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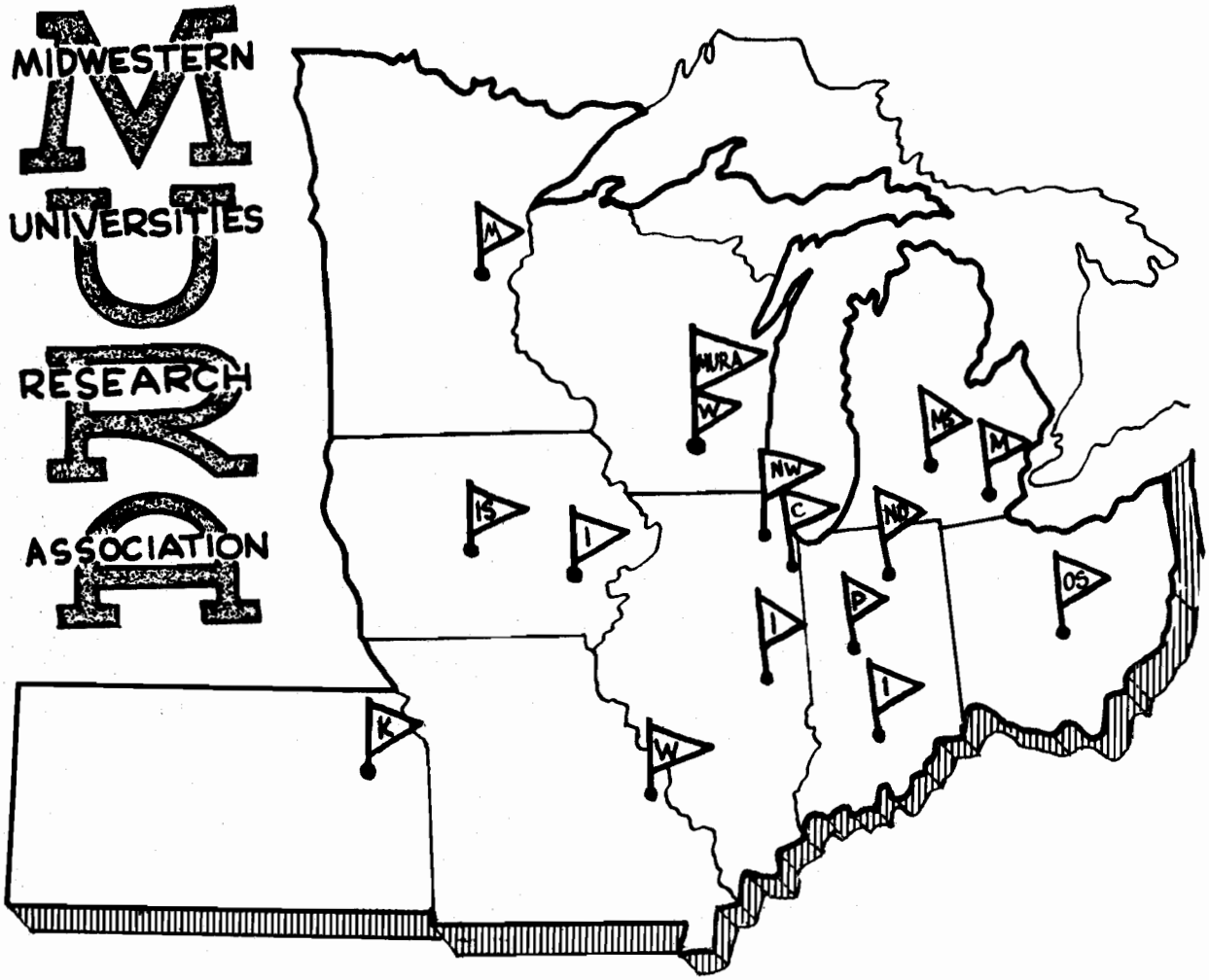

COMPUTATIONAL STUDIES OF COUPLING RESONANCES IN SPIRALLY-RIDGED ACCELERATORS

C. A. Lassettre

REPORT **NUMBER** 595

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MURA-595
UC-28, Particle Accelerators and
High Voltage Machines
TID-4500 (16th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

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COMPUTATIONAL STUDIES OF COUPLING RESONANCES IN SPIRALLY-RIDGED ACCELERATORS

C. A. Lassettre

January 2, 1961

ABSTRACT

Computational results for the $\sigma_x = 2\sigma_y$, $\sigma_x + 2\sigma_y = 2\pi$, $2\sigma_x + 2\sigma_y = 2\pi$ and $3\sigma_x + 2\sigma_y = 2\pi$ resonances are presented for large scaling, FFAG spirally-ridged accelerators. For the two lower order resonances, the results agree well with the theoretical predictions of Sessler and Laslett. For the two higher order resonances, agreement is good only when very close to the resonance line. The anticipated improvement of agreement over the computational results involving small accelerators was not as much as anticipated.

*AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission, Contract No. AT(11-1)-384.

INTRODUCTION

This report is an extension of MURA-263,[1] MURA-320[2] and MURA-596.[3] MURA-263 presented the theoretical treatment of $\sigma_x = 2\sigma_y$, $\sigma_x + 2\sigma_y = 2\pi$, and $2\sigma_x + 2\sigma_y = 2\pi$ resonances in comparison with some computational results for model-sized, spirally-ridged accelerators. MURA-320 and MURA-596 studied the $3\sigma_x + 2\sigma_y = 2\pi$ resonances but the computational results involved a large flutter factor.

However, the equations indicate that if the number of sectors in a machine is large and the flutter factor small, then the non-scaling terms in the equations could be further suppressed, making it possible for agreement between theory and computation to be improved. The purpose of this report was to investigate the above conjecture.

The results showed, however, that little or no improvement was made by increasing the number of sectors. The conclusion is that the non-scaling terms are not as important a source of error as the lower order resonance terms which were ignored in the analysis.

With the thresholds, agreement for the two lower order resonances is good (20% difference or less) below the resonance line as far as the calculations were carried (0.05π units of σ). But beginning at about 0.03π above the resonance, agreement drops off rapidly. Agreement for the two higher order resonances was worse than 50% difference when beyond 0.02π of the resonance.

Lapse-rate agreement for the two lower order resonances was fair (30% difference or less) with the agreement falling off in the same manner as the thresholds. But for the two higher order resonances, agreement was 50% difference or worse, even when close to the resonance.

PROCEDURE

It was felt that $N = 40$ and $f = 1/4$ would adequately suppress the non-scaling terms and that a median plane field of $1 + 1/4 \sin \theta$ should be used. A tune diagram was drawn in order to choose convenient tunes which would cross the resonances in which we were interested, yet would not be influenced by other major resonances. Knowing the tune, preliminary values of $1/w$ and k were obtained from a necktie diagram.[4] The machine parameters were then used to determine the tune more accurately which in turn was used to determine the need for other values of the parameters. The parameters were entered in the TEMPERMESH program.[5]

Approximate stable fixed points were obtained from Phil Morton and entered in Fixed Point Search[6] to determine them more accurately for each $1/w$ and k. Then prescribed deviations (u_o, p_{u_o}) from the fixed points in the initial sector were entered in FORMESH[7] in order to find the deviations (u_1, p_{u_1}) from the homologous fixed point in the succeeding sector. Each set of $(u_o, p_{u_o}, u_1, p_{u_1})$ were in turn used to calculate the matrix coefficients of $\begin{pmatrix} u_1 \\ p_{u_1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u_o \\ p_{u_o} \end{pmatrix}$.[8] Several orders of magnitude of (u_o, p_{u_o}) were used to determine the set of $(u_o, p_{u_o}, u_1, p_{u_1})$ which would make $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ closest to 1.

