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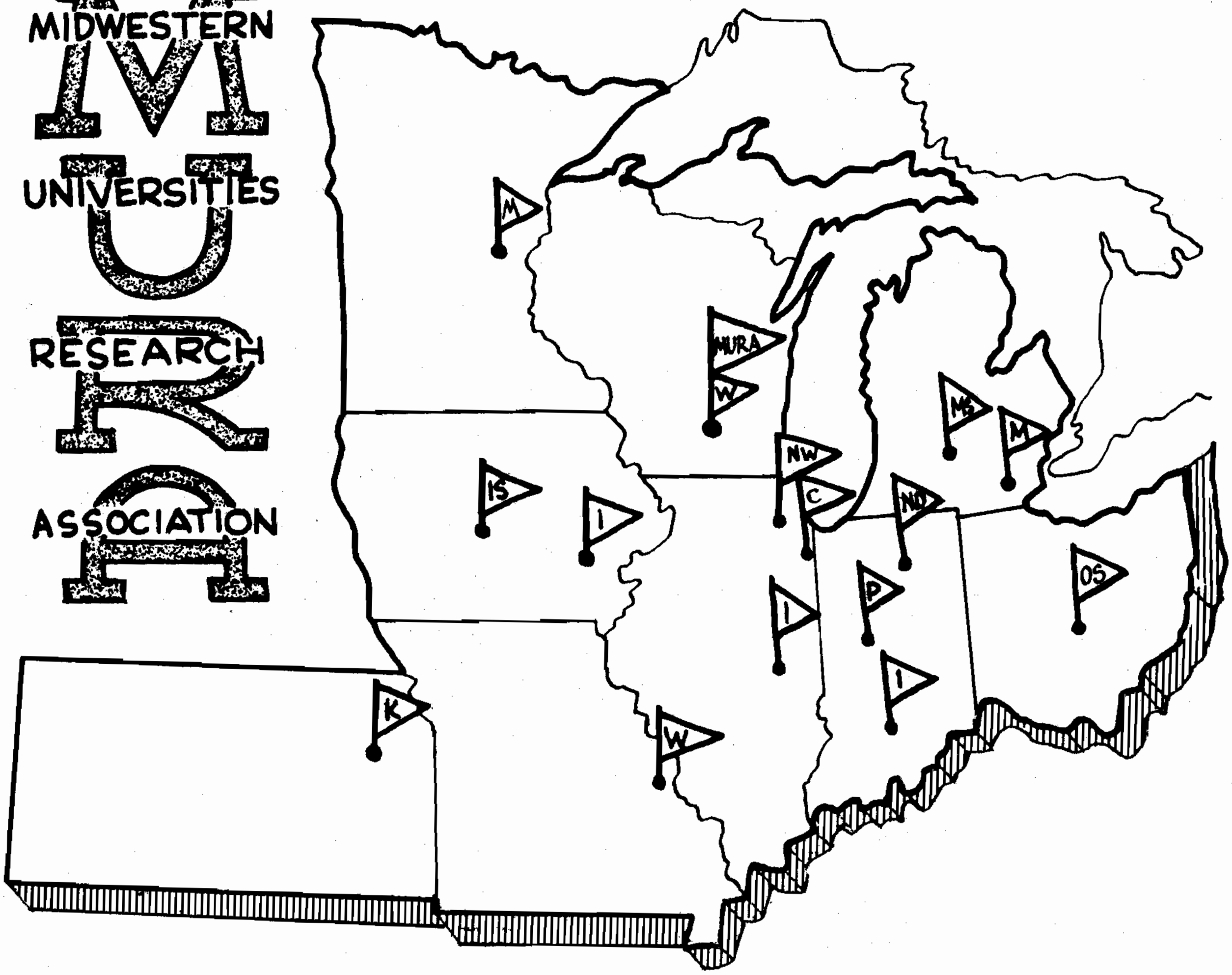

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RELAXATION CALCULATION OF INTEGRAL SCALING MAGNETIC FIELDS PRODUCED BY DISTRIBUTED CURRENTS

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October 17, 1960

ABSTRACT

Algorisms are developed for the solution of an integral scaling magnetostatic problem. The particular problem adapted to relaxation techniques consists of distributed pole-face windings located in spiral grooves periodically connected in radial gaps to form complete current loops. The potentials on the poles thus excited are not graded differentially across the pole face.

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I. INTRODUCTION

Laslett[1] has developed algorisms for the determination of magnetostatic scalar potentials characterizing a scaling spiral sector FFAG magnetic guide field. In addition, Laslett[2] has formulated a modification of this algorism for two-dimensional currents. In a previous paper, this last modification has been generalized to handle three-dimensional scaling currents. The present paper extends both the development of the algorism and its modification to handle the case of three-dimensional integral scaling currents.

