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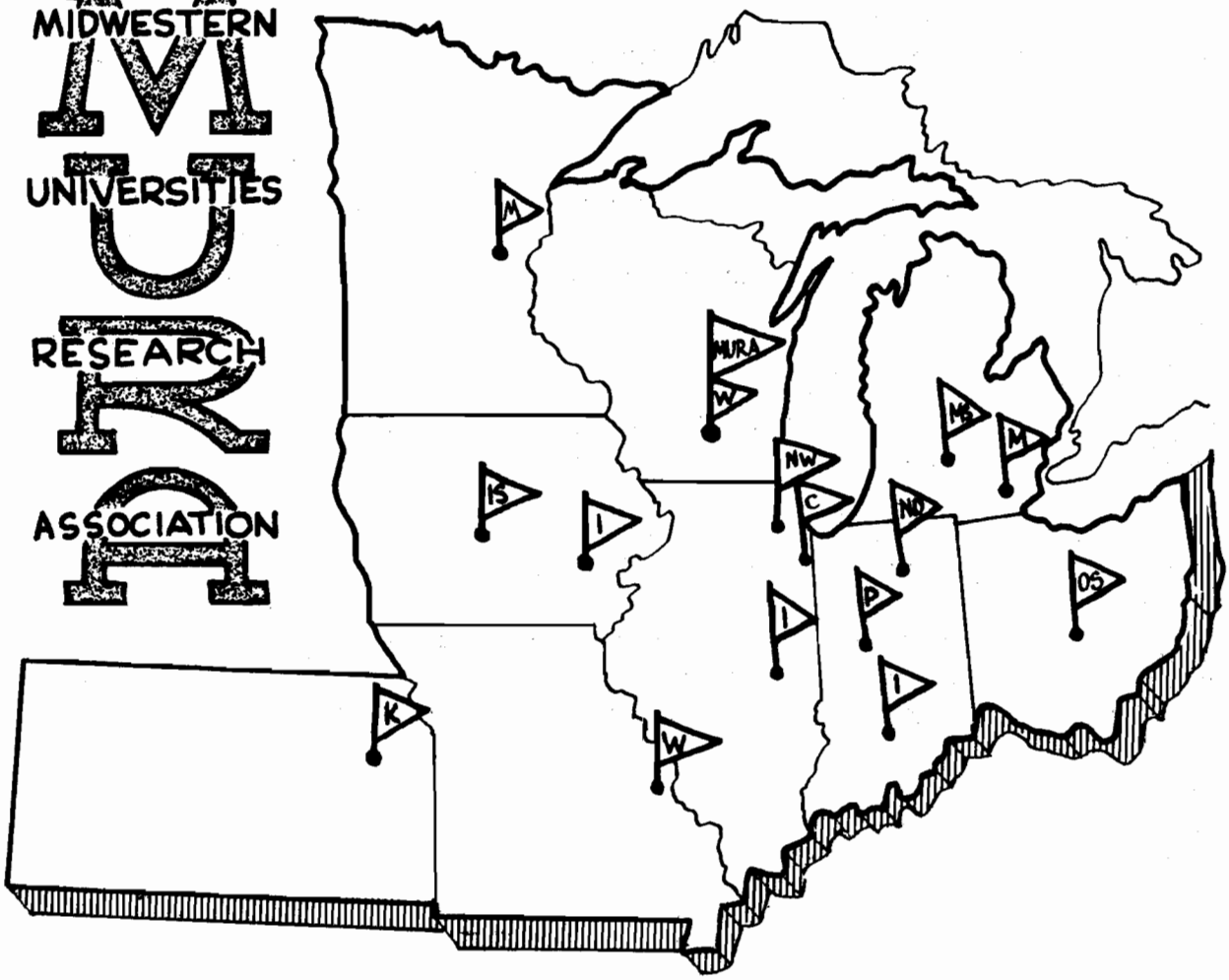

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REPORT **NUMBER** 589

MURA-589
UC-28, Particle Accelerator and
High Voltage Machines
TID-4500 (15th Edition)

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LINEAR OSCILLATIONS ABOUT OFF-CENTER, FIXED POINT ORBITS

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ABSTRACT

In the neighborhood of a nonlinear resonance there exist off-center, closed orbits, representing either stable or unstable fixed points in the phase space diagram. A calculation is given for the linear tune for such orbits for both the $\nu = N/3$ and $N/4$ resonances. The variational method for calculating fixed point orbits directly from the primary equations of motion is discussed.

*AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission, Contract No. AT(11-1)-384.

I. INTRODUCTION

Nonlinear terms in the equations of motion for both AG and FFAG accelerators give rise to nonlinear resonances and the existence of stable and unstable fixed points in the phase space diagram. These fixed points represent off-center closed orbits; that is, there exist periodic solutions of the equations of motion other than the (centered) equilibrium orbit. Just as in the case for the equilibrium orbit it is possible to expand the equations of motion about an off-center fixed point orbit and thereby determine the properties of the linear motion about such an orbit. For a stable fixed point orbit the linear tune will be real, while for an unstable fixed point orbit this tune will be imaginary. We shall present here a simple calculation of the linear tune about both the stable and unstable fixed point orbits in the vicinity of the $\nu = N/3$ (Sect. II) and $\nu = N/4$ (Sect. III) nonlinear resonances.

The calculation of the tune associated with an off-center fixed point orbit can be carried out in the same way that one would calculate the tune associated with the equilibrium (centered) orbit. The problem can be dealt with without using the formalism usually employed in treating nonlinear resonances. The first step is to calculate the periodic solution of the equations of motion corresponding to the fixed point orbit; the next step is to expand the equations of motion to first order about this orbit. The resulting linear equation will be in the form of the general Mathieu-Hill equation and the tune can then be determined. The accuracy of the results can be the same as that obtainable for the comparable case of equilibrium orbits.

In the present note we shall give only a very approximate solution of the problem. The equation of motion we start from is for the oscillations about the equilibrium orbit and even this equation we take in a particularly simple form. The solutions we use for the fixed point orbits are first approximations valid only if these orbits are but slightly off center.

In the last part (Sect. IV) of this note we discuss the determination of fixed point orbits by a variational calculation on the primary equations of motion. This method depends on the fact that fixed point orbits are periodic solutions so that the averaged Lagrangian is stationary.

II. $\nu = N/3$ RESONANCE

For simplicity we shall assume that a transformation in the independent variable θ is made such that $N\theta \longrightarrow 3\,\theta$ so that the tune ν about the equilibrium orbit is then close to unity. We assume the following equation for the oscillations about the equilibrium orbit:

$$x'' + \nu^2 x = C_o x^2 \cos 3\,\theta + C_1 x^3 \,, \tag{2.1}$$

where we have stripped this equation to the bare minimum in order to simplify matters. We assume that the point $\theta = 0$ is so chosen that $C_o > 0$; we assume further that $C_1 > 0$ with no significant loss of generality since reversing the sign of C_1 has the effect of reversing results above ($\nu > 1$) and below ($\nu < 1$) resonance. We write

$$\nu^2 = 1 + 2\,\alpha \tag{2.2}$$

which defines the parameter α ; we assume $|\alpha| \ll 1$, so that the unstable fixed point orbits are only slightly off center. Let $x_o = x_o(\theta)$ be the periodic

solution of (2.1) representing a given fixed point orbit; we then make the substitution $x = x_o + u$ in (2.1) and expand this equation keeping only linear terms in u with the following result:

$$u'' + g(\theta)\, u = 0, \tag{2.3}$$

for the linear oscillations about the fixed point orbit, where

$$g = 1 + 2\alpha - 2 C_o x_o \cos 3\theta - 3 C_1 x_o^2 . \tag{2.4}$$

It now remains to evaluate x_o and hence $g(\theta)$.

