R}\frac{\Omega\lambda}{K^2-\Omega^2} \equiv f(\Omega) \tag{9b}$$

we may then concentrate on the prototype integral equation:

$$(\dot{\theta}(w)-\Omega)\,\psi(w) = g(w)\,f(\Omega)\int_I dw_1\,\dot{\theta}(w_1)\,\psi(w_1) \tag{11a}$$

This procedure of decoupling approximation is of course rather drastic. A better decoupling approximation is to introduce an inhomogeneous function, formally denoted by $\mathcal{J}(w)$, obtained by a second averaging procedure to account for some of the important contributing terms in (8e) other than the D.C. part, so that

$$(\dot{\theta}(w)-\Omega)\,\psi(w) = \mathcal{J}(w) + g(w)\,f(\Omega)\int_I dw_1\,\dot{\theta}(w_1)\,\psi(w_1) \tag{11b}$$

In this exploratory study, we can only afford to treat the simplest decoupling approximation, so (11b) will be disregarded.

In the next section, we are to briefly examine the structure of the function $\dot{\theta}(w)$ and the integration limit I in (11a).

IV. THE LAW OF CONSTRAINT

$\dot{\theta}$ as a function of W, or vice versa, is specified by the law of constraint in the unperturbed problem. Although we do not wish to be particular about a single such law, a realistic example is examined in order to serve as a guiding principle into the analytical properties of the function in which we most likely shall be interested.

Let us say that the external non-energy-contributing constraining agent is equivalent to the effect of a magnetic field B on charged particles, B being perpendicular to the plane of motion of the particles. Then at a radius r, $p = rB$ where p is the scalar momentum of the particle. Further, if we follow the r^k - field law such that $B = B_0 r^k$ with B & k as constants, the latter being very large

then $$p = B_0 r^{k+1}$$

From the defining equation $dW = r\,dp$ and $\dot{\theta} = \frac{dE}{dW} = \frac{p}{rE}$, we first integrate W:

$$W = B_0 \left(\frac{k+1}{k+2}\right) r^{k+2} + \text{Const.}$$

Since W is only differentially defined, the additive constant is completely arbitrary and we would like to put it equal to zero.

Consequently

$$\dot{\theta}(W) = \frac{B_0 \left(\frac{W}{B_0}\,\frac{k+2}{k+1}\right)^{\frac{k}{k+2}}}{\left[B_0^2\left(\frac{W}{B_0}\,\frac{k+2}{k+1}\right)^{\frac{2k+2}{k+2}} + 1\right]^{\frac{1}{2}}} \qquad \text{(particle mass set equal to 1)}$$

$$\xrightarrow{\text{very large } k} \frac{W}{\sqrt{W^2+1}}$$

in which we have made the choice that $\dot{\theta}$ and W are to have the same sign. This sign convention uniquely determines the inverse function

$$W(\dot{\theta}) \longrightarrow \frac{\dot{\theta}}{\sqrt{1-\dot{\theta}^2}}$$

$$\frac{dW}{d\dot{\theta}} \longrightarrow \frac{1}{(1-\dot{\theta}^2)^{3/2}}$$

If the integration interval I in (11a) is for W from o to ∞, it will be for $\dot{\theta}$ from o to 1. For the purpose of studying (11a), it will be seen later that it is more advantageous to use $\dot{\theta}$ as the independent variable rather than W. So returning to the general case, we may say that the law of constraint specifies the function $W(\dot{\theta})$, or what we actually shall need, the function $\frac{dW}{d\dot{\theta}}$ only. We define

$$\dot{\theta}\,\frac{dW}{d\dot{\theta}} \equiv h(\dot{\theta})$$

to be referred to as the constraint function.