The tune was then calculated from $\cos\sigma = 1/2\ (A + D)$ and the invariant coefficients from $\xi = \frac{-CB}{\sin^2\sigma}$, $\eta = \frac{(A-D)B}{\sin^2\sigma}$, $\zeta = \frac{B^2}{\sin^2\sigma}$. The ξ, η, ζ were entered in Formesh Invariants[7] to calculate

$K_x = (\xi_x x^2 + \eta_x x p_x + \zeta_x p_x^2)^{1/2}$ and $K_y = (\xi_y y^2 + \eta_y y p_y + \zeta_y p_y^2)^{1/2}$

which represents the amplitude of the motion from the fixed point.

Interest in this report has been confined to studying y-growth. From an initial (u_{x_o}, u_{y_o}) y-motion grows exponentially when $u_{x_o} > |u_x|_{thr}$ ($|u_x|_{thr}$ is called threshold amplitude). The growth becomes steeper the closer the operating point is to a coupling resonance. And as the operating point is taken to approach the resonance line, the threshold amplitude goes to zero. Below the threshold, y-motion increases and returns to u_{y_o} repetitiously (turnover). In some cases K_y would increase three or four decades, then level off. In this case, the assumption was made that the particle would go unstable since it would likely do so in a real accelerator because of construction errors. In the study of the behavior of unstable particle motion, calculations were carried out until either K_y had increased by a factor of 10^3 to 10^4 where it would go above a designated y_{max} (overflow), or the particle had traversed 250 sectors.

Examination of y-motion was begun by plotting the values of K_y versus the number of sectors traversed for a given x and y amplitude. Though the increase of K_y oscillated, a straight line was drawn provided no turnover was present. This straight line represents the lapse in decades per sector. Then the lapse rate was extrapolated to zero to find the threshold amplitude, where

the particle motion should have become stable, and the observed threshold was compared with the theoretical value. Finally, the observed threshold was substituted into the equation for the theoretical lapse rate and the result compared with the observed lapse rate.

Upon plotting the theoretical and observed thresholds versus the tune, there was evidence that sometimes the tune was displaced, since the observed threshold approached zero at a tune different from that demanded by theory. The assumption was made that the displacement was similar for all points plotted. Runs made on Ill-Tempered Five[9] for comparison with the tune obtained from Formesh revealed good agreement between the σ_y's but poor agreement among the σ_x's.

New theoretical values were therefore computed by arbitrarily changing σ_x in order to coordinate the plots of the observed and theoretical thresholds.

TABLE I. Comparison of Observed and Theoretical Thresholds Near the $\sigma_x = 2\sigma_y$ Resonance for Large and Small Accelerators

F = 1/4, Large Accelerator N = 40									f = 1/4, Small Accelerator N = 5				
pt.	1/w	k	σ_x/π	σ_y/π	$(\sigma_x-2\sigma_y)/\pi$	$\lvert A_{thr}\rvert$ theo	A_{thr}, obs	%diff[10]	pt.	$(\sigma_x-2\sigma_y)/\pi$	$\lvert A_{thr}\rvert$ theo	A_{thr}, obs	%diff[10]
1	896.0	26.52	.2693	.1106	.0481	$6.236\cdot10^{-4}$	$5.33\cdot10^{-4}$	+17					
2	901.3	26.32	.2675	.1170	.0335	$4.392\cdot10^{-4}$	$4.17\cdot10^{-4}$	+ 5					
3	906.6	26.12	.2667	.1232	.0203	$2.683\cdot10^{-4}$	$3.00\cdot10^{-4}$	-11					
47	910.2	25.99	.2662	.1271	.0120	$1.593\cdot10^{-4}$	$1.92\cdot10^{-4}$	-17					
48	913.7	25.85	.2664	.1309	.0046	$6.122\cdot10^{-5}$	$1.00\cdot10^{-4}$	-39					
4	917.3	25.72	.2654	.1347	-.0040	$5.347\cdot10^{-5}$	$1.08\cdot10^{-4}$	-50					
49	920.8	25.59	.2650	.1382	-.0114	$1.527\cdot10^{-4}$	$2.21\cdot10^{-4}$	-31					
50	924.4	25.45	.2644	.1420	-.0196	$2.635\cdot10^{-4}$	$3.66\cdot10^{-4}$	-22	D	-.0199	.0024	.003	-20
5	927.9	25.32	.2643	.1454	-.0265	$3.567\cdot10^{-4}$	$4.86\cdot10^{-4}$	-27	ϵ	-.0322	.0093	.0075	+ 24
6	938.6	24.92	.2631	.1556	-.0481	$6.501\cdot10^{-4}$	$7.88\cdot10^{-4}$	-18	C	-.0494	.0108	.010	+ 8

TABLE II. Comparison of Observed and Theoretical Lapse Rates Near the $\sigma_x = 2\,\sigma_y$ Resonance for Large and Small Accelerators.

