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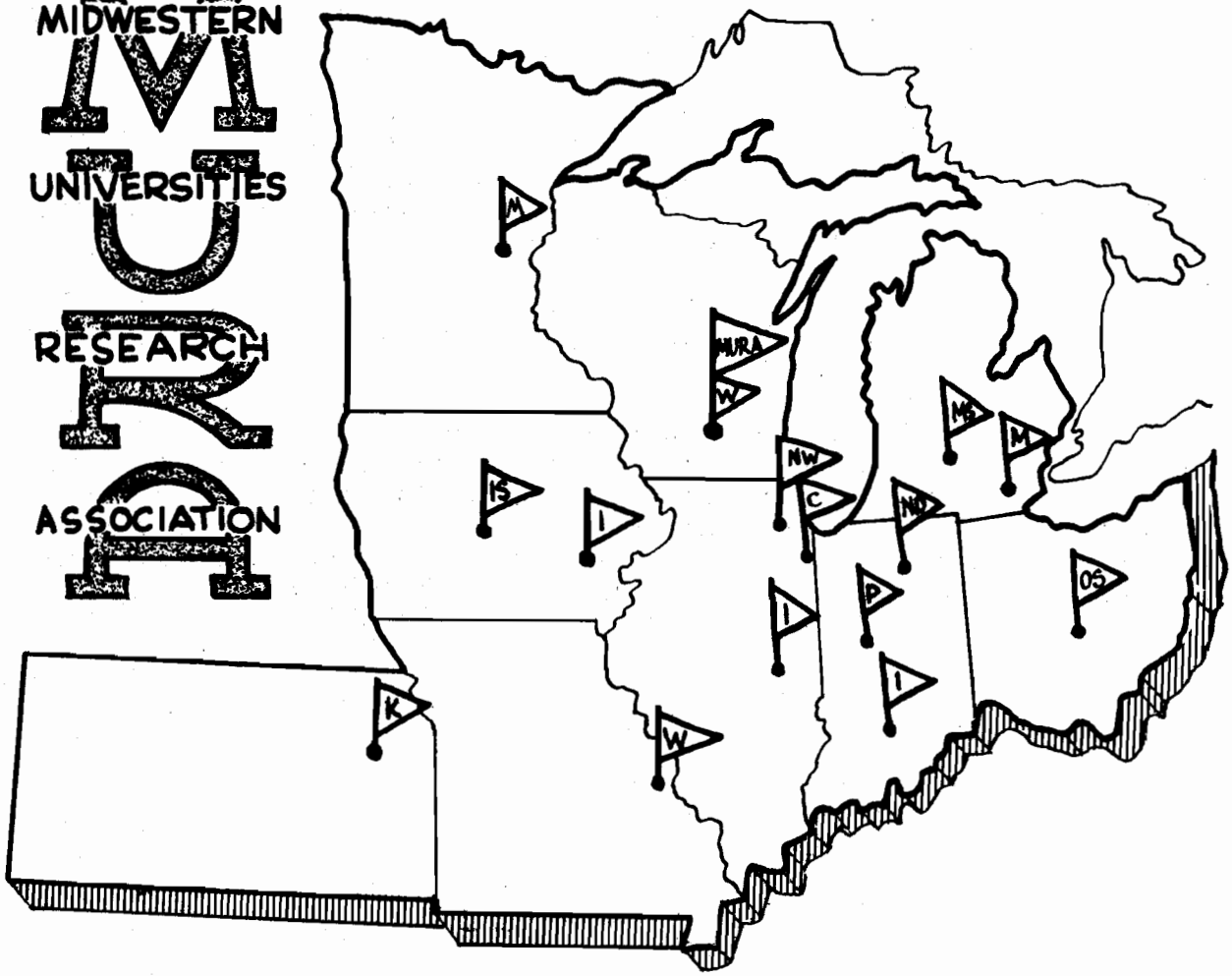

INTEGRAL SCALING MAGNETIC FIELD USING DISTRIBUTED WINDINGS

S. C. Snowdon

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INTEGRAL SCALING MAGNETIC FIELD USING DISTRIBUTED WINDINGS

S. C. Snowdon

August 8, 1960

ABSTRACT

The magnetic field of a spiral sector FFAG magnetic guide possessing integral scaling is calculated with the aid of scalar functions representing the fields and the current sources. In particular, the spiral sector magnet considered possesses equally spaced radial cuts into which RF cavities may be inserted. The distributed pole face windings are located in spiral slots and are connected along the radial cuts.