Equation (11a) in its full generality then assumes the compact form:

$$(\dot{\theta}-\Omega)\,\psi(\dot{\theta}) = g(\dot{\theta})\, f(\Omega)\int_I d\dot{\theta}_1\, h(\dot{\theta}_1)\,\psi(\dot{\theta}_1) \qquad \begin{aligned} \psi(\dot{\theta}) &\equiv \psi(W(\dot{\theta})) \\ g(\dot{\theta}) &\equiv g(W(\dot{\theta})) \end{aligned} \tag{12}$$

V FORMULATION OF THE EIGENVALUE PROBLEM

Let $\mathcal{M}$ be a multiplicative operator and $\mathcal{J}$ be a non-Hermitian integral operator of definite integration limits and let us look at the following operator equation:

$$\mathcal{M}\psi = \Omega\psi + f(\Omega)\mathcal{J}\psi$$

in which $f(\Omega)$ is <u>not</u> a linear function of Ω. The non-linearity in Ω of $f(\Omega)$ greatly complicates the present problem, although it seems that nothing prevents us from asking the same question as we do in an ordinary eigenvalue problem, viz, whether or not there exist particular values of $\Omega = \Omega_\nu$ such that this equation will have solution $\psi = \psi_\nu$. There is no difficulty in obtaining the eigenvalue spectra and the pertinent eigenfunctions (by following K M. Case's method). However, we should expect that the eigenfunctions thus obtained will behave quite differently from what we can expect of those in an ordinary eigenvalue problem, particularly in respect of two questions which profoundly concern our interest. These questions are:

(1) Whether or not there exists a set of functions $\bar{\psi}_\mu$ which have the same spectra $\bar{\Omega}_\mu$ as ψ_ν and which are biorthoganal to ψ_ν over the interval I in the sense that

$$\int_I d\theta\, \bar{\psi}_\mu(\theta)\, \psi_\nu(\theta) \propto \delta(\bar{\Omega}_\mu - \Omega_\nu)$$

(2) Whether or not $\psi_\nu(\theta)$ are complete over the interval I in the sense that any reasonable function of θ defined

on I and belonging to a certain class within our interest can be expanded in terms of them.

The two questions are of course intimately related and without affirmative answers to them our sole knowledge in the spectra Ω_ν and functions ψ_ν has only limited usefulness. By taking it for granted that the answers will be affirmative, various attempts have been made, based on intuition or by sheer trial, to construct an equation which is "adjoint" to (12), or approximately so, in order to generate the required $\bar{\psi}_\mu$, but do not lead to any intelligible result. In the next section we first study the spectra of our problem.

VI THE TIME MODES

In the equation

$$(\dot{\theta}-\Omega)\,\psi(\dot{\theta}) = g(\dot{\theta})\,f(\Omega)\int_I d\dot{\theta}_1\, h(\dot{\theta}_1)\,\psi(\dot{\theta}_1) \quad , \tag{12}$$

we shall always make the normalization convention

$$\int_I d\dot{\theta}_1\, h(\dot{\theta}_1)\,\psi(\dot{\theta}_1) = 1 \tag{12a}$$

and subdivide the spectra into two major classes:

(1) Real Spectrum

(a) $f(\Omega) = 0$ has roots Ω_0 which are in I.

Then $(\dot{\theta}-\Omega_0)\,\psi_0(\dot{\theta}) = 0$ and $\psi_0(\dot{\theta}) = \dfrac{\delta(\dot{\theta}-\Omega_0)}{f(\Omega_0)}$. (13)

(Those roots of $f(\Omega) = 0$, if any, not in I are not eigenvalues since for them ψ is always trivial.)