II. STANDARD ALGORISM

The magnetostatic problem under consideration requires the solution of Poisson's equation in the coordinates ξ, η, φ where

$$\xi = \frac{1}{w} \ln \frac{r}{r_0} - N\theta \; , \quad \eta = \frac{z}{r} \; , \quad \varphi = \theta . \tag{1}$$

In order to permit easy comparison with the scaling problem solved by the FOROCYL program, the following variables are used:

$$X = \frac{1}{2\pi} \xi \; , \quad Y = \frac{\sqrt{\frac{1}{w^2} + N^2}}{2\pi} \eta \; , \quad Z = \frac{M}{2\pi} \varphi . \tag{2}$$

The differential equation in these coordinates becomes:

$$\frac{\partial^2 U}{\partial X^2} + \left(1 + \frac{4\pi^2}{\mathcal{D}_o} Y^2\right)\frac{\partial^2 U}{\partial Y^2} + \frac{M^2}{\mathcal{D}_o}\frac{\partial^2 U}{\partial Z^2}$$

$$- \frac{4\pi}{w\mathcal{D}_o} Y \frac{\partial^2 U}{\partial X \partial Y} - \frac{2MN}{\mathcal{D}_o}\frac{\partial^2 U}{\partial X \partial Z} + \frac{4\pi^2}{\mathcal{D}_o} Y \frac{\partial U}{\partial Y}$$

$$= -\frac{4\pi^2}{\mathcal{D}_o}\begin{cases} \dfrac{V_o}{(X_1 - X_o)(Y_2 - Y_1)} Y\left[X_1 - X + \dfrac{1}{2\pi w}\right] & \text{in I} \\ \dfrac{V_o}{(Z_1 - Z_o)(Y_2 - Y_1)} Y\left[Z_1 - Z\right] & \text{in II} \\ \dfrac{V_o}{(X_1 - X_o)(Y_2 - Y_1)} Y\left[X_1 + X - \dfrac{1}{2\pi w}\right] & \text{in III} \\ \dfrac{V_o}{(Z_1 - Z_o)(Y_2 - Y_1)} Y\left[Z_1 + Z\right] & \text{in IV}, \end{cases} \tag{3}$$

where the quantity $\mathcal{D}_o$ and the regions I, II, III, and IV have been described previously.[3] The approximation of Eq. (3) by a difference equation makes use of the three-mesh dimensions h, ℓ, p such that

$$X = hi, \quad Y = \ell j, \quad \text{and} \quad Z = pk, \tag{4}$$

where i, j, and k are integers.

If the point (o, o, o) is considered to be the standard point, then a second-order approximation to the potential is

$$U_{ijk} = U_{ooo} + ihU_X + j\ell U_Y + kpU_Z$$
$$+ \frac{1}{2!}\left(i^2h^2U_{XX} + j^2\ell^2U_{YY} + k^2p^2U_{ZZ} + 2ijh\ell U_{XY} + 2ikhpU_{XZ} + 2jk\ell pU_{YZ}\right), \tag{5}$$

where the subscript letters indicate the nature of the derivatives.

An algorism for the determination of U_{000} may be found from Eq. (5) by multiplying by a weighting factor N_{ijk} and summing.

$$\begin{aligned}\sum N_{ijk} U_{ijk} = U_{000} \sum N_{ijk} + h U_X \sum i N_{ijk} + \ell U_Y \sum j N_{ijk} + p U_Z \sum k N_{ijk} \\ + \tfrac{1}{2} h^2 U_{XX} \sum i^2 N_{ijk} + \tfrac{1}{2} \ell^2 U_{YY} \sum j^2 N_{ijk} + \tfrac{1}{2} p^2 U_{ZZ} \sum k^2 N_{ijk} \\ + h\ell U_{XY} \sum ij N_{ijk} + hp U_{XZ} \sum ik N_{ijk} + \ell p U_{YZ} \sum jk N_{ijk} \, .\end{aligned} \tag{6}$$

