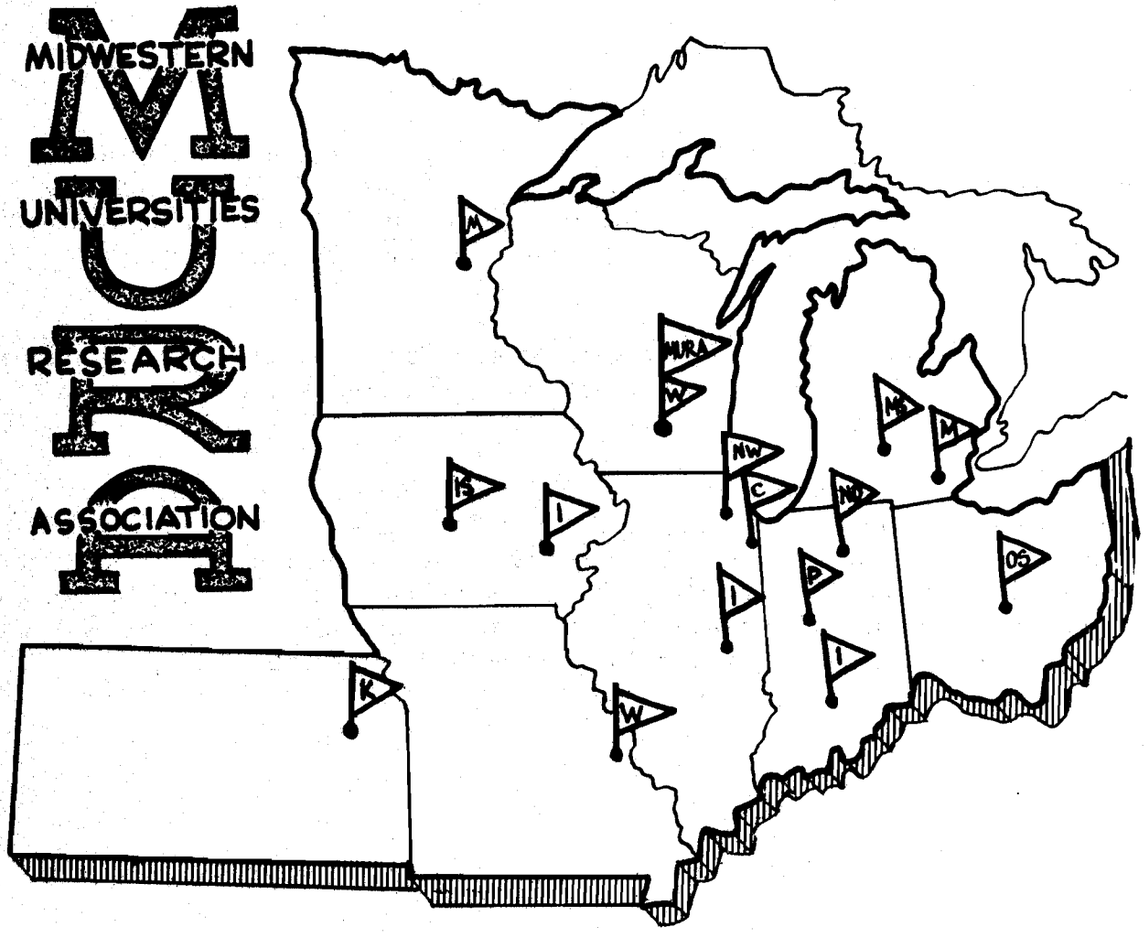
596

Fermilab Library
0 1160 0038875 5

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



THEORETICAL REMARKS CONCERNING THE $3\sigma_x + 2\sigma_y = 2\pi$ RESONANCE IN SPIRAL SECTOR ACCELERATORS

A. M. Sessler

REPORT **NUMBER** 596

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Printed in USA. Price $0.50. Available from the

Office of Technical Services
U. S. Department of Commerce
Washington 25, D. C.