(a) Large Accelerator

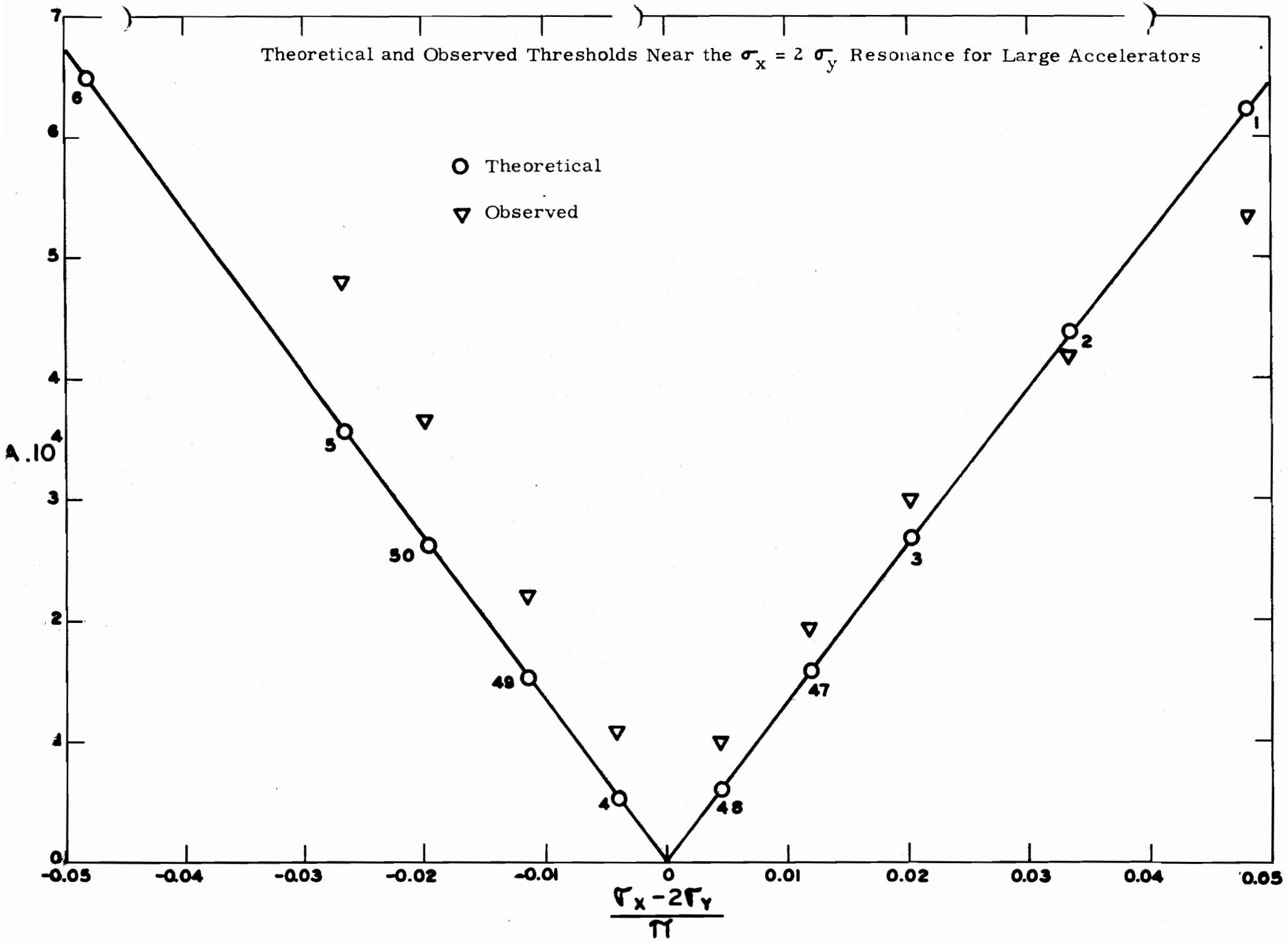
Pt.	$u_o \cdot 10^{+4}$	-10	-9	-8	-7	-6	-5	-4	-3	-2
1	μ_{obs}	.0529	.0448	.0634	.0278	.0167	0			
	μ_{theo}	.0406	.0348	.0268	.0218	.0132				
	%diff	-23	-22	-21	-22	-21				
2	μ_{obs}	.0552	.0473	.0402	.0336	.0251	.0166	0		
	μ_{theo}	.0438	.0383	.0326	.0266	.0199	.0134			
	%diff	-21	-19	-19	-18	-21	-19			
3	μ_{obs}	.0550	.0470	.0431	.0361	.0288	.0233	.0162	0	
	μ_{theo}	.0474	.0421	.0368	.0314	.0258	.0199	.0131	0	
	%diff	-14	-10	-15	-13	-10	-15	-12		
4	μ_{obs}	.0544	.0484	.0427	.0360	.0304	.0250	.0195	.0129	0
	μ_{theo}	.0493	.0442	.0390	.0338	.0286	.0230	.0176	.0116	.0028
	%diff	-11	- 9	- 9	- 6	- 6	- 8	-10	-10	
48	μ_{obs}	.0522	.0448	.0427	.0354	.0311	.0245	.0197	.0140	.0092
	μ_{theo}	.0506	.0455	.0404	.0352	.0301	.0249	.0197	.0144	.0088
	%diff	- 3	+ 2	- 5	- 1	- 3	+ 2	0	+ 3	- 4
4	μ_{obs}	.0499	.0465	.0386	.0341	.0287	.0241	.0188	.0134	.0085
	μ_{theo}	.0511	.0460	.0408	.0356	.0304	.0251	.0198	.0144	.0087
	%diff	+ 2	- 1	+ 6	+ 4	+ 6	+ 4	+ 5	+ 7	+ 2
49	μ_{obs}	.0446	.0441	.0379	.0328	.0268	.0217	.0163	.0099	
	μ_{theo}	.0507	.0454	.0400	.0346	.0290	.0233	.0173	.0106	
	%diff	+13	+ 3	+ 6	+ 5	+ 8	+ 7	+ 6	+ 7	
50	μ_{obs}	.0464	.0406	.0358	.0290	.0231	.0163	.0095		
	μ_{theo}	.0490	.0433	.0375	.0314	.0250	.0179	.0085		
	%diff	+ 6	+ 7	+ 5	+ 8	+ 8	+10	+11		
5	μ_{obs}	.0425	.0375	.0307	.0243	.0185	0			
	μ_{theo}	.0465	.0403	.0338	.0268	.0188	.0063			
	%diff	+ 9	+ 7	+10	+10	+ 2				
6	μ_{obs}	.0266	.0202	0						
	μ_{theo}	.0339	.0240	.0076						
	%diff	+27	+19							

Tables I and II indicate that no improvement has been made by considering an increase of sectors. For the thresholds in the small machine, differences ranged between +24% and -20%, while in the large accelerator, differences were from -18% to -27%. For the lapse rates the differences for N = 5 were between +3% and -13% for point C as compared with +5% to +11% for point 50 representing N = 40. The fact that for small structures the theoretical values are found both above and below the observed indicates that agreement is better than for large machines where the theoretical values are to one side of the observed.

TABLE II. Comparison of Observed and Theoretical Lapse Rates Near the $\sigma_x = 2\sigma_y$ Resonance for Large and Small Accelerators.

(b) Small Accelerator

Pt.	$u_o \cdot 10^{+2}$	-3.75	-3.00	-2.25	-1.50	-0.75	-0.50
	μ_{obs}	.135	.103	.070	.036	0	
C	μ_{theo}	.118	.092	.066	.037		
	%diff	-13	-11	- 6	+ 3		
	μ_{obs}	.101	.080	.055	.029	0	
ϵ	μ_{theo}	.091	.072	.052	.032		
	%diff	-10	-10	- 5	+10		
	μ_{obs}	.16	.13	.08	.05	.02	.01
D	μ_{theo}	.115	.092	.068	.054	.021	.010
	%diff	-28	-29	-15	+ 8	+ 5	0


Theoretical and Observed Thresholds Near the $\sigma_x = 2\,\sigma_y$ Resonance for Large Accelerators
Theoretical
Observed
$A \cdot 10^4$
$\frac{\sigma_x - 2\sigma_y}{\pi}$
-0.05
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04
0.05
0
1
2
3
4
5
6
7
6
5
50
49
4
48
47
3
2
1

$\sigma_x + 2\sigma_y = 2\pi$ Resonance

Table III shows marked improvement of threshold values when N is increased. For N = 5, differences range from + 33% to + 200% as compared to a range of -32% to + 26% for N = 40. Yet Table IV shows a reversal of this improvement for lapse rates. For small N the range is -10% to -16% as compared with -14% to -23% for large N.

TABLE III Comparison of Observed and Theoretical Thresholds Near the $\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large and Small Accelerators

Large Accelerator									Small Accelerator				
pt.	1/w	k	σ_x/π	σ_y/π	$(\sigma_x+2\sigma_y)/\pi$	$\lvert A_{thr}\rvert$ theo	$A_{thr,obs}$	%diff	pt.	$(\sigma_x+2\sigma_y)/\pi$	$\lvert A_{thr}\rvert$ theo	$A_{thr,obs}$	%diff
44	2350	30	.3031	.8245	1.9521	$4.644\cdot10^{-5}$	$4.01\cdot10^{-5}$	+16	3	1.9455	.004	.003	+ 33
101	2355	30	.3025	.8288	1.9601	$3.863\cdot10^{-5}$	$3.27\cdot10^{-5}$	+18					
102	2360	30	.3021	.8333	1.9687	$3.025\cdot10^{-5}$	$2.52\cdot10^{-5}$	+20	4	1.9695	.002	.001	+100
103	2365	30	.3018	.8379	1.9776	$2.162\cdot10^{-5}$	$1.73\cdot10^{-5}$	+25					
104	2370	30	.3017	.8246	1.9869	$1.289\cdot10^{-5}$	$1.13\cdot10^{-5}$	+14					
105	2375	30	.3014	.8474	1.9962	$3.658\cdot10^{-6}$	$5.39\cdot10^{-6}$	-32	5	1.9959	.0003	.0002	+ 50
106	2380	30	.3043	.8523	2.0089	$8.549\cdot10^{-6}$	$8.94\cdot10^{-6}$	- 4					
107	2385	30	.3041	.8575	2.0191	$1.833\cdot10^{-5}$	$1.86\cdot10^{-5}$	- 1					
108	2390	30	.3064	.8627	2.0318	$3.026\cdot10^{-5}$	$2.61\cdot10^{-5}$	+16					
109	2395	30	.3066	.8681	2.0428	$4.070\cdot10^{-5}$	$3.24\cdot10^{-5}$	+26	6	2.0466	.003	.001	+ 200
45	2400	30	.3079	.8739	2.0557	$5.322\cdot10^{-5}$	$3.85\cdot10^{-5}$	+ 38					

TABLE IV. Comparison of Observed and Theoretical Lapse Rates Near the $\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large and Small Accelerators.