To obtain an approximate solution for x_o we assume

$$x_o = A \cos(\theta + \phi) \tag{2.5}$$

where A and ϕ are constants. Using the variational method on Eq. (2.1) with this trial function, we find $\phi = \pm n (60^o)$ with $n = 0, 1, 2, 3$. Above resonance ($\alpha > 0$) the three unstable fixed points have $\phi = 0, \pm 120^o$; below resonance ($\alpha < 0$) they have $\phi = \pm 60^o, 180^o$. The stable fixed points, both above and below resonance, have $\phi = \pm 60^o, 180^o$. The value of A for these fixed point orbits is given by the following:

$$\beta = \frac{2}{b}\left[1 \pm \sqrt{1 + 4\alpha b}\right] \tag{2.6}$$

where β and b are the following dimensionless constants:

$$\beta = -\frac{1}{2} C_o A \cos 3\phi ; \quad b = \frac{24 C_1}{C_o^2} . \tag{2.7}$$

In the equation (2.6) for β the (+) sign applies to the stable fixed points; the (-) sign tothe unstable fixed points. This approximate result for β is valid only if $\beta \ll 1$, that is, the fixed point orbits are not far off center. For the stable fixed points this requires that b be a large number; while for the un-

stable fixed points, $\alpha << 1$, regardless of the magnitude of b. Note that below resonance ($\alpha < 0$) these fixed points disappear (by coalescence) when $4\alpha b \leqq -1$.

If we insert the above expression for $x_o(\theta)$ into (2.4) for $g(\theta)$, we find:

$$g = 1 - 2\alpha - \beta + (\beta - 4\alpha)\cos 2\theta + 2\beta \cos 4\theta ; \tag{2.8}$$

with β and $\alpha << 1$, the value of the tune will be close to or in the stop-band. To convert this expression for $g(\theta)$ into an expression for the corresponding tune, we shall use a simple formula given by G. Parzen.[1] If $g(\theta)$ is given by

$$g = 1 - F - 2G\cos 2\theta,$$

then Parzen's formula for the tune ν is as follows:

$$\nu^2 = 1 - (F^2 - G^2)^{1/2} ; \tag{2.9}$$

which reduces to the "smooth approximation" under appropriate conditions, and also applies in the vicinity of the stop-band with an error of order G^2.

Inserting now the expression (2.8) for g into (2.9) for ν^2, we obtain

$$\nu^2 = 1 - \left(\frac{3}{4}\beta^2 + 6\alpha\beta\right)^{1/2} ; \tag{2.10}$$

where terms of order β^2 or α^2 are neglected. Inserting into this expression the value of β from (2.6) for the stable fixed point orbits, we then find for the corresponding tune ν_s about this orbit

$$\nu_s = 1 - \frac{\sqrt{6}}{2b}\left[(1 + 4\alpha b) + (1 + 2\alpha b)\sqrt{1 + 4\alpha b}\right]^{1/2} ; \tag{2.11}$$

noting that even though $\alpha << 1$, b may be so large that (αb) is comparable to or even greater than unity. If we consider this expression for a fixed value of b, and α decreasing from a positive value, through resonance, to negative values, then we see that ν_s increases from below unity to a value

$\nu_s(\alpha = 0) = 1 - \sqrt{3}/b$ at resonance; and finally, $\nu_s \equiv 1$, for $4\alpha b = -1$, at which point this fixed point coalesces with an unstable fixed point and disappears. Above resonance ($\alpha > 0$) the rotation of phase space points about the stable fixed points is opposite in direction to the rotation of points near the equilibrium orbit so that it is not surprising that $\nu_s < 1$.

Inserting the value of β from (2.6) for the unstable fixed point orbits into (2.10) we obtain the following expression for the tune ν_u about these orbits:

$$\nu_u = 1 - i\,\frac{2\sqrt{3}\,\alpha\,(1 + 4\alpha b)^{1/4}}{1 + \sqrt{1 + 4\alpha b}} \tag{2.12}$$

so that, as expected, ν_u lies in the stop-band. We see that $\nu_u = 1$ when $\alpha = 0$ (on resonance) at which value the unstable fixed points merge with the equilibrium orbit; and also when $4\alpha b = -1$, at which value, as noted before, the unstable and stable fixed points merge and disappear. In the limit where $4\alpha b \ll 1$, which means either the case where the stable fixed points are at very large displacements, or else $\alpha \to 0$ (near resonance) the value of ν_u becomes $\nu_u = 1 - i\sqrt{3}\,\alpha$.