MURA-596
UC-28, Particle Accelerators and
High Voltage Machines
TID-4500 (16th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

THEORETICAL REMARKS CONCERNING THE $3\sigma_x + 2\sigma_y = 2\pi$ RESONANCE IN SPIRAL SECTOR ACCELERATORS

A. M. Sessler**

January 2, 1961
Manuscript completed July, 1957.
Report printed January 2, 1961†

ABSTRACT

Previously developed methods[1] in which the equation for axial motion is treated as linear and the radial motion is independently prescribed are used to exhibit a resonance in a spiral sector accelerator in the neighborhood of $3\sigma_x + 2\sigma_y = 2\pi$. Quantitative results are compared with the computational results of Laslett.[2]

* AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission, Contract No. AT(11-1)-384.

**On leave from the Ohio State University, Columbus, Ohio.

† Note added December, 1960: This report has been delayed for three and one-half years because there was a sufficient disagreement with experiment (factor of 15, typically) to discourage even this most optimistic of workers. It now appears (see MURA-595 by C. A. Lassettre) that the agreement between theory and experiment can be significantly improved by limiting comparison to accelerators with small flutter factors. In view of this development, and despite the fact that a similar (but independent) analysis has since been presented by N. Marshall King, A. E. R. E. - TP/R 2342, the report is hereby released in the hope that it may still be of some small interest.

I. INTRODUCTION

The computational examination of the $3\sigma_x + 2\sigma_y = 2\pi$ resonance by Laslett[2] makes a theoretical treatment of the same resonance of some interest. In this report the methods previously developed[1] are extended to the study of this fifth-order resonance. The treatment is straightforward, except for concern over the solution of the non-linear radial equation. The analysis is limited to an accelerator with sinusoidally varying median-plane field in the azimuthal direction. The computational example is for a field with about 30% second harmonic, and consequently the numerical comparison is only to be taken as an indication as to order of magnitudes. The functional form of the resonance width and lapse rate seems to be correct.

The interesting prediction of Hagedorn and Moser[3] that fifth-order sum resonances are "harmless" does not seem to be equivalent to the absence of "y-growth." To check the possibility that the motion will always "turn over" is unfortunately beyond the scope of the present method, and must await analysis by more powerful treatments.

II. COMPUTATION

1) Comments

Since the calculation follows so closely that in L-S, the same notation will be used without being explained in this report. We start with the equations suitable for large-scale, sinusoidal field accelerators:[4]

$$u'' + \left[a_x + b_x \cos N\theta\right] u = -\frac{f}{2\,w^2} \sin N\theta\, u^2 + \frac{f}{6\,w^3} \cos N\theta\, u^3$$

$$y'' + \left[a_y + b_y \cos N\theta\right] y = \frac{f}{w^2} \sin N\theta\, u\, y - \frac{f}{2\,w^3} \cos N\theta\, u^2 y$$

$$- \frac{f}{6\,w^4} \sin N\theta\, u^3 y \quad . \tag{1}$$

The term $u^3 y$ is most important for the resonance under consideration, and must be added to Cole's work. The coefficient can be derived trivially by an examination of Cole's report.[5] Notice that the axial equation has been kept linear in y, while coupling terms in the x equation have been ignored.

The resonance under consideration will be driven by terms in the y-equation of frequencies $3\,\nu_x$, $3\,\nu_x \pm N$, $3\,\nu_x \pm 2N \cdots$. If we ignore all but the first three, we may concentrate on finding a solution to the x equation which will lead to the frequencies $3\,\nu_x$ and $3\,\nu_x \pm N$. If we (arbitrarily) choose a solution to the x equation of odd parity, then the linearized y equation will be of the form:

$$y'' + \left[a_y + b_y \cos N\theta + c \sin 3\nu_x\theta \sin N\theta + d \cos 3\nu_x\theta\right] y = 0 \tag{2}$$

where only the most important resonant terms have been retained. Letting:

$$\nu_o = N - 3\,\nu_x$$

$$q = N/\nu_o \quad , \tag{3}$$

this transforms into:

$$y'' + \left[a_y + b_y \cos q\,\nu_o\theta + \frac{c}{2} \cos \nu_o\theta - \frac{c}{2} \cos (2q - 1)\,\nu_o\theta \right.$$

$$\left. + d \cos (q - 1)\,\nu_o\theta\right] y = 0 \tag{4}$$

which by Appendix III C of L-S exhibits a resonance with boundaries given by:

$$\left| \nu_o^2 - (2\,\nu_{y_o})^2 \right| \doteq \left| c + \frac{2\,b_y\,d}{N^2} \right| . \tag{5}$$