(a) Large Accelerator

Pt.	$u_o \cdot 10^{+5}$	-10.0	- 9.5	- 9.0	- 8.5	- 8.0	- 7.5	- 7.0	- 6.5	- 6.0
	μ_{obs}	.0926	.0711	.0656	.0610	.0581	.0507	.0455	.0401	.0341
44	μ_{theo}	.0636	.0597	.0559	.0520	.0480	.0440	.0398	.0355	.0310
	%diff	-31	-16	-15	-15	-17	-13	-13	-11	- 9
	μ_{obs}	.0800	.0739	.0690	.0659	.0610	.0562	.0509	.0467	.0428
101	μ_{theo}	.0658	.0621	.0584	.0546	.0508	.0470	.0431	.0391	.0350
	%diff	-18	-16	-15	-17	-17	-16	-15	-16	-18
	μ_{obs}	.0833	.0766	.0734	.0698	.0649	.0600	.0556	.0515	.0461
102	μ_{theo}	.0677	.0640	.0604	.0568	.0531	.0494	.0457	.0419	.0381
	%diff	-19	-16	-18	-19	-18	-18	-18	-19	-17
	μ_{obs}	.0862	.0813	.0766	.0723	.0682	.0635	.0580	.0543	.0502
103	μ_{theo}	.0691	.0656	.0620	.0584	.0548	.0512	.0476	.0440	.0403
	%diff	-20	-19	-19	-19	-20	-19	-18	-19	-20
	μ_{obs}	.0888	.0860	.0789	.0750	.0701	.0655	.0612	.0562	.0525
104	μ_{theo}	.0700	.0665	.0629	.0594	.0558	.0523	.0487	.0451	.0415
	%diff	-21	-23	-20	-21	-20	-20	-20	-20	-21
	μ_{obs}	.0912	.0850	.0818	.0765	.0727	.0678	.0625	.0578	.0538
105	μ_{theo}	.0706	.0671	.0635	.0600	.0565	.0529	.0494	.0458	.0423
	%diff	-23	-21	-22	-22	-22	-22	-21	-21	-21
	μ_{obs}	.0929	.0875	.0822	.0779	.0739	.0688	.0635	.0595	.0550
106	μ_{theo}	.0709	.0673	.0638	.0602	.0566	.0530	.0494	.0458	.0422
	%diff	-24	-23	-22	-23	-23	-23	-22	-23	-23
	μ_{obs}	.0968	.0900	.0850	.0802	.0749	.0696	.0638	.0595	.0547
107	μ_{theo}	.0702	.0666	.0630	.0593	.0556	.0519	.0483	.0445	.0408
	%diff	-27	-26	-26	-26	-26	-25	-24	-25	-25
	μ_{obs}	.0959	.0900	.0853	.0798	.0760	.0700	.0638	.0591	.0532
108	μ_{theo}	.0694	.0657	.0619	.0582	.0544	.0506	.0467	.0428	.0389
	%diff	-28	-28	-28	-27	-29	-28	-27	-28	-27
	μ_{obs}	.1017	.0924	.0852	.0802	.0764	.0694	.0640	.0587	.0527
109	μ_{theo}	.0683	.0645	.0606	.0567	.0528	.0489	.0448	.0407	.0365
	%diff	-33	-30	-29	-29	-31	-30	-30	-31	-31
	μ_{obs}	.0952	.0920	.0860	.0813	.0750	.0692	.0622	.0567	.0503
45	μ_{theo}	.0670	.0630	.0590	.0550	.0509	.0467	.0424	.0380	.0334
	%diff	-30	-32	-31	-32	-32	-33	-32	-33	-34

TABLE IV. Comparison of Observed and Theoretical Lapse Rates Near the $\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large and Small Accelerators.

(a) Large Accelerator

Pt.	$u_0 \cdot 10^{+5}$	- 5.5	- 5.0	- 4.5	- 4.0	- 3.5	- 3.0	- 2.5	- 2.0	- 1.5	- 1.0
	μ_{obs}	.0283	.0232	.0160	0						
44	μ_{theo}	.0261	.0207	.0142							
	%diff	- 8	-11	-11							
	μ_{obs}	.0377	.0310	.0261	.0188	.0101	0				
101	μ_{theo}	.0308	.0263	.0215	.0161	.0087					
	%diff	-18	-15	-18	-14	-14					
	μ_{obs}	.0417	.0357	.0308	.0271	.0216	.0128	0			
102	μ_{theo}	.0342	.0302	.0261	.0217	.0170	.0114				
	%diff	-18	-15	-15	-20	-21	-11				
	μ_{obs}	.0451	.0408	.0339	.0292	.0241	.0201	.0152	.0086	0	
103	μ_{theo}	.0367	.0329	.0292	.0253	.0214	.0172	.0127	.0071		
	%diff	-19	-19	-14	-13	-11	-14	-16	-17		
	μ_{obs}	.0479	.0439	.0372	.0326	.0283	.0236	.0188	.0145	.0090	0
104	μ_{theo}	.0380	.0343	.0307	.0271	.0234	.0196	.0157	.0117	.0070	
	%diff	-21	-22	-17	-17	-17	-17	-17	-19	-20	
	μ_{obs}	.0485	.0444	.0395	.0336	.0292	.0248	.0206	.0163	.0115	.0070
105	μ_{theo}	.0387	.0352	.0316	.0280	.0245	.0209	.0173	.0136	.0099	.0060
	%diff	-20	-21	-20	-17	-16	-16	-16	-17	-14	-14
	μ_{obs}	.0500	.0453	.0398	.0347	.0299	.0243	.0198	.0147	.0099	
106	μ_{theo}	.0386	.0350	.0314	.0278	.0241	.0204	.0166	.0127	.0086	
	%diff	-23	-23	-21	-20	-19	-16	-16	-14	-13	
	μ_{obs}	.0503	.0450	.0391	.0337	.0273	.0216	.0153	.0068		
107	μ_{theo}	.0370	.0332	.0293	.0253	.0212	.0168	.0120	.0053		
	%diff	-26	-26	-25	-25	-22	-22	-22	-23		
	μ_{obs}	.0489	.0437	.0372	.0300	.0236	.0149	0			
108	μ_{theo}	.0348	.0307	.0264	.0218	.0168	.0107				
	%diff	-29	-30	-29	-27	-29	-28				
	μ_{obs}	.0474	.0408	.0337	.0240	.0143	0				
109	μ_{theo}	.0321	.0275	.0226	.0170	.0096					
	%diff	-32	-33	-33	-29	-33					
	μ_{obs}	.0443	.0368	.0227	.0106	0					
	μ_{theo}	.0285	.0231	.0169	.0078						
	%diff	-36	-37	-26	-26						

TABLE IV. Comparison of Observed and Theoretical Lapse Rates Near the $\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large and Small Accelerators.

(b) Small Accelerator

Pt.	$u_o \cdot 10^{+2}$	-1.500	-.0750	-.0375
	μ_{obs}	.179	.086	.042
5	μ_{theo}	.15	.076	.038
	%diff	-16	-12	-10

$2\sigma_x + 2\sigma_y = 2\pi$ Resonance

The evidence, although meager for small machines, indicates that agreement is worsened by increasing the number of sectors. The relative differences among thresholds is + 71% as against + 114% and among lapse rates ranges - 42% to - 58% as compared with - 71% to - 74% for small and large machines respectively.