III. = N/4 RESONANCE

We assume here that a transformation of the independent variable θ is made such that $N\theta \to 4\theta$; as a result the linear tune ν for the equilibrium orbit will be close to unity. To describe the nonlinear oscillations about the equilibrium orbit we assume the following equation:

$$x'' + \nu^2 x = D_o x^3 \cos 4\theta + D_1 x^3 \tag{3.1}$$

where, as before, we shall set $\nu^2 = 1 + 2\alpha$. Using the same justification as before we shall assume both $D_o > 0$, $D_1 > 0$. If $x_o(\theta)$ is the periodic solution of (3.1) for the fixed point orbit, then substituting $x = x_o + u$ into (3.1) and expanding to first order in u, we obtain $u'' + g u = 0$ for the linear oscillations about the given fixed point orbit. In this case we find for $g(\theta)$:

$$g = 1 + 2\alpha - 3 D_o x_o^2 \cos 4\theta - 3 D_1 x_o^2 . \tag{3.2}$$

It now remains to evaluate x_o. As before, we assume $x_o = A \cos(\theta + \phi)$ as a suitable trial function for a variational calculation based on the Lagranian for (3.1).

The resulting possible values of ϕ are $\phi = \pm n (45^o)$, with $n = 0, 1, 2, 3, 4$. The resulting values of A are given by the following formula:

$$A^2 = \frac{16\alpha}{6 D_1 \pm D_o} \tag{3.3}$$

where the (+) sign holds for $\phi = 0, \pm 90^o, 180^o$; and the (-) sign holds for $\phi = \pm 45^o, \pm 135^o$. There are two cases to consider. In the first case, $6 D_1 > D_o$ for which, above resonance ($\alpha > 0$), there are four unstable fixed points at $\phi = 0, \pm 90^o, 180^o$, and four stable fixed points at $\phi = \pm 45^o, \pm 135^o$; while below resonance ($\alpha < 0$), there are no such fixed points at all. In the second case, $6 D_1 < D_o$, for which there are only unstable fixed points having $\phi = 0, \pm 90^o, 180^o$ above resonance (as in the first case) and $\phi = \pm 45^o, \pm 135^o$ below resonance. This approximate solution is valid only if $\alpha \ll 1$ and $|6 D_1 - D_o|$ is not small; that is the fixed point orbits are displaced by only a small amount.

Inserting the above expression for $x_o(\theta)$ into (3.2) for $g(\theta)$, we find

$$g = 1 + 2\alpha - \frac{3}{2} D_1 A^2 + \frac{3}{4} A^2 (2 D_1 \pm D_o) \cos 2\theta , \tag{3.4}$$

where we have omitted 4θ nad 6θ harmonics inasmuch as they contribute a negligible amount to the resultant ν^2 under the assumptions we are using. Applying Parzen's formula (2.9) and inserting A^2 from (3.3), we obtain the following results: for the four unstable fixed points having $\phi = 0, \pm 90^o, 180^o$ above resonance ($\alpha > 0$),

$$\nu_u = 1 - 2 i \alpha \left[\frac{2 D_o}{6 D_1 + D_o} \right]^{1/2} , \tag{3.5}$$

for the four stable fixed points having $\phi = \pm 45^o, \pm 135^o$ which occur when $6 D_1 > D_o$ and $\alpha > 0$,

$$\nu_s = 1 - 2 \alpha \left[\frac{2 D_o}{6 D_1 - D_o} \right]^{1/2} , \tag{3.6}$$

and for the four unstable fixed points having $\phi = \pm 45^o, \pm 135^o$ which occur when $6 D_1 < D_o$ and $\alpha < 0$, we have

$$\nu_u' = 1 - 2 i \alpha \left[\frac{D_o}{D_o - 6 D_1} \right]^{1/2} . \tag{3.7}$$

In the case $6 D_1 > D_o$, as $\alpha \to 0$, all the fixed points merge with the equilibrium orbit and $\nu_s \to \nu_u \to 1$, as expected. Note that $\nu_s < 1$ so that the direction of rotation of phase points around this fixed point is opposite to that for the equilibrium orbit, again as expected. In the particular case where $D_1 \to 0$, we find $\nu_u = \nu_u' = 1 - 2\sqrt{2}\, i \alpha$.