The weights are then arranged so that the use of Eq. (3) eliminates all the derivative terms. Thus, if the coefficients of the derivative terms are each made proportional to the corresponding coefficients in Eq. (3)

$$\sum i N_{ijk} = 0 \, , \tag{7}$$

$$\ell \sum j N_{ijk} = \frac{4\pi^2 \alpha}{D_0} Y \, , \tag{8}$$

$$\sum k N_{ijk} = 0 \, , \tag{9}$$

$$\tfrac{1}{2} h^2 \sum i^2 N_{ijk} = \alpha \, , \tag{10}$$

$$\tfrac{1}{2} \ell^2 \sum j^2 N_{ijk} = \alpha \left(1 + \frac{4\pi^2}{D_0} Y^2 \right) , \tag{11}$$

$$\tfrac{1}{2} p^2 \sum k^2 N_{ijk} = \frac{M^2}{D_0} \alpha \, , \tag{12}$$

$$h\ell \sum ij N_{ijk} = - \frac{4\pi\alpha}{w D_0} Y \, , \tag{13}$$

$$hp \sum ik N_{ijk} = - \frac{2MN}{D_0} \alpha \, , \tag{14}$$

and

$$\sum jk\, N_{ijk} = 0 \,. \tag{15}$$

There are 26 values of the number triples (i, j , k) for the integers ranging from - 1 to 1 if the value (o, o, o) is omitted. Equations (7 - 15) supply nine relations. It follows then that the determination of the algorism is not unique. To permit comparison with the two-dimensional grid algorism of the FOROCYL program as modified previously,[4] the following distribution of weights is adopted.

$$N_{100} = \frac{1}{2} r^2 = N_{-100} \,, \tag{16}$$

$$N_{010} = \frac{1}{2}\left(1 + \frac{4\pi^2}{D_o} Y^2\right) + \frac{\pi^2 l}{D_o} Y \,, \tag{17}$$

$$N_{0-10} = \frac{1}{2}\left(1 + \frac{4\pi^2}{D_o} Y^2\right) - \frac{\pi^2 l}{D_o} Y \,, \tag{18}$$

$$N_{001} = \frac{M^2 s^2}{2 D_o} = N_{00-1} \,, \tag{19}$$

$$N_{110} = -\frac{\pi r}{2 w D_o} Y = N_{-1-10} = -N_{1-10} = -N_{-110} \,, \tag{20}$$

and

$$N_{101} = -\frac{M N r s}{4 D_o} = N_{-10-1} = -N_{10-1} = -N_{-101} \,, \tag{21}$$

where

$$r = \frac{l}{h} \,, \quad \text{and} \quad s = \frac{l}{p} \,. \tag{22}$$

All other values of N_{ijk} are taken to be zero. Equations (16 - 22) satisfy Eqs. (7 - 15) identically if $\alpha = \frac{\ell^2}{2}$. Furthermore, if $S = 0$, the weights reduce to those used previously[4] for the case in which the differential scaling parameter k is set equal to - 1. If

$$\mathcal{D} \equiv \sum N_{ijk} = 1 + r^2 + \frac{M^2 s^2}{\mathcal{D}_0} + \frac{4\pi^2}{\mathcal{D}_0} Y^2, \tag{23}$$

Eq. (6) gives for the standard 14-point algorism

$$\mathcal{D} U_{000} = \sum N_{ijk} U_{ijk} + \mathcal{E}, \tag{24}$$

where

$$\mathcal{E} = \frac{2\pi^2 \ell^2}{\mathcal{D}_0} \begin{cases} \frac{V_0}{WH} J\left[I_4 - I + \frac{1}{2\pi h w}\right] & \text{in I} \\ \frac{V_0}{BH} J\left[K_4 - K\right] & \text{in II} \\ \frac{V_0}{WH} J\left[I - I_1 - \frac{1}{2\pi h w}\right] & \text{in III} \\ \frac{V_0}{BH} J\left[K - K_1\right] & \text{in IV}. \end{cases} \tag{25}$$

In Eq. (25) the notation

$$X = (I - I_0) h, \tag{26}$$

$$Y = J\ell, \tag{27}$$

$$Z = (K - K_0) p, \tag{28}$$

$$W = I_2 - I_1 = I_4 - I_3 \,, \tag{29}$$

$$H = J_2 - J_1 \,, \tag{30}$$

and

$$B = K_2 - K_1 = K_4 - K_3 \tag{31}$$

is used. Figure 1 illustrates the new origin of coordinates used in Eqs. (26 - 31).

In the new coordinate system the boundary conditions[3] become

$$U(I_2JK) = U(I_3JK) = U(IJK_2) = U(IJK_3) = V_0 \frac{J_2 - J}{H} \,, \tag{32}$$

and

$$U(I_1JK) = U(I_4JK) = U(IJ_2K) = 0 \tag{33}$$

at the copper-iron interfaces. Likewise, at the copper-air interfaces

$$V(I_1JK) - U(I_1JK) = V(I_4JK) - U(I_4JK) = 0 \,, \tag{34}$$

$$V(IJK_1) - U(IJK_1) = V(IJK_4) - U(IJK_4) = 0 \,, \tag{35}$$

and

$$V(IJ_1K) - U(IJ_1K) = V(IJ_2K) - U(IJ_2K) = 0 \,. \tag{36}$$

At the interfaces I-II, II-III, III-IV, and IV-I both the potential function and its normal derivative are continuous.