(b) $f(\Omega_c) \neq 0$, then from $(\dot{\theta}-\Omega_c) = g(\dot{\theta})\,f(\Omega_c)$

$$\psi_c(\dot{\theta}) = \mathcal{P}\,\frac{g(\dot{\theta})\,f(\Omega_c)}{\dot{\theta}-\Omega_c} + \rho(\Omega_c)\,\delta(\dot{\theta}-\Omega_c) \tag{14}$$

The first term is the principal value which does not define the function ψ_c at $\dot{\theta} = \Omega_c$. The second term defines the value of ψ_c at $\dot{\theta}-\Omega_c$ The weight function $\rho(\Omega_c)$, is obtained from the normalization convention which yields

$$\rho(\Omega_c) = \frac{1}{h(\Omega_c)} - \frac{f(\Omega_c)}{h(\Omega_c)}\,\mathcal{P}\int_I d\dot{\theta}\,\frac{h(\dot{\theta})\,g(\dot{\theta})}{\dot{\theta}-\Omega_c} \tag{14a}$$

(c) If in (14), $g(\Omega_{c=s})=0$, then the principal value sign would not be necessary. In addition, if $f(\Omega_s)=0$, then (14a) yields

$$1 = \int_I d\dot{\theta}\, \frac{h(\dot{\theta})\, g(\dot{\theta})\, f(\Omega_s)}{\dot{\theta}-\Omega_s} \tag{15a}$$

Solution of this algebraic equation will give all Ω_s's.

(11) <u>Complex Spectrum</u>

Since $\dot{\theta}$ is real, if Ω_i is complex, we have right away

$$\psi_i(\dot{\theta}) = \frac{g(\dot{\theta})\, f(\Omega_i)}{\dot{\theta}-\Omega_i}$$

Normalization supplies the algebraic equation in Ω_i:

$$1 = \int_I d\dot{\theta}\, \frac{h(\dot{\theta})\, g(\dot{\theta})\, f(\Omega_i)}{\dot{\theta}-\Omega_i} \tag{15b}$$

which is identical to (15). Since h, g & f are all real functions, this algebraic equation should supply complex roots in conjugate pairs. These real roots are just the Ω_s's in (Ic).

The degree of complication of equation (15) of course depends on the structure of the functions h & g and it is almost certain that in general it will be transcendental in nature.

Of the three functions involved in (15), f (Ω) depends on a large number of parameters. Its structure (9b) already bespeaks that (15), as an algebraic equation in Ω, will always come out higher than cubic in degree of Ω. It is of considerable interest for us to examine whether there exist any realistic

simplifying assumptions to be imposed upon $h(\dot{\theta})$ or $g(\dot{\theta})$, so that the integration in (15) will make the final algebraic equation in Ω not higher than biquadratic, which is the highest degree equation known to be analytically solvable. Such an example is studied in the next section for the purpose of gaining some insight rather than taking it seriously as representing the true physical situation.

VII. SHAPE-INDEPENDENT APPROXIMATIONS

Great simplification is attained by taking a Gaussian-like curve as the initial distribution (treated as a function of $\dot{\theta}$ rather than of W at the very beginning). Furthermore if this Gaussian curve is narrow and peaked to such an extent that its detailed shape could be overlooked so that a plateau-like pulse such as pictured in Fig. 1 could crudely represent the situation, the difficulty of our problem is eased no matter what the constraint function $h(\dot{\theta})$ is.