IV. DETERMINATION OF FIXED POINT ORBITS FROM PRIMARY EQUATIONS OF MOTION

The usual method for determining fixed point orbits (and stability limits) involves several steps. First there is the determination of the equilibrium orbit; then the linear oscillation properties about this orbit; then there is the expansion of the equations of motion about the equilibrium orbit maintaining the significant nonlinear terms; it is from this last equation then that the off-center fixed point orbits (and resultant stability limits) are determined. We want to point out here that the off-center fixed point orbits can, of course, be determined directly from the primary equations of motion in a manner directly analogous to that used for determining equilibrium orbits themselves. That is, all fixed point orbits, including the equilibrium orbit, represent periodic solutions of the equations of motion and can be treated on an equal footing. The usual method of expanding about the equilibrium orbit is clearly preferable only if the scalloping of this orbit is large in comparison to the relative displacement of the other fixed point orbits.

The equations of motion can be derived from the Lagrangian $L(r, \dot{r}, \theta)$ given as follows:

$$L = p\,(r^2 + \dot{r}^2)^{1/2} - \int r\,B(r, \theta)\,dr\,. \tag{4.1}$$

For any fixed point orbit r is a periodic function of θ; it follows then that a variational equation for determining such orbits is

$$\delta \langle L \rangle = 0\,; \tag{4.2}$$

where the angular brackets signify an average over θ. To cast this equation

into a more useful form we make the substitution $r = R\,(1 + u)$, where $\langle u \rangle = 0$ so that R is the mean radius of the given fixed point orbit; and we assume that $|u|$ is small compared to unity. For the Lagrangian $L\,(u, \dot{u}, \theta)$ we can then write

$$L = \left[(1 + u)^2 + \dot{u}^2\right]^{1/2} - \int Q\,(u, \theta)\, d\,u \tag{4.3}$$

where

$$Q = r\,B\,(r, \theta)/p$$

and we assume that Q may be expanded into a (rapidly) converging series in u as follows:

$$Q = \sum_{n=0} \frac{u^n}{n!}\; Q^n\,(R, \theta) \tag{4.4}$$

where

$$Q^n\,(R, \theta) = \frac{R^n}{p}\,\frac{d^n}{dR^n}\left[R\,B\,(R, \theta)\right] .$$

It is most convenient to use R rather than p as an independent variable and then determine $p\,(R)$ after calculating u; note that R is not the same, in general, for all fixed point orbits at a given p, which may be somewhat of a nuisance.

Assuming $\dot{u}^2$, as well as u, is small compared to unity, we may then expand $L\,(u, \dot{u}, \theta)$ in powers of u and $\dot{u}^2$ as follows:

$$L = \left[\frac{1}{2}\,\dot{u}^2 - \frac{1}{2}\,u\,\dot{u}^2 + \cdots\right] - \sum_{n=0} \frac{Q^n\,(R, \theta)}{(n + 1)\,!}\; u^{n+1} \quad , \tag{4.5}$$

where those terms which obviously average to a constant have been dropped. The choice of a trial function for u will depend on the character of $B\,(r, \theta)$ and its derivatives; when a suitable trial function is found, it is inserted in L,

which is then averaged, and then inserted into (4.2) to determine the parameters involved.