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I. INTRODUCTION

Since radial cuts must be provided in an FFAG magnetic guide to permit the insertion of rf acceleration cavities, it is of interest to calculate the magnetic field of an integral scaling structure in which the scaling property is maintained periodically but violated locally. This magnetic field will then permit a careful study of the orbit dynamics to ascertain the usefulness of the structure. In the following, a spiral sector structure is used in which the pole face windings are located in spiral slots and are connected periodically along the radial cuts to form complete loops.

II. GENERAL MAGNETOSTATIC PROBLEM

Figure 1. illustrates schematically the general nature of the pole face windings and of the radial cuts. The surface shown is a section of the top pole face where any flat surface imagined actually represents a surface of a cone. The pole face windings are not necessarily flush with the iron surface. Similarly the zero potential poles between the excited poles may lie on different conical surfaces. All windings, however, lie between two distinct cones. The nature of the integral scaling contemplated is such that the field under any cell is known once it has been found within one cell. A typical unit cell is given in Fig. 2, in order to show the coordinates employed. Since the field is to possess scaling properties on the average, the coordinates employed are those used in similar calculations of differential scaling problems.[1,2]

The method of scalar functions employed previously is again used here, for the magnetic field

$$\vec{H} = 4\pi\vec{\mathcal{L}}\tau + \nabla\times\vec{\mathcal{L}}U, \tag{1}$$

and for the current density

$$\vec{J} = \vec{\mathcal{L}}\sigma + \nabla\times\vec{\mathcal{L}}\tau, \tag{2}$$

where

$$\nabla^2 U = -4\pi\sigma. \tag{3}$$

The operator $\vec{\mathcal{L}}$ is taken to be

$$\vec{\mathcal{L}} = \vec{k}\times\nabla, \tag{4}$$

where $\vec{k}$ is the unit vector along the cylindrical coordinate z-axis. In this case, Eq. (1) has the form

$$\vec{H} = 4\pi\vec{\mathcal{L}}\tau - 4\pi\vec{k}\sigma - \nabla\frac{\partial U}{\partial z} \tag{5}$$

from which it is natural to choose

$$W = \frac{\partial U}{\partial z} \tag{6}$$

as the potential function. Since, in the free space region, W reduces to the magnetostatic potential.

The coordinates useful in scaling problems are

$$\xi = \frac{1}{w}\ln\frac{r}{r_0} - N\theta\ , \quad \varphi = \theta\ , \quad \eta = \frac{z}{r}\ , \tag{7}$$

where r, θ, z are the cylindrical coordinates and the z-axis is the axis of the magnet. Thus

$$r\frac{\partial}{\partial r} = \frac{1}{w}\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\ , \tag{8}$$

$$\frac{\partial}{\partial\theta} = -N\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\varphi}\ , \tag{9}$$

and

$$r\frac{\partial}{\partial z} = \frac{\partial}{\partial \eta} \, . \tag{10}$$

The operator $\vec{\mathcal{L}}$ of Eq. (4) becomes

$$\vec{\mathcal{L}} = \frac{1}{r_0} e^{-w(\xi + N\varphi)} \left[\vec{i}_r \left(N\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \varphi} \right) + \vec{i}_\theta \left(\frac{1}{w}\frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right) \right] , \tag{11}$$

and

$$r^2 \mathcal{L}^2 = \left(\frac{1}{w^2} + N^2 \right) \frac{\partial^2}{\partial \xi^2} - \frac{2\eta}{w} \frac{\partial^2}{\partial \xi \partial \eta} - 2N \frac{\partial^2}{\partial \xi \partial \varphi} + \frac{\partial^2}{\partial \varphi^2} + \eta \frac{\partial}{\partial \eta} \left(\eta \frac{\partial}{\partial \eta} \right) . \tag{12}$$