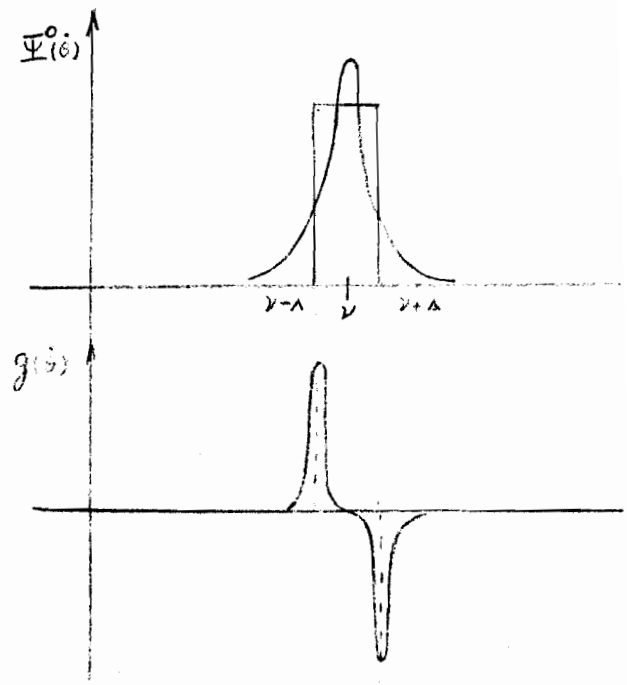

FIG 1

This shape-independent treatment yields the gradient function

$$g(\dot{\theta}) = \frac{A}{2}\left[\delta(\dot{\theta}-\nu+\Delta) - \delta(\dot{\theta}-\nu-\Delta)\right]$$

where Δ is the half-width of the plateau and ν is the center point at which the Gaussian curve has vanishing gradient. A is an amplitude constant. Then equation (15) leads to

$$\frac{1}{f(\Omega)} = \frac{A}{2}\left[\frac{h(\nu-\Delta)}{\nu-\Delta-\Omega} - \frac{h(\nu+\Delta)}{\nu+\Delta-\Omega}\right] \tag{16}$$

If Δ, as a small quantity, is considered significant only to its first order, we obtain

$$\frac{1}{f(\Omega)} \overset{\Delta^2 \to 0}{\simeq} \frac{A\Delta}{(\nu-\Omega)^2}\left[h(\nu)-\nu h'(\nu)+\Omega h'(\nu)\right] \overset{A\Delta \equiv N}{=} \frac{N}{(\nu-\Omega)^2}\left[h(\nu)-\nu h'(\nu)+\Omega h'(\nu)\right] \tag{17}$$

This last result is identical to that of an "Impulse Distribution", i.e., if the original distribution is exaggerated to be just an impulse like δ-function, so that the gradient function is simply

$$N\,\delta'(\dot{\theta}-\nu)$$

Equation (17) is a fifth-degree equation in Ω and we get assurance that at least one root will be real.

Written out explicitly in terms of the parameters $\gamma, \lambda, \kappa, \tilde{R}$ which are dependent on the spatial quantum numbers n, m, σ, (17), after rearranging some of the factors, looks like:

$$\Omega(\kappa^2-\Omega^2)(\nu-\Omega) = \frac{N}{\gamma+\tilde{R}\lambda}\left[h'(\nu)\Omega+h(\nu)-\nu h'(\nu)\right]\frac{\frac{\gamma\kappa^2}{\gamma+\tilde{R}\lambda}-\Omega^2}{\nu-\Omega} \tag{17a}$$

which is in the appropriate form for further discussion.

For a given spatial mode $(n\,m\,\sigma)$, $\gamma, \lambda, \kappa, \tilde{R}$ are determined or the quantity $\kappa\sqrt{\frac{\gamma}{\gamma+\tilde{R}\lambda}}$ is fixed. Conversely if the quantity $\kappa\sqrt{\frac{\gamma}{\gamma+\tilde{R}\lambda}}$ is given as a single entity by itself, it is possible, although quite difficult, to find a set $(n\,m\,\sigma)$ or more probably, an infinite number of such sets out of the three-fold infinite manifold spanned by all possible $n\,m\,\sigma$, which will give the corresponding $\kappa, \gamma, \lambda, \tilde{R}$ such that

the product $\kappa\sqrt{\frac{\gamma}{\gamma+\tilde{R}\lambda}}$ is a preassigned quantity. The defining constant ν in $g(\dot{\theta})$ can be looked upon as a preassigned quantity, so that it is always possible to set

$$\nu = \kappa\sqrt{\frac{\gamma}{\gamma+\tilde{R}\lambda}} \tag{18}$$

thus selecting out those spatial modes with which the assumed initial distribution has direct concern.