Theoretical and Observed Thresholds Near the $\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large Accelerators

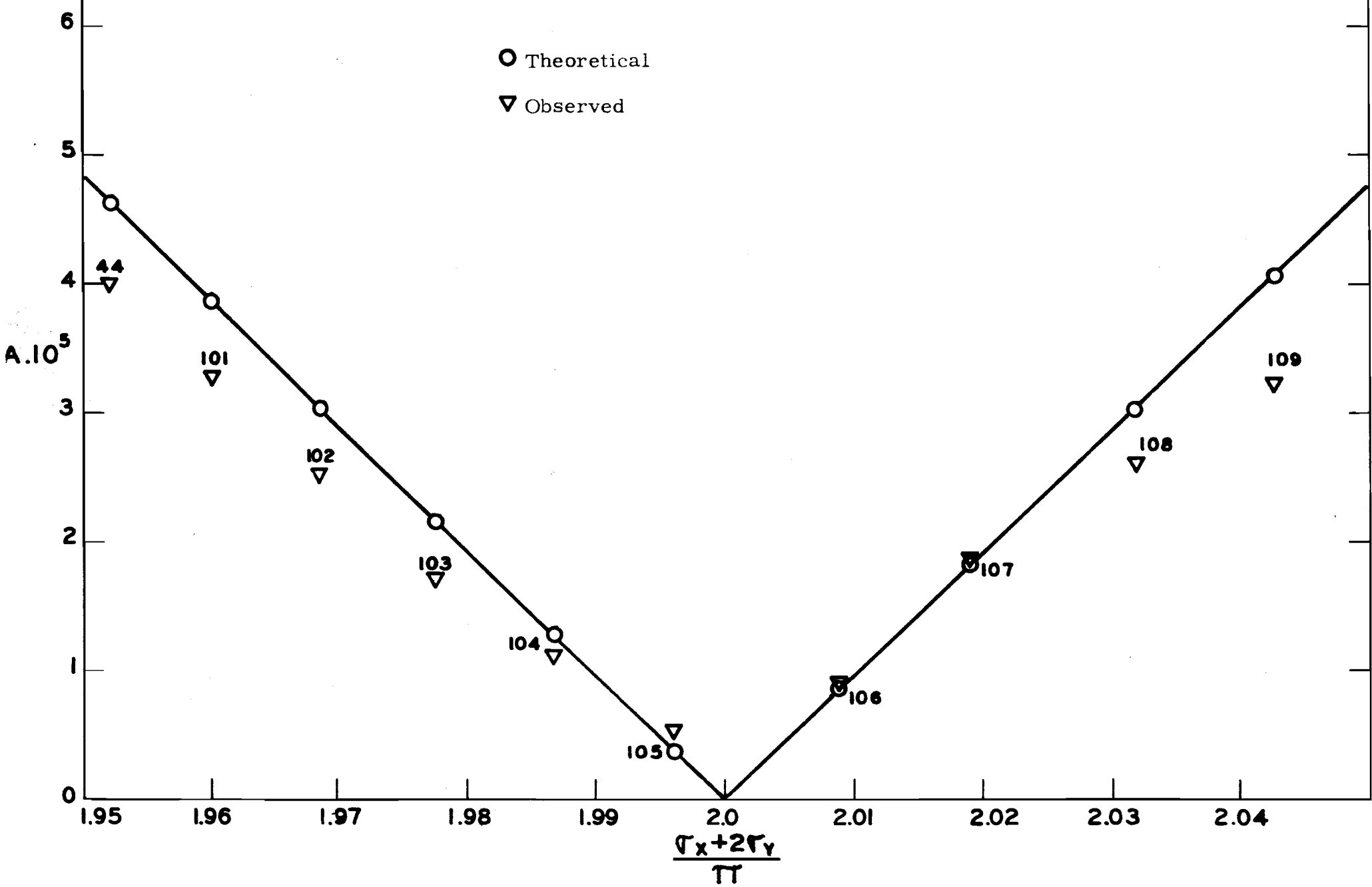


TABLE V. Comparison of Observed and Theoretical Thresholds Near the $2\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large and Small Accelerators

Large Accelerator									Small Accelerator				
pt.	1/w	k	σ_x/π	σ_y/π	$(2\sigma_x+2\sigma_y)/\pi$	$\lvert A_{thr}\rvert$ theo	$A_{thr,obs}$	%diff	pt	$(2\sigma_x+2\sigma_y)/\pi$	$\lvert A_{thr}\rvert$ theo	$A_{thr,obs}$	%diff
127	2142	23.23	.2611	.7079	1.9380	$3.418\cdot10^{-4}$	$1.6\cdot10^{-4}$	+114	13	1.9348	.029	.017	+71
128	2157	23.81	.2685	.7146	1.9672	$2.497\cdot10^{-4}$	$1.3\cdot10^{-4}$	+ 92					
18	2172	24.38	.2752	.7215	1.9934	$1.092\cdot10^{-4}$	$9.00\cdot10^{-5}$	+ 21	12	1.9908	.010	.010	0
129	2180	24.69	.2780	.7252	2.0064	$1.070\cdot10^{-4}$	$8.93\cdot10^{-5}$	+ 20					
130	2188	25.00	.2809	.7290	2.0198	$1.872\cdot10^{-4}$	$1.3\cdot10^{-4}$	+ 44					
19	2196	25.31	.2858	.7328	2.0372	$2.551\cdot10^{-4}$	$1.7\cdot10^{-4}$	+ 50					
131	2207	25.70	.2890	.7382	2.0544	$3.064\cdot10^{-4}$	$1.9\cdot10^{-4}$	+ 61					
20	2217	26.09	.2877	.7431	2.0610	$3.245\cdot10^{-4}$	$2.0\cdot10^{-4}$	+ 62					
132	2225	26.40	.2915	.7471	2.0772	$3.614\cdot10^{-4}$	$2.2\cdot10^{-4}$	+ 64					
133	2233	26.72	.2945	.7516	2.0912	$3.930\cdot10^{-4}$	$2.3\cdot10^{-4}$	+ 71					
21	2241	27.03	.2982	.7552	2.1068	$4.207\cdot10^{-4}$	$2.4\cdot10^{-4}$	+ 75					

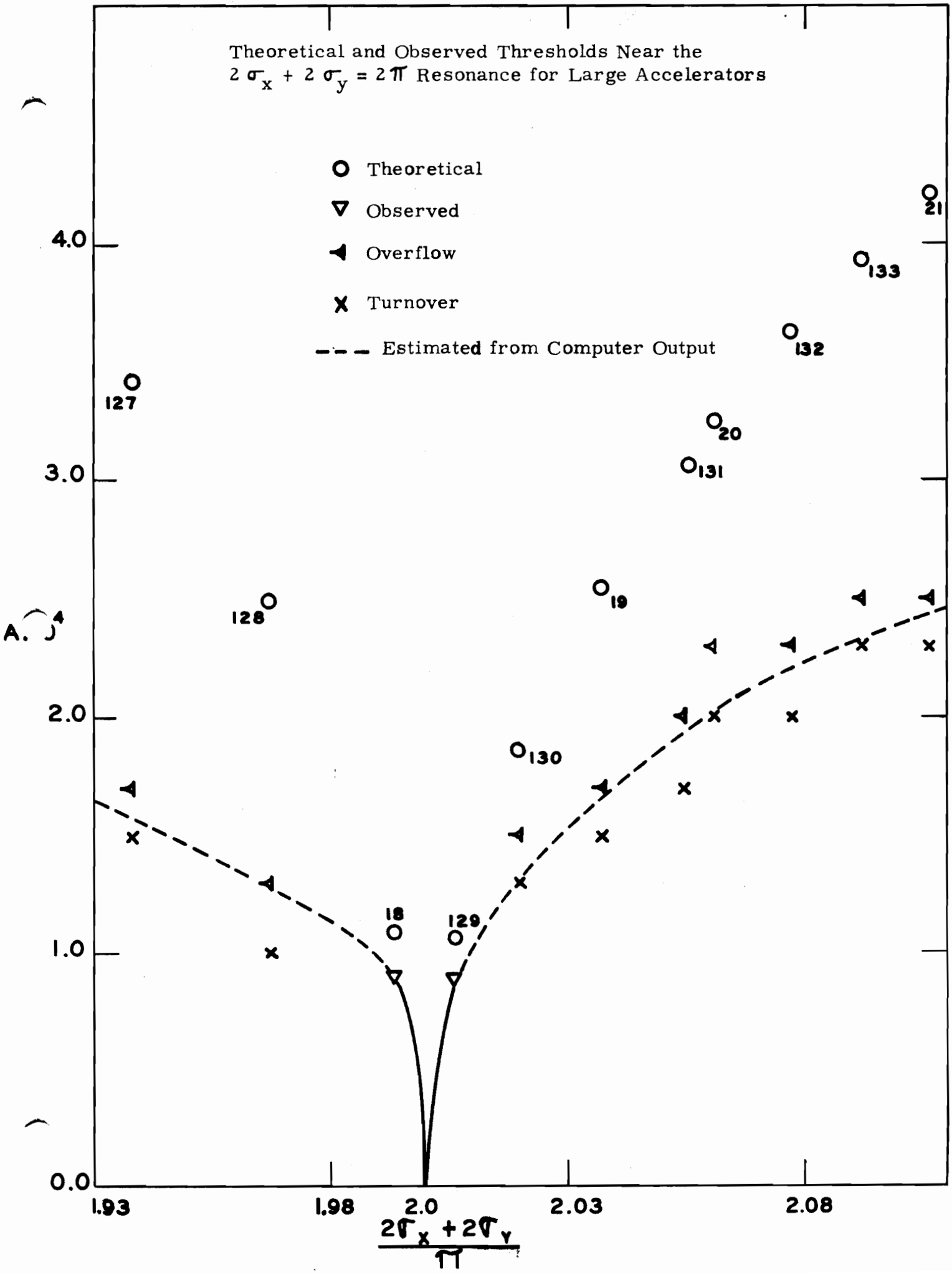
TABLE VI. Comparison of Observed and Theoretical Lapse Rates Near the $2\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large and Small Accelerators.