In addition

$$r^2 \nabla \times \vec{\mathcal{L}} = -\vec{i}_r \left[\frac{1}{w} \frac{\partial^2}{\partial \xi \partial \eta} - \frac{\partial}{\partial \eta} \left(\eta \frac{\partial}{\partial \eta} \right) \right] + \vec{i}_\theta \left[N \frac{\partial^2}{\partial \xi \partial \eta} - \frac{\partial^2}{\partial \varphi \partial \eta} \right] + \vec{k} \, r^2 \mathcal{L}^2 . \tag{13}$$

As in the case of the previous differential scaling problems,[1,2] sufficient flexibility in the expression for the current density is obtained if the source densities σ and τ are restricted to

$$\sigma = r_0 e^{-w(\xi + N\varphi)} f(\xi \varphi) , \tag{14}$$

and

$$\tau = r_0^2 \eta \, g(\xi \varphi) , \tag{15}$$

where it will be noted that the constant k has been set equal to -1 indicating the absence of potential grading on the pole face. If, in addition, the current in the slots is restricted to flow in the conical surfaces,

$$J_z = \eta J_r . \tag{16}$$

Equation (16), using Eqs. (11) and (13) for the operators and Eqs. (14) and (15) for the source densities, results in the following restriction between the functions f and g.

$$N\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \varphi} + \frac{1}{w}\frac{\partial g}{\partial \xi} - \left(\frac{1}{w^2} + N^2 \right) \frac{\partial^2 g}{\partial \xi^2} - \frac{\partial^2 g}{\partial \varphi^2} + 2N \frac{\partial^2 g}{\partial \xi \partial \varphi} = 0 . \tag{17}$$

The current density, under the conditions imposed, becomes

$$\vec{J} = e^{-2w(\xi+N\varphi)}\Big\{(\vec{i}_r+\eta\vec{k})\Big[N\frac{\partial f}{\partial\xi}-\frac{\partial f}{\partial\varphi}+g-\frac{1}{w}\frac{\partial g}{\partial\xi}\Big] + \vec{i}_\theta\Big[-f+\frac{1}{w}\frac{\partial f}{\partial\xi}+N\frac{\partial g}{\partial\xi}-\frac{\partial g}{\partial\varphi}\Big]\Big\}, \tag{18}$$

or

$$\vec{J} = e^{-2w(\xi+N\varphi)}\Big\{(\vec{i}_r+\frac{1}{wN}\vec{i}_\theta+\eta\vec{k})\Big[N\frac{\partial f}{\partial\xi}-\frac{\partial f}{\partial\varphi}-\frac{1}{w}\frac{\partial g}{\partial\xi}+g\Big] - \vec{i}_\theta\Big[f+\frac{g}{wN}-\frac{1}{N}\Big(\frac{1}{w^2}+N^2\Big)\frac{\partial g}{\partial\xi}-\frac{1}{wN}\frac{\partial f}{\partial\varphi}+\frac{\partial g}{\partial\varphi}\Big]\Big\}. \tag{19}$$

A degree of flexibility in the formulation of the magnetostatic problem is introduced if the potential W is written as

$$W = U - C, \tag{20}$$

where the U has no direct relation with the U in Eq. (1). The function C may be chosen to simplify the matching of the tangential components of $\vec{H}$ at the copper-air and copper-iron interfaces.[3] Equation (5) becomes

$$\vec{H} = \frac{1}{r_0}e^{-w(\xi+N\varphi)}\Big\{\vec{i}_r\Big[4\pi r_0^2\eta\Big(N\frac{\partial g}{\partial\xi}-\frac{\partial g}{\partial\varphi}\Big)+\frac{1}{w}\frac{\partial C}{\partial\xi}-\eta\frac{\partial C}{\partial\eta}-\frac{1}{w}\frac{\partial U}{\partial\xi}+\eta\frac{\partial U}{\partial\eta}\Big] + \vec{i}_\theta\Big[4\pi r_0^2\eta\Big(-g+\frac{1}{w}\frac{\partial g}{\partial\xi}\Big)-N\frac{\partial C}{\partial\xi}+\frac{\partial C}{\partial\varphi}+N\frac{\partial U}{\partial\xi}-\frac{\partial U}{\partial\varphi}\Big] - \vec{k}\Big[4\pi r_0^2 f-\frac{\partial C}{\partial\eta}+\frac{\partial U}{\partial\eta}\Big]\Big\}, \tag{21}$$