III. CURRENT VALUES

The only remaining discontinuities that occur are discontinuities in the normal components of $\vec{H}$ at the copper-air interfaces characterized by $J = J_1$ and $J = J_2$. If the difference between the function V extrapolated into the copper region and U in the same region is designated by

$$\Delta = V - U , \tag{37}$$

then the value of Δ at the point (i , 1 , k) is

$$\Delta_{i1k} = \ell \Delta_Y + \tfrac{1}{2}\ell^2 \Delta_{YY} + ih\ell \Delta_{XY} + k\ell p \Delta_{XZ} , \tag{38}$$

and at the point (i , -1 , k) is

$$\Delta_{i-1k} = -\ell \Delta_Y + \tfrac{1}{2}\ell^2 \Delta_{YY} - ih\ell \Delta_{XY} - k\ell p \Delta_{YZ} . \tag{39}$$

In general, since $\mathcal{E}$ is zero in the air, Eq. (24) gives for the standard algorism

$$\mathcal{D} V_{000} = \sum N_{ijk} V_{ijk} . \tag{40}$$

The correction to Eq. (40) required on the cones $J = J_1$ and $J = J_2$ is

$$V_{000} = \left[\text{Std. Algorism}\right] + \left[CV\right] , \tag{41}$$

where

$$\mathcal{D}[CV] = \sum N_{i1k} \Delta_{i1k} \tag{42}$$

for $J = J_1$ and

$$\mathcal{D}[CV] = \sum N_{i-1k} \Delta_{i-1k} \tag{43}$$

for $J = J_2$. Equations (16 - 22) may be used to reduce Eqs. (42) and (43) to

$$\mathcal{D}[CV] = N_{010}\,\Delta_{010} + N_{110}\left(\Delta_{110} - \Delta_{-110}\right) \tag{44}$$

for $J = J_1$ and

$$\mathcal{D}[CV] = N_{0-10}\,\Delta_{0-10} + N_{110}\left(\Delta_{-1-10} - \Delta_{1-10}\right) \tag{45}$$

for $J = J_2$.

Equation (56) of MURA-583 gives the discontinuity in the η or Υ derivative of the potential. From this and the differential equation in Eq. (3), it is possible to evaluate the various Δ's in Eqs. (45) and (46). Thus,

$$\Delta_{010} = \begin{Bmatrix} \dfrac{V_0}{WH}\left(I_4 - I\right) \\ \dfrac{V_0}{BH}\left(K_4 - K\right) \\ \dfrac{V_0}{WH}\left(I - I_1\right) \\ \dfrac{V_0}{BH}\left(K - K_1\right) \end{Bmatrix} + \frac{1}{2}\,\frac{\ell^2 J_1 V_0}{WH\left[\dfrac{\mathcal{D}_0^2}{4\pi^2} + \ell^2 J_1^2\right]} \begin{Bmatrix} -\dfrac{1}{2\pi h w} \\ 0 \\ \dfrac{1}{2\pi h w} \\ 0 \end{Bmatrix} \tag{46}$$

$$\Delta_{110} - \Delta_{-110} = 2 \begin{Bmatrix} -\dfrac{V_0}{WH} \\ 0 \\ \dfrac{V_0}{WH} \\ 0 \end{Bmatrix}, \tag{47}$$

$$\Delta_{0-10} = -\begin{Bmatrix} \dfrac{V_0}{WH}(I_4 - I) \\[2ex] \dfrac{V_0}{BH}(K_4 - K) \\[2ex] \dfrac{V_0}{WH}(I - I_1) \\[2ex] \dfrac{V_0}{BH}(K - K_1) \end{Bmatrix} + \frac{1}{2}\,\frac{\ell^2 J_2 V_0}{WH\left[\dfrac{\mathcal{D}_e^2}{4\pi^2} + \ell^2 J_2^2\right]} \begin{Bmatrix} -\dfrac{1}{2\pi h w} \\[2ex] 0 \\[2ex] \dfrac{1}{2\pi h w} \\[2ex] 0 \end{Bmatrix} \tag{48}$$

and

$$\Delta_{-1-10} - \Delta_{1-10} = 2\begin{Bmatrix} -\dfrac{V_0}{WH} \\[2ex] 0 \\[2ex] \dfrac{V_0}{WH} \\[2ex] 0 \end{Bmatrix}, \tag{49}$$

where the inserts in the column symbols from top to bottom refer to the regions I, II, III, and IV.