(a) Large Accelerator

Pt	$u_o \cdot 10^4$	-2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2	-1.1	-1.0
	μ_{obs}	.0555	.0497	.0439	.0375	.0334	.0282	.0241	.0203	.0170	.0117	
18	μ_{theo}	.0147	.0132	.0118	.0104	.0091	.0079	.0067	.0056	.0045	.0034	
	%diff	-74	-74	-73	-72	-73	-72	-72	-73	-74	-71	
	μ_{obs}	.0537	.0474	.0415		.0302	.0254	.0206	.0167	.0134	.0102	.0074
129	μ_{theo}	.0149	.0134	.0120	.0106	.0093	.0080	.0068	.0057	.0046	.0035	.0023
	%diff	-72	-72	-71		-69	-69	-67	-66	-66	-66	-69

(b) Small Accelerator

Pt.	u_o	-.0300	-.0225	-.0150
	μ_{obs}	12	.05	.02
12	μ_{theo}	.051	.029	.011
	%diff	-58	-42	-45


Theoretical and Observed Thresholds Near the
$2\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large Accelerators
Theoretical
Observed
Overflow
Turnover
Estimated from Computer Output
4.0
3.0
2.0
1.0
0.0
1.93
1.98
2.0
2.03
2.08
$\frac{2\sigma_x + 2\sigma_y}{\pi}$
127
128
18
129
130
19
131
20
132
133
21

$3\sigma_x + 2\sigma_y = 2\pi$ Resonance

The computations for this resonance are compared with earlier calculations which involved a large flutter factor. Sessler showed[3] that the width of the resonance should vary as the cube of the radial amplitude where the proportionality constant should be $2.4 \cdot 10^6$. Laslett's calculations[2] (using large f) showed that although the width did vary as the cube of the amplitude, the proportionality constant was 3.5 to $4.0 \cdot 10^7$. It was hoped that the smaller flutter would improve agreement, and this appears to be true numerically, although the dependence of width on amplitude was difficult to ascertain experimentally, and in fact appeared to be more linear than cubic.

TABLE VII. Comparison of Observed and Theoretical Thresholds Near the $3\sigma_x + 2\sigma_y = 2\pi$ Resonance for a Large Accelerator

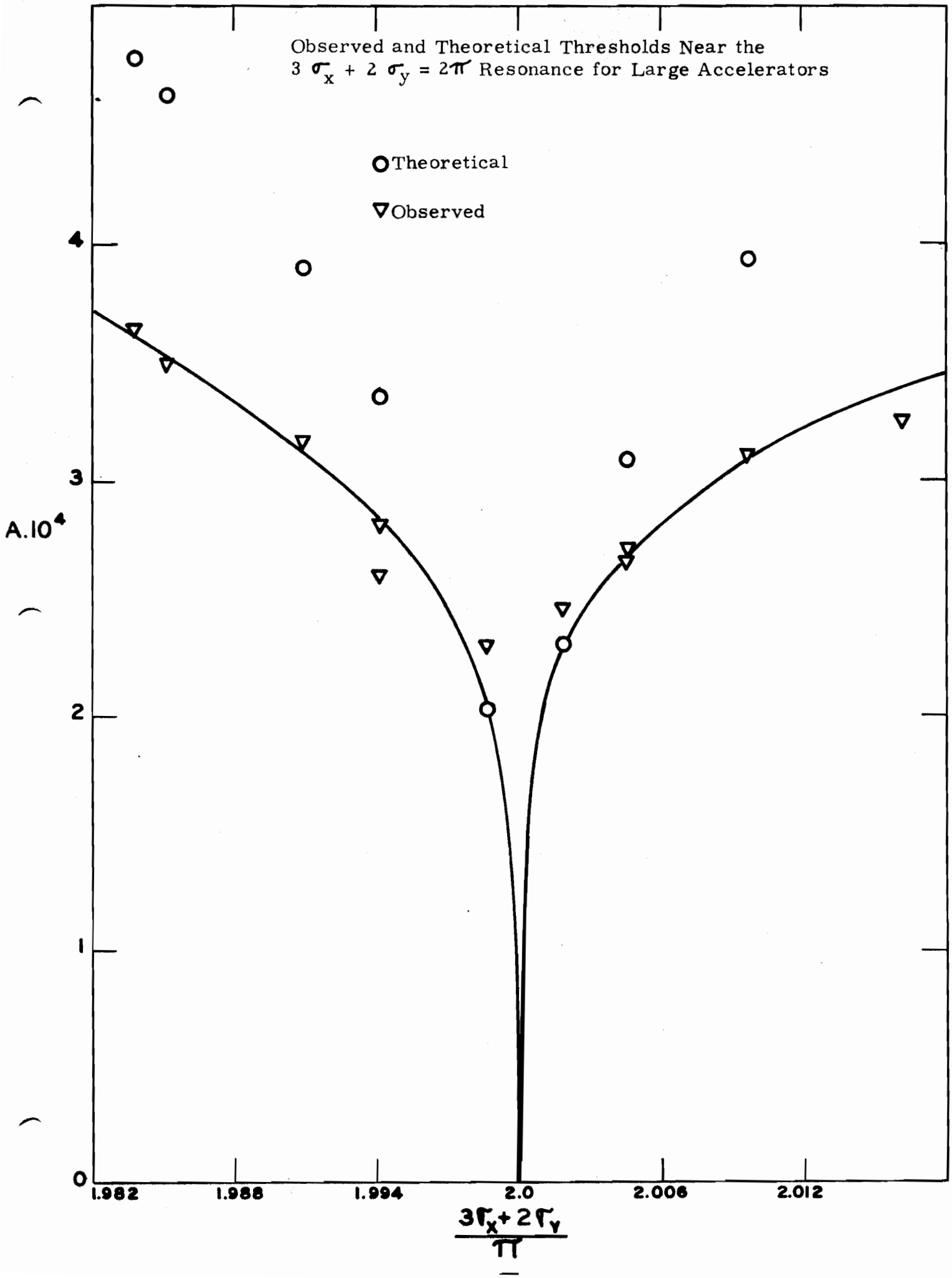
Pt.	1/w	k	σ_x/π	σ_y/π	$(3\sigma_x + 2\sigma_y)/\pi$	$\lvert A_{thr}\rvert_{theo}$	$A_{thr,\,obs}$	%diff
26	1632	64.0	.4298	.3259	1.9412	$7.569 \cdot 10^{-4}$		
51	1648	65.4	.4365	.3288	1.9671	$6.215 \cdot 10^{-4}$		
52	1656	66.1	.4394	.3302	1.9786	$5.274 \cdot 10^{-4}$		
112	1660.8	66.46	.4404	.3313	1.9838	$4.787 \cdot 10^{-4}$	$3.65 \cdot 10^{-4}$	+31
27	1664	66.7	.4404	.3320	1.9852	$4.634 \cdot 10^{-4}$	$3.49 \cdot 10^{-4}$	+33
114	1671	67.3	.4414	.3334	1.9910	$3.904 \cdot 10^{-4}$	$3.17 \cdot 10^{-4}$	+23
53	1678	67.9	.4416	.3347	1.9942	$3.355 \cdot 10^{-4}$	$2.81 \cdot 10^{-4}$	+19
117	1682	68.2	.4425	.3356	1.9987	$2.031 \cdot 10^{-4}$	$2.32 \cdot 10^{-4}$	-12
118	1684	68.3	.4431	.3363	2.0019	$2.301 \cdot 10^{-4}$	$2.45 \cdot 10^{-4}$	- 6
120	1686	68.5	.4438	.3366	2.0046	$3.084 \cdot 10^{-4}$	$2.66 \cdot 10^{-4}$	+16
121	1688	68.7	.4436	.3369	2.0046	$3.080 \cdot 10^{-4}$	$2.71 \cdot 10^{-4}$	+14
124	1692	69.0	.4447	.3378	2.0097	$3.936 \cdot 10^{-4}$	$3.11 \cdot 10^{-4}$	+27
126	1695	69.2	.4463	.3387	2.0163	$4.665 \cdot 10^{-4}$	$3.25 \cdot 10^{-4}$	+44
28	1696	69.3	.4478	.3388	2.0210	$5.058 \cdot 10^{-4}$		
54	1704	69.9	.4503	.3409	2.0327	$5.831 \cdot 10^{-4}$		
55	1712	70.6	.4512	.3423	2.0382	$6.103 \cdot 10^{-4}$		
56	1720	71.3	.4542	.3439	2.0504	$6.642 \cdot 10^{-4}$		
29	1728	72.0	.4574	.3455	2.0632	$7.107 \cdot 10^{-4}$		