where U, from Eqs. (3), (6), (14), and (20), satisfies

$$\mathcal{M}U = \mathcal{M}C. \tag{22}$$

In Eq. (22) the function C is assumed to be given and the linear operator $\mathcal{M}$ is

$$\mathcal{M} = \eta\frac{\partial}{\partial\eta} - \frac{2\eta}{w}\frac{\partial^2}{\partial\xi\partial\eta} + \left(\frac{1}{w^2}+N^2\right)\frac{\partial^2}{\partial\xi^2} + (1+\eta^2)\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\varphi^2} - 2N\frac{\partial^2}{\partial\xi\partial\varphi} . \tag{23}$$

It is desirable to select the functions f and g in order that the current flow in the spiral slot is along the direction

$$\vec{n} = \frac{1}{\sqrt{1+\left(\frac{1}{wN}\right)^2+\eta^2}}\left(\vec{i}_r + \frac{1}{wN}\vec{i}_\theta + \eta\vec{k}\right), \tag{24}$$

and in the radial returns along the direction

$$\vec{m} = \frac{1}{\sqrt{1+\eta^2}}\left(\vec{i}_r + \eta\vec{k}\right). \tag{25}$$

Thus from Eq. (18), if the excess θ- component is set equal to zero, the f and g in the radially directed regions are subject to

$$\frac{1}{w}\frac{\partial f}{\partial\xi} - f + N\frac{\partial g}{\partial\xi} - \frac{\partial g}{\partial\varphi} = 0. \tag{26}$$

From Eq. (19), if the excess θ- component is set equal to zero, the f and g in the spiral slots are subject to

$$f + \frac{g}{wN} - \frac{1}{N}\left(\frac{1}{w^2}+N^2\right)\frac{\partial g}{\partial\xi} - \frac{1}{wN}\frac{\partial f}{\partial\varphi} + \frac{\partial g}{\partial\varphi} = 0 . \tag{27}$$

The simultaneous solutions of Eqs. (17) and (26) and of Eqs. (17) and (27) subject to the condition that $\vec{m}\cdot\vec{J}$ and $\vec{n}\cdot\vec{J}$ depend only on η except for inverse square factors gives

$$f = \frac{g_0}{wN} \begin{cases} 1 & \text{in region I} \\ 0 & \text{in region II} \\ -1 & \text{in region III} \\ 0 & \text{in region IV,} \end{cases} \tag{28}$$

and

$$g = \begin{cases} -g_0 & \text{in region I} \\ g_1 & \text{in region II} \\ g_0 & \text{in region III} \\ -g_1 & \text{in region IV} \end{cases} \tag{29}$$

The current flowing in the spiral slot is given by

$$I = -\iint \vec{i}_\theta \cdot \vec{J}\, dr\, dz = -w r_0^2 \iint \vec{i}_\theta \cdot \vec{J}\, e^{2w(\xi + N\varphi)} d\xi\, d\eta , \tag{30}$$

which after using Eq. (19) may be written as

$$I = -w r_0^2 \iint \left[-f + \frac{1}{w}\frac{\partial f}{\partial \xi} + N\frac{\partial g}{\partial \xi} - \frac{\partial g}{\partial \varphi} \right] d\xi\, d\eta . \tag{31}$$

Inserting Eqs. (28) and (29) into Eq. (31) gives

$$I = \frac{r_0^2 g_0}{N} (\xi_1 - \xi_0)(\eta_2 - \eta_1) . \tag{32}$$

In a similar manner the current in the radial return is

$$I = \iint \vec{i}_r \cdot \vec{J}\, r\, d\theta\, dz = r_0^2 g_1 (\varphi_1 - \varphi_0)(\eta_2 - \eta_1) . \tag{33}$$

