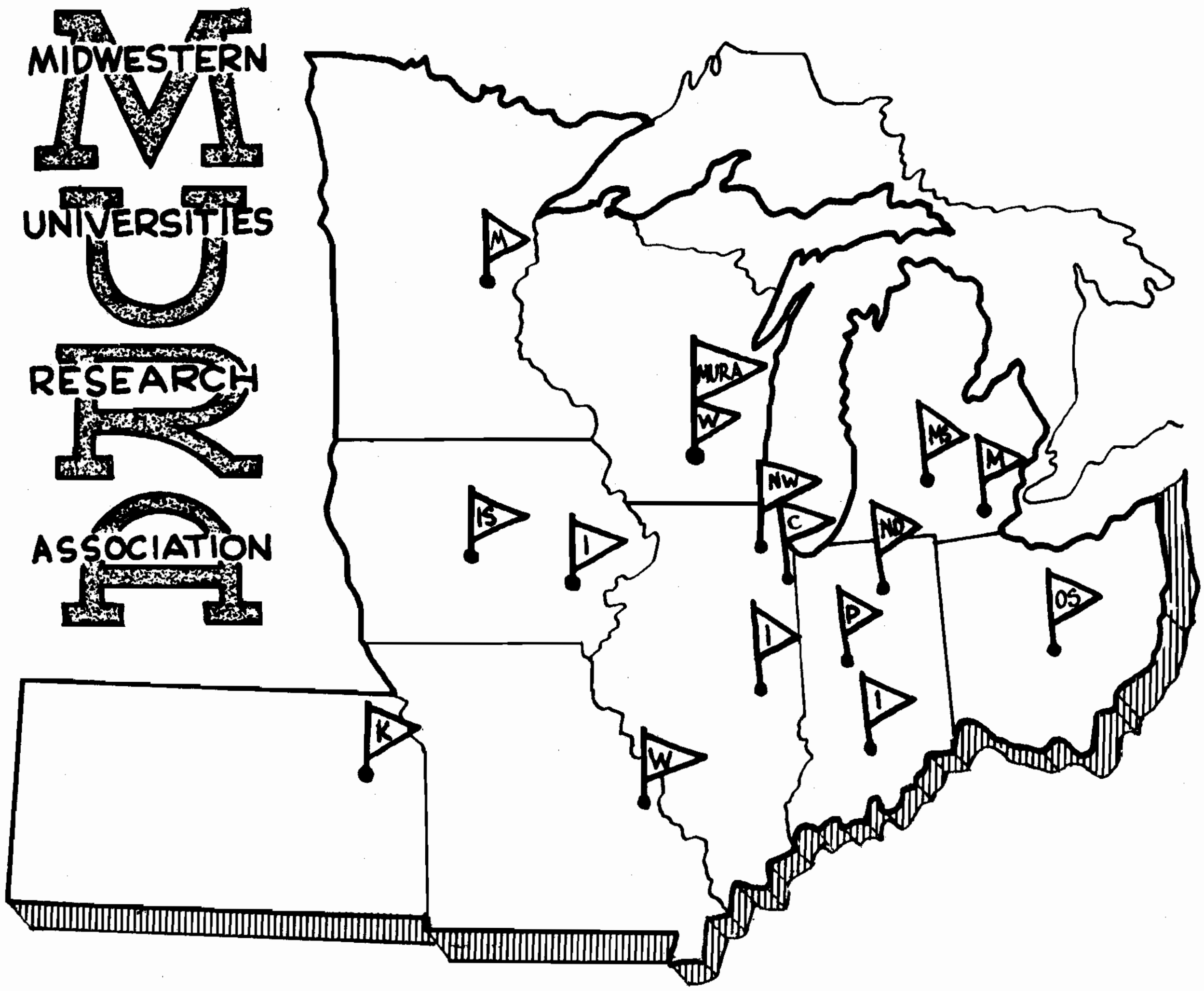

A PROPOSAL FOR THE STUDY OF LONG-TIME STABILITY

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A PROPOSAL FOR THE STUDY OF LONG-TIME STABILITY

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ABSTRACT

A method is proposed for studying the stability of a classically moving particle over long times. The method is based on some observations made by R. P. Feynman in presenting a space-time formulation of quantum mechanics. The stability is related to a multi-dimensional integral which, it is suggested, may be evaluated using a Monte Carlo procedure.

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The question of stability over a long time of classically moving particles is difficult to treat even with a computer. The motion is usually found by solving the differential equations by steps in time which have to be small compared to the periods of the dominant frequencies present. After about a thousand cycles of the motion, the accuracy of the results of the integration becomes poor. This report presents a proposal for studying the long-time stability of the classical motion of particles.

This proposal is based on some observations made by R. P. Feynman in presenting a space-time formulation of quantum mechanics.[1]

Consider the integral

$$S = \int_{t_o}^{t} L(x, \dot{x}, t)\, dt \quad , \tag{1}$$

where $L(x, \dot{x}, t)$ is the classical Lagrangian. We consider a one-dimensional system for the sake of simplicity and the integral is done along some path $x = x(t)$. The quantity S depends on the path $x(t)$. To indicate this, we will write

$$S = S\,(\text{path}) \ . \tag{2}$$

To investigate the stability, we may introduce the following quantity:

$$X^2 = \frac{1}{t - t_o} \int_{t_o}^{t} x^2(t)\, dt \quad . \tag{3}$$

X^2 depends on the path $x(t)$, and we would like to know its value over the path which is the solution of the classical equations of motion. To indicate that X^2 depends on the path, we will write

$$X^2 = X^2 \text{ (path)} . \tag{4}$$

Now consider the quantity

$$\overline{X^2} = \frac{\sum_{\text{paths}} X^2 \text{ (path) } e^{i S \text{ (path)}/\hbar}}{\sum_{\text{paths}} e^{i S \text{ (path)}/\hbar}} , \tag{5}$$

where we sum over all paths that pass through x_o t_o, and x, t.