We thus must find the coefficients c and d, which means first solving the u equation [Eq. (1)] with concern for all terms which may yield terms of frequency $3\,\nu_x$ or $3\,\nu_x \pm N$ in the y equation. This task is considerably simplified by the observation that the resonance width depends directly on c, so that any term leading to a term small compared to the dominant term in c may be immediately neglected. In particular it will be observed that the simple approximate solution to the linear radial equation, namely

$$u = A \sin \nu_x \theta , \tag{6}$$

when cubed and inserted into the fourth-order term in the y equation, yields

$$y'' + \left[a_y + b_y \cos N\theta + \frac{f\,A^3}{6\,w^4} \sin^3 \nu_x \theta \sin N\theta\right] y = 0 . \tag{7}$$

This then leads to

$$c = -\frac{f\,A^3}{24\,w^4} , \tag{8}$$

which is in fact the dominate term in c, as we will explicitly show.

2. Radial Solution

We may solve the u equation by perturbation theory in which we expand in powers of the amplitude of the radial motion. Let:

$$u = A\,u_o + A^2\,u_1 + A^3\,u_2 . \tag{9}$$

Then Eq. (1) becomes:

$$u'' + \left[a_x + b_x \cos N\theta\right] u_o = 0$$

$$u_1'' + \left[a_x + b_x \cos N\theta\right] u_1 = \frac{-f}{2w^2} \sin N\theta \, u_o^2$$

$$u_2'' + \left[a_x + b_x \cos N\theta\right] u_2 = \frac{-f}{w^2} \sin N\theta \, u_o u_1 + \frac{f}{6w^3} \cos N\theta \, u_o^3 . \tag{10}$$

The equations for u_o may be solved as in L-S Appendix II to yield:

$$u_o = \left[\sin \nu_x \theta + \left(\frac{f}{wN^2}\right) \sin \nu_x \theta \cos N\theta - 2\left(\frac{f}{wN^2}\right) \frac{\nu_x}{N} \cos \nu_x \theta \sin N\theta\right] \tag{11}$$

where terms of order $\left(\frac{f}{wN^2}\right)^2$ have been ignored. This may be inserted into the equation for u_1, which if we only desire terms of order $\left(\frac{f}{wN^2}\right)$ in the solution, may be simplified to

$$u_1'' + \nu_x^2 u_1 = -\frac{f}{2w^2} \sin N\theta \sin^2 \nu_x \theta . \tag{12}$$

Thus, as in L-S Section III, if we ignore terms in $\left(\frac{\nu_x}{N}\right)^2$,

$$u_1 = \frac{f}{4w^2N^2} \left[\sin N\theta - \sin N\theta \cos 2\nu_x \theta + \frac{4\nu_x}{N} \cos N\theta \sin 2\nu_x \theta\right]. \tag{13}$$

This may now be used, along with u_o, to obtain the forcing terms in the equation for u_2. Once again, if we concentrate only on terms of order $\left(\frac{f}{wN^2}\right)$, we obtain a simple equation:

$$u_2'' + \nu_x^2 u_2 = -\frac{f}{w^2} \sin N\theta \, u_o u_1 + \frac{f}{6w^3} \cos N\theta \sin^3 \nu_x \theta \tag{14}$$

yielding for u_2:

$$u_2 = \frac{f}{48w^3N^2} \left[\sin 3\nu_x \theta \cos N\theta - \frac{6\nu_x}{N} \cos 3\nu_x \theta \sin N\theta\right]. \tag{15}$$