Since the two currents are equal, the relation between g_0 and g_1 becomes

$$g_0 (\xi_1 - \xi_0) = g_1 N (\varphi_1 - \varphi_0) . \tag{34}$$

III. BOUNDARY VALUE PROBLEM

The tangential components of $\vec{H}$ at the interfaces between the various regions must be continuous. In addition, the normal component of $\vec{B}$ must be continuous at the copper-air interface. From Eq. (21) and Eqs. (24) and (25) the various components are:

$$\vec{l}_\theta \cdot \vec{H} = \frac{1}{r_0} e^{-w(\xi + N\varphi)} \begin{cases} 4\pi r_0^2 g_0 \eta - N\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial \varphi} + N\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \varphi} & \text{in I} \\ -4\pi r_0^2 g_1 \eta - N\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial \varphi} + N\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \varphi} & \text{in II} \\ -4\pi r_0^2 g_0 \eta - N\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial \varphi} + N\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \varphi} & \text{in III} \\ 4\pi r_0^2 g_1 \eta - N\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial \varphi} + N\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \varphi} & \text{in IV} \end{cases}, \tag{35}$$

$$\vec{k} \cdot \vec{H} = -\frac{1}{r_0} e^{-w(\xi + N\varphi)} \begin{cases} \frac{4\pi r_0^2 g_0}{wN} - \frac{\partial C}{\partial \eta} + \frac{\partial U}{\partial \eta} & \text{in I} \\ -\frac{\partial C}{\partial \eta} + \frac{\partial U}{\partial \eta} & \text{in II} \\ -\frac{4\pi r_0^2 g_0}{wN} - \frac{\partial C}{\partial \eta} + \frac{\partial U}{\partial \eta} & \text{in III} \\ -\frac{\partial C}{\partial \eta} + \frac{\partial U}{\partial \eta} & \text{in IV} \end{cases}, \tag{36}$$

$$\vec{n}\cdot\vec{H} = \frac{e^{-w(\xi+N\varphi)}}{r_0\sqrt{1+\left(\frac{1}{wN}\right)^2+\eta^2}}\begin{cases} \frac{1}{wN}\frac{\partial C}{\partial\varphi} - \frac{1}{wN}\frac{\partial U}{\partial\varphi} & \text{in I} \\ -\frac{4\pi r_0^2 g_1\eta}{wN} + \frac{1}{wN}\frac{\partial C}{\partial\varphi} - \frac{1}{wN}\frac{\partial U}{\partial\varphi} & \text{in II} \\ \frac{1}{wN}\frac{\partial C}{\partial\varphi} - \frac{1}{wN}\frac{\partial U}{\partial\varphi} & \text{in III} \\ \frac{4\pi r_0^2 g_1\eta}{wN} + \frac{1}{wN}\frac{\partial C}{\partial\varphi} - \frac{1}{wN}\frac{\partial U}{\partial\varphi} & \text{in IV} \end{cases}, \tag{37}$$

and

$$\vec{m}\cdot\vec{H} = \frac{e^{-w(\xi+N\varphi)}}{r_0\sqrt{1+\eta^2}}\begin{cases} -\frac{4\pi r_0^2 g_0\eta}{wN} + \frac{1}{w}\frac{\partial C}{\partial\xi} - \frac{1}{w}\frac{\partial U}{\partial\xi} & \text{in I} \\ \frac{1}{w}\frac{\partial C}{\partial\xi} - \frac{1}{w}\frac{\partial U}{\partial\xi} & \text{in II} \\ \frac{4\pi r_0^2 g_0\eta}{wN} + \frac{1}{w}\frac{\partial C}{\partial\xi} - \frac{1}{w}\frac{\partial U}{\partial\xi} & \text{in III} \\ \frac{1}{w}\frac{\partial C}{\partial\xi} - \frac{1}{w}\frac{\partial U}{\partial\xi} & \text{in IV} \end{cases}. \tag{38}$$

An examination of Eqs. (35-38) aided by a comparison with the corresponding scaling problem[3] shows that much simplification can be introduced if C is chosen to be

$$C = -4\pi r_0^2\eta\begin{cases} \frac{g_0}{N}\left(\xi_1-\xi-\frac{1}{w}\right) & \text{in I} \\ g_1(\varphi_1-\varphi) & \text{in II} \\ \frac{g_0}{N}\left(\xi_1+\xi+\frac{1}{w}\right) & \text{in III} \\ g_1(\varphi_1+\varphi) & \text{in IV} \end{cases}. \tag{39}$$