This procedure reduces (17a) to a biquadratic equation in Ω, the real root $\Omega=\nu$ being separated out:

$$\Omega^4-\nu\Omega^3-\Omega^2\left[\kappa^2+\frac{N\nu^2}{\kappa^2\gamma}h'(\nu)\right]-\Omega\left[\nu\kappa^2+\frac{N\nu^2}{\kappa^2\gamma}h(\nu)\right]-\frac{N\nu^2}{\kappa^2\gamma}\left[\nu h(\nu)-\nu^2h'(\nu)\right]=0 \tag{19}$$

This equation can be analytically solved by means of Ferrari's method.[7] The results are,

defining $B\equiv-\left[\kappa^2+\frac{N\nu^2}{\kappa^2\gamma}h'(\nu)\right]$, $C\equiv-\left[\nu\kappa^2+\frac{N\nu^2}{\kappa^2\gamma}h(\nu)\right]$, $D\equiv-\frac{N\nu^2}{\kappa^2\gamma}\left[\nu h(\nu)-\nu^2h'(\nu)\right]$,

and

$$X\equiv-\nu C-4D-\frac{1}{3}B^2$$

$$Y\equiv 4BD-\nu^2D-C^2-\frac{1}{3}B\left[\nu C+4D\right]-\frac{2}{27}B^3$$

Constructing the resolvent

$$\mathcal{R}\equiv\left\{-\frac{Y}{2}+\sqrt{\frac{Y^2}{4}+\frac{X^3}{27}}\right\}^{1/3}+\left\{-\frac{Y}{2}-\sqrt{\frac{Y^2}{4}+\frac{X^3}{27}}\right\}-\frac{1}{3}B$$

The roots being:

$$\Omega=\frac{\nu}{4}\overset{+}{\underset{-}{\pm}}\frac{1}{2}\left[\frac{\nu^2}{4}-B+\mathcal{R}\right]^{1/2}\overset{+}{\underset{-}{\pm}}\frac{1}{2}\left\{\frac{\nu^2}{2}-B+\mathcal{R}\overset{+}{\underset{-}{\pm}}\nu\left[\frac{\nu^2}{4}-B+\mathcal{R}\right]^{\frac{1}{2}}\overset{-}{\underset{-}{\pm}}2\left[\mathcal{R}\overset{+}{\underset{-}{\pm}}\frac{C+\frac{\nu}{2}\mathcal{R}}{\sqrt{\frac{\nu^2}{4}-B+\mathcal{R}}}\right]\right\}^{1/2}$$

subject to the condition (18).

One might wonder whether such an approach to handling the fifth-degree equation (17), by confining to only those selected spatial modes that satisfy the rather arbitrary auxiliary condition (18), is logical, since (17) should yield five roots for every possible spatial mode if it could be solved without any artifice. This is intuitively answered by referring back to Section VI. There it is noted that any real root of (15) belonging to class (Ic) are special cases of class (Ib) in which the gradient function $g(\dot{\theta})=0$ at $\dot{\theta}=\Omega$. From Fig. 1, we can see that only if $\Omega=\nu$ do we have $g(\Omega)=0$. Condition (18) precisely insures that $\Omega=\nu$ is a real root of (17). For those modes which do not satisfy (18), no real root of (17) will be $\Omega=\nu$ and $g(\Omega)=0$ is never possible in this exaggeratedly simple approximation. Should we be a little more reasonable to use the true plateau distribution in which the height A and half-width Δ are separately finite instead of the impulse distribution in which $A \rightarrow \infty$, $\Delta \rightarrow 0$ while $A\Delta \equiv N$ is finite, the situation would be different since then Ω may have values ranging from $\nu-\Delta$ to $\nu+\Delta$ within which $g(\Omega)=0$ can be satisfied. The problem would amount to a true fifth-degree equation.