TABLE VIII. Comparison of Observed and Theoretical Lapse Rates Near the $3\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large Accelerator.

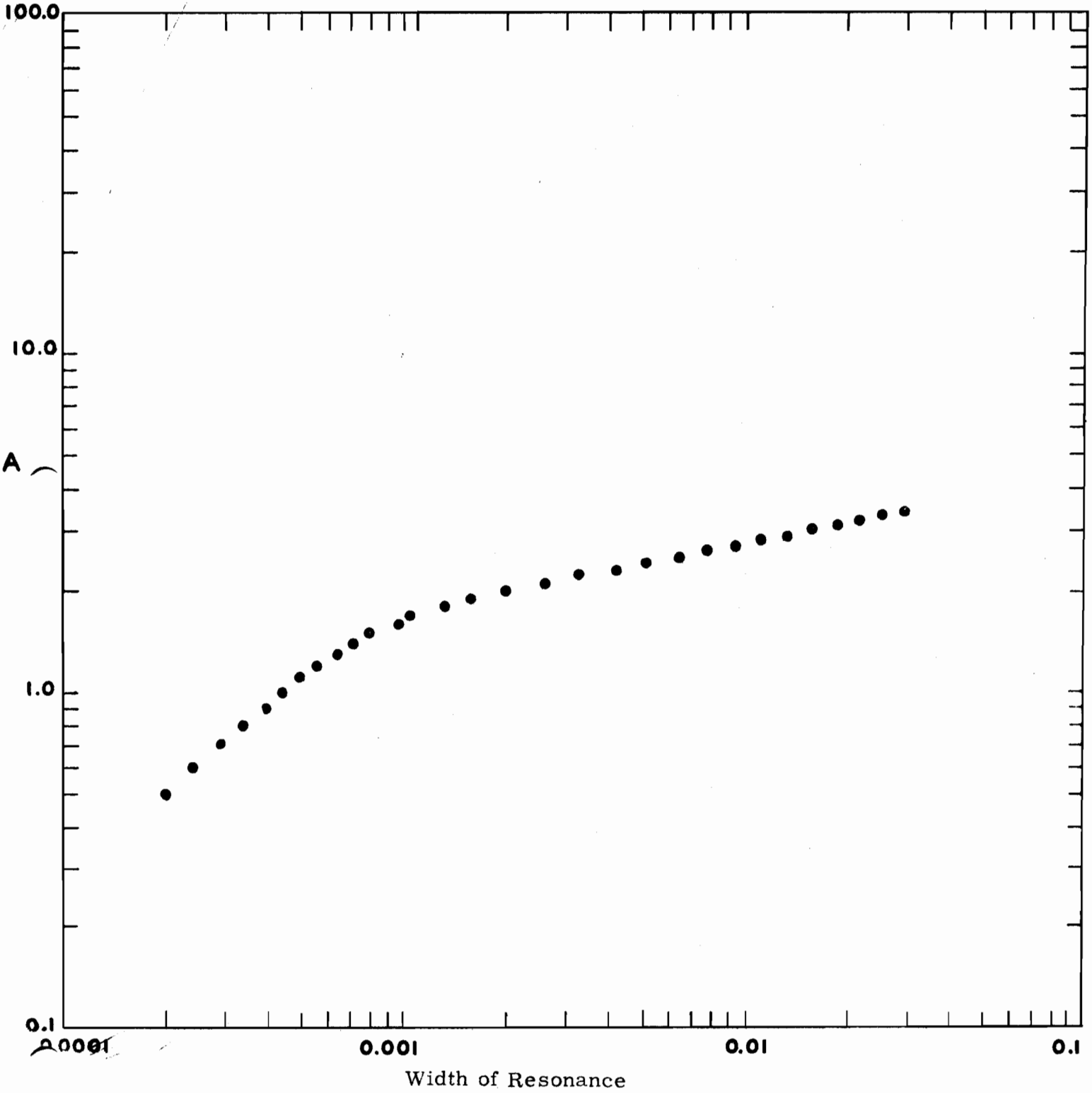
Pt.	$u_o \cdot 10^4$	-5.0	-4.9	-4.8	-4.7	-4.6	-4.5	-4.4	-4.3	-4.2	-4.1	-4.0
112	μ_{obs}											.0227
	μ_{theo}											.0104
	%diff											-54
27	μ_{obs}	.0561	.0542	.0545	.0517	.0487	.0425	.0404	.0376	.0333	.0300	.0270
	μ_{theo}	.0295	.0275	.0256	.0238	.0220	.0202	.0185	.0169	.0152	.0136	.0120
	%diff	-48	-49	-53	-54	-55	-53	-54	-55	-54	-55	-56
114	μ_{obs}						.0462	.0408	.0377	.0345	.0309	.0302
	μ_{theo}						.0219	.0202	.0187	.0171	.0157	.0142
	%diff						-53	-51	-50	-51	-49	-53
53	μ_{obs}											.0317
	μ_{theo}											.0156
	%diff											-51
11.	μ_{obs}											
	μ_{theo}											
	%diff											
118	μ_{obs}											
	μ_{theo}											
	%diff											
120	μ_{obs}											.0314
	μ_{theo}											.0164
	%diff											-48
121	μ_{obs}											.0312
	μ_{theo}											.0164
	%diff											-47
124	μ_{obs}						.0452	.0424	.0405	.0347	.0325	.0284
	μ_{theo}						.0231	.0218	.0201	.0185	.0170	.0155
	%diff						-49	-49	-50	-47	-48	-45
126	μ_{obs}	.0659	.0644	.0591	.0591	.0508	.0474	.0428	.0396	.0362	.0324	.0294
	μ_{theo}	.0333	.0312	.0292	.0272	.0253	.0234	.0216	.0199	.0182	.0166	.0150
	%diff	-49	-52	-51	-50	-50	-51	-50	-50	-50	-49	-49

TABLE VIII. Comparison of Observed and Theoretical Lapse Rates Near the $3\sigma_x + 2\sigma_y = 2\pi$ Resonance for Large Accelerator.