Equation (39) substituted into Eqs. (35-38) results in

$$\vec{\ell}_\theta \cdot \vec{H} = \frac{1}{r_0} e^{-w(\xi + N\varphi)} \left(N \frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \varphi} \right) \quad \text{in I, II, III, and IV,} \tag{40}$$

$$\vec{k} \cdot \vec{H} = -\frac{1}{r_0} e^{-w(\xi + N\varphi)} \begin{cases} \dfrac{4\pi r_0^2 g_0}{N} (\xi_1 - \xi) + \dfrac{\partial U}{\partial \eta} & \text{in I} \\ 4\pi r_0^2 g_1 (\varphi_1 - \varphi) + \dfrac{\partial U}{\partial \eta} & \text{in II} \\ \dfrac{4\pi r_0^2 g_0}{N} (\xi_1 + \xi) + \dfrac{\partial U}{\partial \eta} & \text{in III} \\ 4\pi r_0^2 g_1 (\varphi_1 + \varphi) + \dfrac{\partial U}{\partial \eta} & \text{in IV} \end{cases} , \tag{41}$$

$$\vec{n} \cdot \vec{H} = -\frac{e^{-w(\xi + N\varphi)}}{r_0 \sqrt{1 + \left(\frac{1}{wN}\right)^2 + \eta^2}} \cdot \frac{1}{wN} \cdot \frac{\partial U}{\partial \varphi} \quad \text{in I, II, III, and IV,} \tag{42}$$

and

$$\vec{m} \cdot \vec{H} = -\frac{e^{-w(\xi + N\varphi)}}{r_0 \sqrt{1 + \eta^2}} \cdot \frac{1}{w} \cdot \frac{\partial U}{\partial \xi} \quad \text{in I, II, III, and IV.} \tag{43}$$

In the space outside the currents and the iron, the distinction between the various regions disappears because the source terms are zero. Letting V designate the magnetostatic potential in this region, the various components become

$$\vec{\ell}_\theta \cdot \vec{H} = \frac{1}{r_0} e^{-w(\xi + N\varphi)} \left(N \frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \varphi} \right) , \tag{44}$$

$$\vec{k} \cdot \vec{H} = -\frac{1}{r_0} e^{-w(\xi + N\varphi)} \frac{\partial V}{\partial \eta} , \tag{45}$$

$$\vec{n}\cdot\vec{H} = -\frac{e^{-w(\xi+N\varphi)}}{r_0\sqrt{1+\left(\frac{1}{wN}\right)^2+\eta^2}}\cdot\frac{1}{wN}\frac{\partial V}{\partial\varphi} \quad , \tag{46}$$

and

$$\vec{m}\cdot\vec{H} = -\frac{e^{-w(\xi+N\varphi)}}{r_0\sqrt{1+\eta^2}}\cdot\frac{1}{w}\frac{\partial V}{\partial\xi} \quad . \tag{47}$$

Equations (40-43) used as tangential components are equated to the corresponding Eqs. (44-47) at the copper-air interfaces or to zero at the copper-iron interfaces. The resulting relations may be integrated and, if the pole potential $V_0 = 4\pi I$ is used through Eqs. (32) and (33) to replace g_0 and g_1, the following boundary conditions obtain

$$U(\pm\xi_0\,\eta\,\varphi) = U(\xi\,\eta\,\pm\varphi_0) = V_0\,\frac{\eta_2-\eta}{\eta_2-\eta_1} \quad , \tag{48}$$

and

$$U(\pm\xi_1\,\eta\,\varphi) = U(\xi\,\eta_2\,\varphi) = 0 \tag{49}$$

at the copper-iron interfaces.

$$V(\pm\xi_1\,\eta\,\varphi) - U(\pm\xi_1\,\eta\,\varphi) = 0 \quad , \tag{50}$$

$$V(\xi\,\eta\,\pm\varphi_1) - U(\xi\,\eta\,\pm\varphi_1) = 0 \quad , \tag{51}$$

$$V(\xi\,\eta_1\,\varphi) - U(\xi\,\eta_1\,\varphi) = 0 \quad , \tag{52}$$

and

$$V(\xi\,\eta_2\,\varphi) - U(\xi\,\eta_2\,\varphi) = 0 \tag{53}$$

at the copper-air interfaces.