It should be noted that the term in the equation for u_2, arising from $u_o u_1$ will lead to secular terms. In particular it contains

$$-\frac{f}{w^2} \sin N\theta \, (\sin \nu_x \theta) \left(\frac{f}{4w^2N^2} \sin N\theta\right) \tag{16}$$

which will lead (amongst other terms) to:

$$-\frac{f^2}{8\,w^4\,N^2}\,\sin\nu_x\,\theta\,. \tag{17}$$

This then means that u_2 has amongst the terms in its solution the secular term,

$$-\frac{f^2}{16\,w^4 N^2}\,\theta\cos\nu_x\,\theta\,. \tag{18}$$

Although this term is secular, its origin can be clearly understood by imagining solving the radial equation by harmonic balance. In that case, a solution (assumed, and hence forced to be periodic) of the form

$$u = A\sin\nu_x\,\theta + B\sin(\nu_x + N)\,\theta + \cdots\; c\sin 2\,\nu_x\,\theta + \cdots$$

would, when cubic terms are included, only be soluble by allowing ν_x to be a function of amplitude. In fact one would find that to first approximation $\nu_x \rightarrow \nu_x + \frac{f^2 A^3}{16\,w^4 N^2}$, so that if the solution were expanded in powers of A, the term in $u_o = \sin\nu_x\,\theta$ would yield

$$\sin\nu_x\,\theta\left[1 - 1/2\left(\frac{f^2 A^3}{16\,w^4 N^2}\right)^2\theta^2 + \cdots\right] + \\ + \cos\nu_x\,\theta\left[\frac{f^2 A^3}{16\,w^4 N^2}\,\theta + \cdots\right] \tag{19}$$

in agreement with Eq. (18). Thus one can ignore the secular term in u_2, and simply remember in the final result that ν_x is the frequency of the radial motion corresponding to amplitude A, in distinction to the linear tune.

3. y motion

If we insert the solution for u in the y equation, we obtain after the simplification of retaining only $3\nu_x$ and $3\nu_x \pm N$ frequency terms:

$$y'' + \left\{ a_y + b_y \cos N\theta + \frac{f^2 A^3}{w^5 N^3}\left(\frac{14\nu_x}{32}\right)\cos 3\nu_x\theta - \frac{f A^3}{24 w^4}\sin 3\nu_x\theta \sin N\theta \right\} y = 0. \quad (20)$$

Thus by comparison with Eq. (2):

$$c = -\frac{f A^3}{24 w^4}$$

$$d = \frac{7}{16}\frac{f^2 A^3 \nu_x}{w^5 N^3} . \quad (21)$$

This is then correct ignoring terms of order $\left(\frac{f}{w N^2}\right)^3$ which are completely negligible. By Eq. (5), dropping the small term in $\frac{2 b_y d}{N^2}$, we obtain

$$\left|(N - 3\nu_x)^2 - (2\nu_{y_o})^2\right| = \frac{f A^3}{24 w^4} . \quad (22)$$

This formula predicts a resonance which is symmetrical and very narrow since it varies as A^3.

Using the methods of L-S, we may readily calculate the lapse rate for the y motion in the unstable band of the resonance:

$$\mu = \frac{f}{96 w^4 (N - 3\nu_x)}\left[A^6 - A_{thr}^6\right]^{1/2} \quad \text{nepers/radian} \quad (23)$$

$$\mu = (.0284)\left(\frac{f}{w N^2}\right)\frac{1}{\left[1 - 3\nu_x/N\right]}\left[\left(\frac{A}{w}\right)^6 - \left(\frac{A_{thr}}{w}\right)^6\right]^{1/2} \quad \text{decades/sector} \quad (24)$$

III. COMPARISON WITH COMPUTATIONAL EXAMPLE

Comparing with Laslett's calculation,[2] one finds the functional form of the resonance width and lapse rate in excellent agreement. However, the calculated coefficient relating the width to the cube of the amplitude is 2.4×10^6 rather than $3.5 - 4.0 \times 10^7$ as found experimentally. The lapse rate, for typical parameters ($u_o = -.001$), is 8.1×10^{-4} decades/sector rather than .0125 decades/sector as found experimentally.

The source of this disagreement is not yet understood.

IV. ACKNOWLEDGEMENT

The author is indebted to members of the MURA Technical Group for stimulating comments, and in particular to L. Jackson Laslett for a careful reading of the manuscript and discovery of a computational error.

REFERENCES

1. L. J. Laslett and A. M. Sessler, MURA-263, hereafter called L-S.
2. L. J. Laslett, MURA-320.
3. R. Hagedorn, CERN 57-1, and papers cited therein.
4. F. T. Cole, MURA-95.
5. I am indebted to F. T. Cole for this remark.