VIII. CONCLUSION

The result of this study, if any, instead of constructive, are rather discouraging. We have attempted to appreciate the various difficulties confronting us at each stage of the approximation in an effort to proceed further on in a simple manner by sacrificing such considerations as should otherwise be extended to physical reality. Nevertheless, even after many such desperate procedures, we are still so remote from attaining a very crude idea regarding the ultimate purpose for studying the behavior of a coasting beam, viz, what conditions, no matter how approximate, should be imposed upon the initial distribution so that the occurrence of those predominant unstable modes can be minimized in the temporal evolution of a beam subsequent to injection. At least in the kind of approach assumed here, this intricate problem will be very very hard to analyze, if the two basic questions raised in Section V cannot be affirmatively settled.

We therefore would like to conclude this study with a very brief discussion on what might be called "the initial-value problem."

In the perturbed distribution (cf (8c) , N introduced because of the normalization convention (12a))

$$\Psi^*(\theta,\dot{\theta},x,y,t) = \sum_{\Omega n m \sigma} N_{\Omega n m \sigma}\, \psi_{\Omega n m \sigma}(\dot{\theta})\, \langle \theta, x, y | n m \sigma \rangle$$

let us extract out one spatial mode so that

$$\Psi^*_{nm\sigma}(\dot{\theta},t) = \sum_{\Omega} N_{\Omega nm\sigma}\, \psi_{\Omega nm\sigma}(\dot{\theta})\, e^{-in\Omega t}$$

As $t \to 0$, we should have

$$\Psi^*_{nm\sigma}(\dot{\theta}, t \to 0) \longrightarrow q^{(0)}_{nm\sigma}\, \Psi^0(\dot{\theta}) = \sum_{\Omega} N_{\Omega nm\sigma}\, \psi_{\Omega nm\sigma}(\dot{\theta})$$

If the set of "adjoint functions" $\bar{\psi}_{\Omega nm\sigma}$, as mentioned in Section V, is proved to exist and actually available, then the coefficients N_{Ω} can be evaluated. To minimize the occurrence of a certain time mode Ω' , we have to make the pertinent coefficient $N_{\Omega'}$ small. With the structure of N_{Ω} available in relation to $\Psi^0(\dot{\theta})$ and its gradient $g(\dot{\theta})$, we at least have something to start with to investigate the above-mentioned problem, although the prospect is not very optimistic because of the further intricacy that the eigenfunctions ψ_{Ω} themselves actually also depend on $g(\dot{\theta})$ as evidenced by (12) and all those expressions in Section VI. To know more about the operator equation cited in Section V seems to be the key to a better appreciation of these difficulties we have to cope with.

REFERENCES

(1) K.M. Case: "Plasma Oscillations", Annals of Physics Vol. 7. No. 3, p. 369 (1959).

(2) Same coordinate system as used in MURA-555.

(3) This is the differential form of equation of motion equivalent to the variational form given in MURA-556 (15) for first-order perturbation theory.

(4) The appropriate del-operator is given in Ref. 2 p. 5.

(5) Except for notation and minor details, this equation amounts to the same content as equation (39) in: C.E. Nielsen, A.M. Sessler & K.R. Symon, International Conference on High Energy Accelerators & Instrumentation, CERN 1959, p. 239.

(6) In Ref. 2 equations (19a) & (19b), Γ and Λ are used instead of γ and λ, the difference being that in the latter ω-dependent factors have been extracted, so that

$$\Gamma = \frac{n}{\omega}\gamma \quad , \qquad \Lambda = \frac{n^2}{k^2-\omega^2}\lambda$$

$\omega = o$ case is not discussed in Ref. 2. When $\omega = o$, the Maxwell equations are decoupled into two separate problems: an electrostatic problem and a steady current problem. Then (8b) is still good but (8a) needs modification.

(7) For example, Uspensky: Theory of Equations p. 94 (McGraw-Hill).

(8) Laslett, Neil and Sessler: Coherent Electromagnetic Effects in High-Current Particle Accelerators III. - Electromagnetic-Coupling Instabilities in a Coasting Beam.