In addition, matching the normal components of $\vec{H}$ at the copper-air interfaces gives for $\xi = \pm \xi_1$ after using Eq. (50),

$$\frac{\partial V}{\partial \xi} - \frac{\partial U}{\partial \xi} = 0 . \tag{54}$$

Similarly, along the faces $\varphi = \pm \varphi_1$ after using Eq. (51),

$$\frac{\partial V}{\partial \varphi} - \frac{\partial U}{\partial \varphi} = 0 . \tag{55}$$

Finally, at $\eta = \eta_1$ and $\eta = \eta_2$ after using Eq. (43),

$$\frac{\partial V}{\partial \eta} - \frac{\partial U}{\partial \eta} = 4\pi r_0^2 \begin{cases} \frac{g_0}{N}(\xi_1 - \xi) & \text{in I} \\ g_1(\varphi_1 - \varphi) & \text{in II} \\ \frac{g_0}{N}(\xi_1 + \xi) & \text{in III} \\ g_1(\varphi_1 + \varphi) & \text{in IV} \end{cases} . \tag{56}$$

In order to match the normal and tangential components of $\vec{H}$ at the interfaces between the spiral and radial current flow, it is first necessary to choose these surfaces in a definite manner. From Eq. (41) it will be noted that

$$\frac{\partial V}{\partial \eta} \quad \text{is continuous} \tag{57}$$

through the surfaces

$$F = \xi_1 + \begin{Bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{Bmatrix} \xi - \beta \sqrt{\frac{1}{w^2} + N^2} \left[\varphi_1 + \begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix} \varphi \right] = 0 \tag{58}$$

if the relation between g_0 and g_1 is chosen to be

$$\beta \sqrt{\frac{1}{w^2} + N^2}\, g_0 = N g_1 , \tag{59}$$

where the column matrix elements to be used refer to the surfaces in the following order.

$$\begin{Bmatrix} \text{I - II} \\ \text{II - III} \\ \text{III - IV} \\ \text{IV - I} \end{Bmatrix} \tag{60}$$

Thus Eq. (58) is a convenient definition of these interfaces. The quantity β may be chosen to vary the width of the radial return current region relative to the spiral slot. When $\beta = 1$, these two widths are approximately equal. The other tangential component of $\vec{H}$ orthogonal to $\vec{k}$ on these surfaces is proportional to

$$-\frac{1}{w}\begin{Bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{Bmatrix}\left[N\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \phi}\right] - \left[N\begin{Bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{Bmatrix} + \beta\sqrt{\frac{1}{w^2}+N^2}\begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix}\right]\cdot\left[4\pi r_0^2\eta\begin{Bmatrix} \frac{g_0}{N}(\xi_1-\xi) \\ g_1(\phi_1-\phi) \\ \frac{g_0}{N}(\xi_1+\xi) \\ g_1(\phi_1+\phi) \end{Bmatrix} - \frac{1}{w}\frac{\partial U}{\partial \xi} + \eta\frac{\partial U}{\partial \eta}\right] \tag{61}$$

where the last column matrix refers to the regions I, II, III, and IV in descending order. Equations (57), (58), and (59) reduce the expression (61) to

$$\frac{1}{w}\begin{Bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{Bmatrix}\frac{\partial U}{\partial \phi} + \frac{\beta}{w}\sqrt{\frac{1}{w^2}+N^2}\begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix}\frac{\partial U}{\partial \xi} \quad , \tag{62}$$

which must be continuous at the surfaces of Eq. (58). Expressions (57) and (60) may both be satisfied by the requirement that

U is continuous function across surfaces, $F = 0$. (63)

Continuity of the normal component of $\vec{H}$ at the surfaces, $F = 0$, requires that

$$\frac{1}{w}\begin{Bmatrix}-1\\1\\1\\-1\end{Bmatrix}\left[4\pi r_0^2\eta\begin{Bmatrix}\frac{g_0}{N}(\xi_1-\xi)\\ g_1(\varphi_1-\varphi)\\ \frac{g_0}{N}(\xi_1+\xi)\\ g_1(\varphi_1+\varphi)\end{Bmatrix}-\frac{1}{w}\frac{\partial U}{\partial\xi}+\eta\frac{\partial U}{\partial\eta}\right]-\left[N\begin{Bmatrix}-1\\1\\1\\-1\end{Bmatrix}+\beta\sqrt{\frac{1}{w^2}+N^2}\begin{Bmatrix}-1\\-1\\1\\1\end{Bmatrix}\right]\left[N\frac{\partial U}{\partial\xi}-\frac{\partial U}{\partial\varphi}\right] \tag{64}$$