As was pointed out by Feynman, when $\hbar \longrightarrow 0$, the main contribution to the sum will come from the paths close to the classical path, since $\delta \cdot \int L\, dt = 0$ for the classical path. Thus we may expect that as $\hbar \longrightarrow 0$

$$\overline{X^2} \xrightarrow[\hbar \to 0]{} X^2 \text{ (classical path).} \tag{6}$$

Thus to investigate the stability of the classical motion, it is proposed that $\overline{X^2}$ be calculated. The questions that remain are how to evaluate the sum over paths, and how small to choose $\hbar$.

<u>Evaluating the Sum over Paths.</u>

Let us break the interval, $t_o \longrightarrow t$, up into intervals Δt. Then we can regard the functional S (path) as a function of the great number of variables x_1, x_2, x_3, ... where

$$x_n = x\,(t_n) , \tag{7}$$

$$t_n = t_o + n\, \Delta t , \tag{8}$$

We write then

$$S = S\,(x_1, x_2 \ldots x_n \ldots) , \tag{9}$$

$$X^2 = X^2\,(x_1, x_2 \ldots . x_n) , \tag{10}$$

and

$$\overline{X^2} = \frac{\sum_{x_1 x_2 \ldots x_n \ldots} X^2 \, e^{iS/\hbar}}{\sum_{x_1 x_2 \ldots x_n \ldots} e^{iS/\hbar}} \tag{11}$$

The multi-dimensional integrals occurring in $\overline{X^2}$ may be evaluated by a Monte Carlo procedure. Thus, let us consider

$$K = \sum_{x_1 x_2 \ldots x_n \ldots} \cos\left(\frac{1}{b} S\right) \quad . \tag{12}$$

The variables $x_1 x_2 \ldots x_n \ldots$ are restricted to lie in a region which we believe is large enough to contain the classical path. We suppose that we require

$$x_{min} \leqslant x_n \leqslant x_{max} \; , \tag{13}$$

and we divide the interval $x_{min} \rightarrow x_{max}$ into N_x intervals of length Δx. Let N_t be the number of intervals from t_o to t of length Δt. Then $N_t - 1$ is the number of x variables.

We can write for K,

$$K = \frac{1}{(\Delta x)^{N_t - 1}} \int dx_1 \, dx_2 \ldots dx_n \ldots dy \; , \tag{14}$$

where the integration over y goes from 0 to $\cos\left(\frac{1}{b} S\right)$.

In the Monte Carlo procedure, to evaluate K, we make M trials in each of which we chose values of $x_1 x_2 \ldots x_n \ldots y$, which lie in the region $x_{min} < x_n < x_{max}$, $-1 < y < 1$. We divide the y interval into N_y intervals of length Δy.

There are $N = (N_y + 1)(N_x + 1)(N_t - 1)$ points to choose from. Of the M trials, let M_+ trials result in a choice of $x_1 x_2 \ldots x_n \ldots y$ such that $\cos\left(\frac{1}{b} S\right) > 0$ and $0 < y < \cos\left(\frac{1}{b} S\right)$ and M_- such that $\cos\left(\frac{1}{b} S\right) < 0$ and $\cos\left(\frac{1}{b} S\right) < y < 0$.

Then

$$K = \left[\frac{M_+}{M} - \frac{M_-}{M}\right] N \quad . \tag{15}$$

An Example.

As an example of the above procedure, the case of the harmonic oscillator will be treated. In this example, the procedure for choosing b will be indicated.

For the harmonic oscillator

$$L(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - \frac{m \omega^2}{2} x^2 , \tag{16}$$

$$S = \int_{t_o}^{t} \frac{m}{2} (\dot{x}^2 - \omega^2 x^2) \, dt \quad . \tag{17}$$

We can also write for S

$$S = \frac{m}{2} \sum_{i=1} \left[\frac{1}{\varepsilon} (x_i - x_{i-1})^2 - \omega^2 \varepsilon x_i^2\right] , \tag{18}$$

where $\varepsilon = \Delta t$.

What shall we choose for b? Feynman pointed out that if $b = \hbar$, Planck's constant, then $\sum_{path} \exp(i S/\hbar)$ is related to the quantum mechanical-wave function. This suggests that the choice of b determines the uncertainty in the energy ΔE of the path we are examining. That is, we may suppose that if we are examining the motion over the time $\Delta T = t - t_0$, then the $\overline{X^2}$ we find will be for a path whose energy may differ from the classical path by ΔE where

$$\Delta E \ \Delta T \simeq b \ . \tag{19}$$

Thus, b is determined by the length of time, ΔT, over which we examine the path and by the uncertainty in the energy ΔE that we allow.

The above relationship, $\Delta E \ \Delta T \simeq b$, and the relation $\overline{X^2} \xrightarrow[b \to 0]{} X^2$ (classical path) have only been suggested by the above. They can be tested by actual calculation for some examples where the classical path is known.

One may observe that should this method prove workable, it would also be applicable to other systems governed by a variational principle.

REFERENCE

[1]R. P. Feynman. Rev. Mod. Phys. 20, 367 (1948).