When a value of u has been determined from the variational equation, one can then calculate $p = p(R)$. This can be done in a number of ways; the one we have found easiest to execute is by use of the following equation:

$$p = \left\langle \sqrt{r^2 + \dot{r}^2}\, B(r, \theta) \right\rangle ,$$

or

$$\left\langle \sqrt{1 + \dot{r}^2/r^2}\, Q(r, \theta) \right\rangle = 1 . \tag{4.6}$$

As noted before, one will obtain in general a different expression $p(R)$ for each fixed point orbit.

For the equilibrium orbit the periodicity of u will be the same as that for $B(r, \theta)$ itself; for an off-center fixed point orbit the periodicity of u will be some subharmonic of this. The accuracy with which fixed point orbits may be determined using such a method depends on the complexity of B, the size of $|u|$, and, of course, on the amount of effort invested.

We illustrate this procedure by a simple example. We choose $B(r, \theta)$ to have the following form:

$$B = B_0(r) + B_1(r) \cos 3\theta$$

and we can then write

$$Q^n(R, \theta) = Q_0^n(R) + Q_1^n(R) \cos 3\theta . \tag{4.7}$$

We shall assume that Q_0^1 is close to unity, that Q_1^2 is large, and that the fixed point orbit can adequately be given by

$$u = A \cos(\theta + \phi) , \tag{4.8}$$

where A is small, but significantly larger than the equilibrium orbit scalloping ($\cong Q_1^o/9$). If we neglect all terms in L of higher order than A^3, then we obtain

$$\langle L \rangle = \frac{1}{4} A^2 (1 - Q_o^1) - \frac{Q_1^2}{48} A^3 \cos 3\phi . \tag{4.9}$$

Taking the variation with respect to ϕ, we obtain $\sin 3\phi = 0$, while, taking the variation with respect to A, we find

$$A \cos 3\phi = \frac{8 (1 - Q_o^1)}{Q_1^2} , \tag{4.10}$$

which determines then the three unstable fixed point orbits. To obtain an implicit relation $p = p(R)$ we insert u into (4.6) and obtain to order A^2,

$$Q_o^o (1 + \frac{1}{4} A^2) + \frac{1}{4} A^2 Q_o^2 = 1 , \tag{4.11}$$

which, for small A gives, as a first approximation, $Q_o^o = 1$, or, $p = R B_o(R)$.

The equation (2.1), used in Sect. II for the nonlinear oscillations about the equilibrium orbit, will yield results comparable to those found above when $C_1 = 0$. If then, $x_o = A \cos(\theta + \phi)$ is the solution to (2.1) for the unstable fixed point orbits, we find there

$$(C_1 = 0)$$

$$A = \frac{4 (\nu^2 - 1)}{C_o}$$

which can be compared with Eq. (4.10). The tune ν, which occurs in (2.1) is the true tune for the oscillations about the equilibrium orbit, including AG effects; that is, in setting up (2.1) we assume that a suitable transformation has already been made to eliminate the explicit appearance of AG terms. We certainly expect that as $\nu \rightarrow 1$, the unstable fixed point orbits must merge

with the equilibrium orbit. The above variational method for calculating the unstable fixed point orbits, however, makes no direct reference to the equilibrium orbit (or ν), and it is not at all apparent that, even if carried out with precision, this method will yield such a result. If we examine the approximate results obtained above, we see that $Q_o^1 \cong 1 + k$, and since $\nu^2 \cong 1 + k$, then the numerator in the expression (4.10) for A is then approximately proportional to $(\nu^2 - 1)$.

The situation where C_1 of Eq. (2.1) is large corresponds above to a large Q_o^3; when this is true, it is then necessary to maintain terms to order A^4 in the calculation of $\langle L \rangle$. The results obtained then will be quite comparable to those found in Sect. II, and will not be discussed further.

The above calculation of u could, of course, be improved by including higher harmonics in the trial function as well as more terms from L. An accurate calculation of the unstable fixed point orbits for the $\nu \rightarrow N/3$ resonance has been given by L. J. Laslett[2] working from an equation similar to (2.1).

REFERENCES

[1]G. Parzen, MURA Report No. 451 (April, 1959).

[2]L. J. Laslett, MURA Report No. 452 (April, 1959).