The solutions of the boundary value problem may be completed by noting that the integral scaling boundary conditions at the extremities of the unit cell are

$$V(IJK_\delta) = e^{2\pi(k+1)w\frac{N}{M}}\,V(IJ0), \tag{50}$$

and

$$V(I_\delta JK) = e^{2\pi(k+1)w}\,V(0JK). \tag{51}$$

Finally V is to be made an odd function of J.

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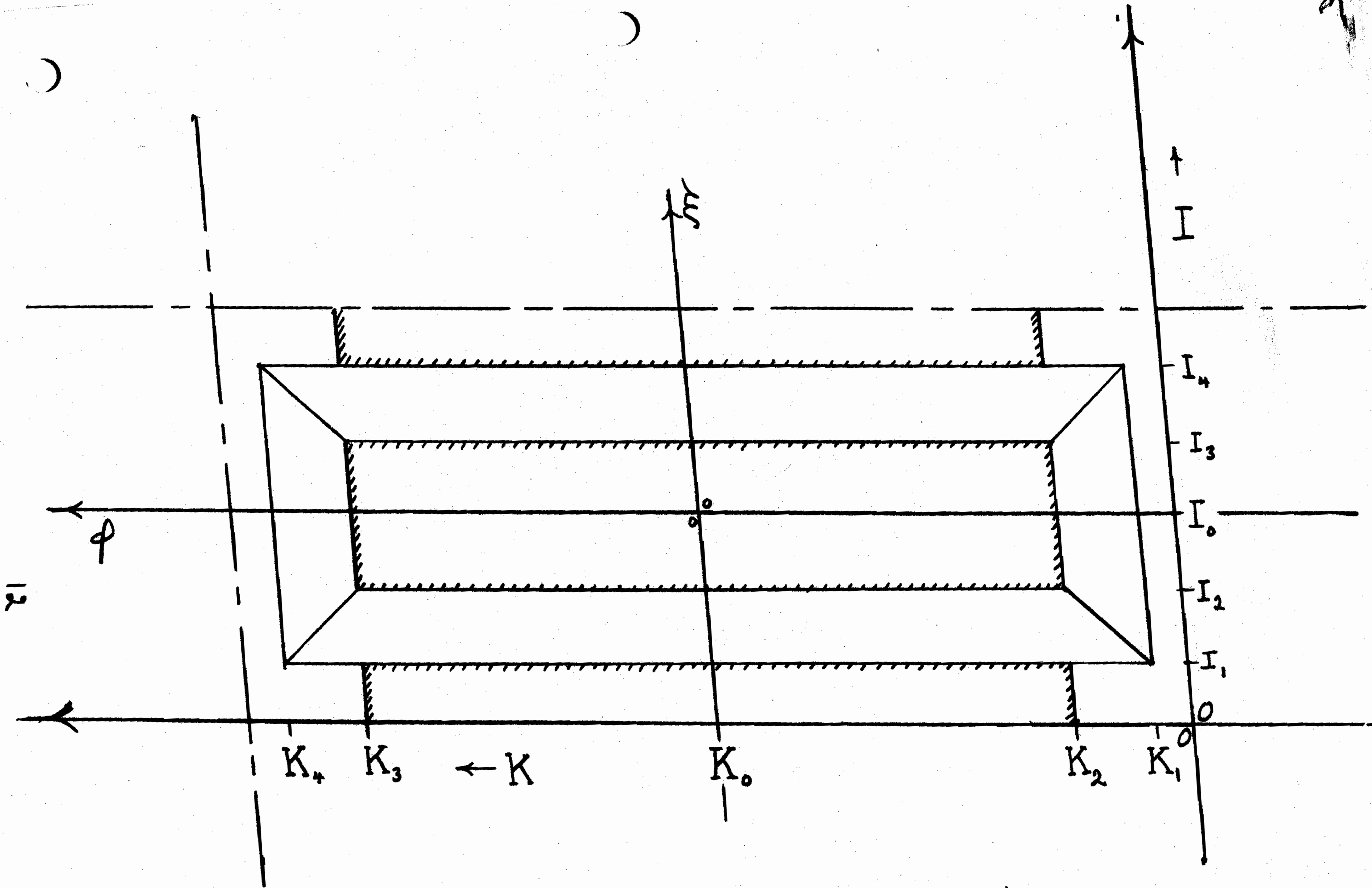

FIG.1 POLE FACE UNIT CELL