Pt.	$u_o \cdot 10^4$	-3.9	-3.8	-3.7	-3.6	-3.5	-3.4	-3.3	-3.2	-3.1	-3.0
	μ_{obs}	.0183	.0142	.0082	0						
112	μ_{theo}	.0085	.0064	.0037							
	%diff	-54	-55	-55							
	μ_{obs}	.0231	.0186	.0143	.0112	.0031					
27	μ_{theo}	.0104	.0087	.0069	.0049	.0015					
	%diff	-55	-53	-52	-56	-52					
	μ_{obs}	.0274	.0244	.0220	.0183	.0151	.0121	.0077	.0046		
114	μ_{theo}	.0128	.0114	.0101	.0087	.0073	.0059	.0042	.0020		
	%diff	-53	-53	-54	-53	-52	-51	-46	-57		
	μ_{obs}	.0293	.0268	.0246	.0229	.0211	.0191	.0166	.0142	.0128	
53	μ_{theo}	.0143	.0131	.0119	.0107	.0096	.0084	.0074	.0063	.0052	
	%diff	-51	-51	-52	-53	-55	-56	-56	-56	-59	
	μ_{obs}					.0195	.0177	.0159	.0145	.0128	.0113
1	μ_{theo}					.0108	.0099	.0089	.0080	.0072	.0063
	%diff					-45	-44	-44	-45	-44	-44
	μ_{obs}					.0193	.0171	.0154	.0136	.0122	.0106
118	μ_{theo}					.0107	.0097	.0087	.0078	.0069	.0060
	%diff					-45	-43	-44	-43	-43	-43
	μ_{obs}	.0290	.0251	.0234	.0219	.0189	.0174	.0153	.0136	.0121	
120	μ_{theo}	.0151	.0138	.0126	.0115	.0104	.0093	.0082	.0072	.0062	
	%diff	-48	-45	-46	-50	-45	-47	-46	-47	-49	
	μ_{obs}	.0285	.0261	.0231	.0197	.0180	.0158	.0145	.0120		
121	μ_{theo}	.0151	.0138	.0125	.0114	.0102	.0091	.0081	.0070		
	%diff	-47	-47	-46	-42	-43	-42	-45	-42		
	μ_{obs}	.0265	.0238	.0212	.0185	.0155	.0138				
124	μ_{theo}	.0140	.0126	.0112	.0098	.0084	.0069				
	%diff	-47	-47	-47	-47	-46	-50				
	μ_{obs}	.0265	.0232	.0202	.0172	.0136	.0113				
126	μ_{theo}	.0134	.0119	.0104	.0082	.0072	.0054				
	%diff	-49	-49	-49	-49	-47	-52				


Observed and Theoretical Thresholds Near the
3 σx + 2 σy = 2π Resonance for Large Accelerators
Theoretical
Observed
A.10⁴
4
3
2
1
0
1.982
1.988
1.994
2.0
2.006
2.012
(3σx + 2σy)/π

Logarithmic Plot of Width of Resonance vs. Amplitude
for $3\sigma_x + 2\sigma_y = 2\pi$ Resonance



EQUATIONS

The equations used in this report to determine the theoretical thresholds and lapse rates were:

$\sigma_x = 2\,\sigma_y$ Resonance[1]

$$|A_{thr}|_{theo} \; \frac{1}{16}\; w \left(\frac{wN^2}{f}\right)^2 \frac{N}{\nu_x} \left| \left(\frac{\sigma_x}{\pi}\right)^2 - \left(2\,\frac{\sigma_x}{\pi}\right)^2 \right|$$

$$\mu_{theo} = 2.73\; \frac{1}{w} \left(\frac{f}{wN^2}\right)^2 \left(A^2 - A^2_{thr,\,obs}\right)^{1/2}$$

$\sigma_x + 2\,\sigma_y = 2\pi$ Resonance[1]

$$|A_{thr}|_{theo} \; \frac{w}{4} \left(\frac{wN^2}{f}\right) \left| \left(2 - \frac{\sigma_x}{\pi}\right)^2 - \left(2\,\frac{\sigma_y}{\pi}\right)^2 \right|$$

$$\mu_{theo} = 0.682\; \frac{1}{w} \left(\frac{f}{wN^2}\right) \frac{1}{1 - \nu_x/N} \left(A^2 - A^2_{thr,\,obs}\right)^{1/2}$$

$2\,\sigma_x + 2\,\sigma_y = 2\pi$ Resonance[1]

$$|A_{thr}|_{theo} = 2\,w \left[\frac{wN^2}{f} \left| \left(1 - \frac{\sigma_x}{\pi}\right)^2 - \left(\frac{\sigma_y}{\pi}\right)^2 \right| \right]^{1/2}$$

$$\mu_{theo} = 0.17\; \frac{1}{w^2} \left(\frac{f}{wN^2}\right) \frac{1}{1 - 2\,\nu_x/N} \left(A^4 - A^4_{thr,\,obs}\right)^{1/2}$$

$3\,\sigma_x + 2\,\sigma_y = 2\pi$ Resonance[3]

$$|A_{thr}|_{theo} = \left[\frac{24\,w^4}{f} \left| (N - 3\,\nu_x)^2 - (2\,\nu_y)^2 \right| \right]^{1/3}$$

$$\mu_{theo} = 0.0284 \left(\frac{1}{w}\right)^3 \left(\frac{f}{wN^2}\right) \frac{1}{1 - 3\,\nu_x/N} \left(A^6 - A^6_{thr,\,obs}\right)^{1/2}$$

NOTES AND REFERNCES

1 L. J. Laslett and A. M. Sessler, "Concerning Coupling Resonances in the Spirally-Ridged FFAG Accelerator," May 6, 1957.

2. L. J. Laslett, "Computational Evidence for a Spirally-Ridged FFAG Resonance Near $3\sigma_x + 2\sigma_y = 2\pi$," July 5, 1957.

3. A. M. Sessler, "Theoretical Remarks Concerning the $3\sigma_x + 2\sigma_y = 2\pi$ Resonance in Spiral Sector Accelerators."

4. K. R. Symon, D. W. Kerst, L. W. Jones, L. J. Laslett and K. M. Terwilliger, Phys. Rev. 103, 1837 (1956).

5. L. D. Fosdick, MURA-241, March, 1957. This program computes the magnetic field components (H_x, H_y, H_θ) at prescribed mesh points. The mesh dimensions were a = 40, b = 10.

6. J. N. Snyder, MURA-231, February-March 1957. The accuracy employed in this program for these points was $\epsilon = 10^{-8}$.

7. MURA-222, February-March 1957. This program solves the differential equations for the trajectory of a particle in an FFAG accelerator, employing fields stored on a two-dimensional mesh.

8. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ carries a particle through successive sectors from one homologous point to the next.

9. E. Z. Chapman, MURA-457, May, 1959. This program solves the equations of motion describing the betatron oscillations of particles in a fixed field alternating gradient accelerator.

10. Percent difference is relative to the observed values.