be continuous. Expressions (57), (58), and (59) reduce Eq. (64) to the requirement that

$$\sqrt{\frac{1}{w^2}+N^2}\begin{Bmatrix}-1\\1\\1\\-1\end{Bmatrix}\frac{\partial U}{\partial\xi}-\beta\begin{Bmatrix}-1\\-1\\1\\1\end{Bmatrix}\frac{\partial U}{\partial\varphi} \tag{65}$$

be continuous across the surfaces, F = 0.

Finally the differential equation for U becomes

$$\mathcal{M}U=-\begin{cases}\dfrac{V_0}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\,\eta\left(\xi_1-\xi+\frac{1}{w}\right) & \text{in I}\\[2ex] \dfrac{V_0}{(\varphi_1-\varphi_0)(\eta_2-\eta_1)}\,\eta\left(\varphi_1-\varphi\right) & \text{in II}\\[2ex] \dfrac{V_0}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\,\eta\left(\xi_1+\xi-\frac{1}{w}\right) & \text{in III}\\[2ex] \dfrac{V_0}{(\varphi_1-\varphi_0)(\eta_2-\eta_1)}\,\eta\left(\varphi_1+\varphi\right) & \text{in IV}\end{cases} \tag{66}$$

In principle the boundary value problem is now completely specified if $v\left(\varphi-\frac{2\pi i}{M},\xi-2\pi j,\eta\right)$ designates the solution in which the potential is set equal to zero on all surfaces outside of the cell (i,j) including the median plane ($\eta=0$) and is set equal to unity on the (i,j) pole. The resultant potential is then given by

$$V(\varphi\xi\eta)=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}V_{ij}\,v\left(\varphi-\frac{2\pi i}{M},\xi-2\pi j,\eta\right), \tag{67}$$

where V_{ij} is the potential on the (i,j) pole. The φ -coordinate at the center of the pole is

$$\Phi_i = \frac{2\pi}{M} i \quad . \tag{68}$$

The ξ -coordinate at the center of the (i,j) pole is

$$\Xi_j = 2\pi j \, , \tag{69}$$

and the r -coordinate at the center of the (i,j) cell is R_{ij}, where

$$2\pi j = \frac{1}{w} \ln \frac{R_{ij}}{r_0} - N \frac{2\pi i}{M} \quad . \tag{70}$$

The solution in Eq. (67) will possess integral scaling if

$$V_{ij} = V_0 \, e^{2\pi(k+1)w\left(j + \frac{N}{M} i\right)} \quad . \tag{71}$$

This property of integral scaling may be used to simplify the boundary value problem by noting that

$$V\left(\varphi + \frac{2\pi m}{M}, \xi + 2\pi n, \eta\right) = V_0 \sum_{i,j} e^{2\pi(k+1)w\left(j + \frac{N}{M} i\right)} v\left[\varphi - \frac{2\pi}{M}(i-m), \xi - 2\pi(j-n), \eta\right] \tag{72}$$

or

$$V\left(\varphi + \frac{2\pi m}{M}, \xi + 2\pi n, \eta\right) = e^{2\pi(k+1)w\left(n + \frac{N}{M} m\right)} V(\varphi, \xi, \eta) \, .$$

The integral scaling property expressed in Eq. (72) may be used to specify a boundary value problem in one cell, say the (0,0), the solution of which via Eq. (72) permits the solution in any cell to be obtained. Thus, instead of using the solution v_{ij} in Eq. (67), one can obtain a solution by requiring on the cell boundaries that

$$V\left(\frac{\pi}{M}, \xi, \eta\right) = e^{2\pi(k+1)w\frac{N}{M}} V\left(-\frac{\pi}{M}, \xi, \eta\right), \tag{73}$$

and that

$$V(\varphi, \pi, \eta) = e^{2\pi(k+1)w} V(\varphi, -\pi, \eta) \quad . \tag{74}$$

As in previous problems the solutions are required to be odd functions of η .

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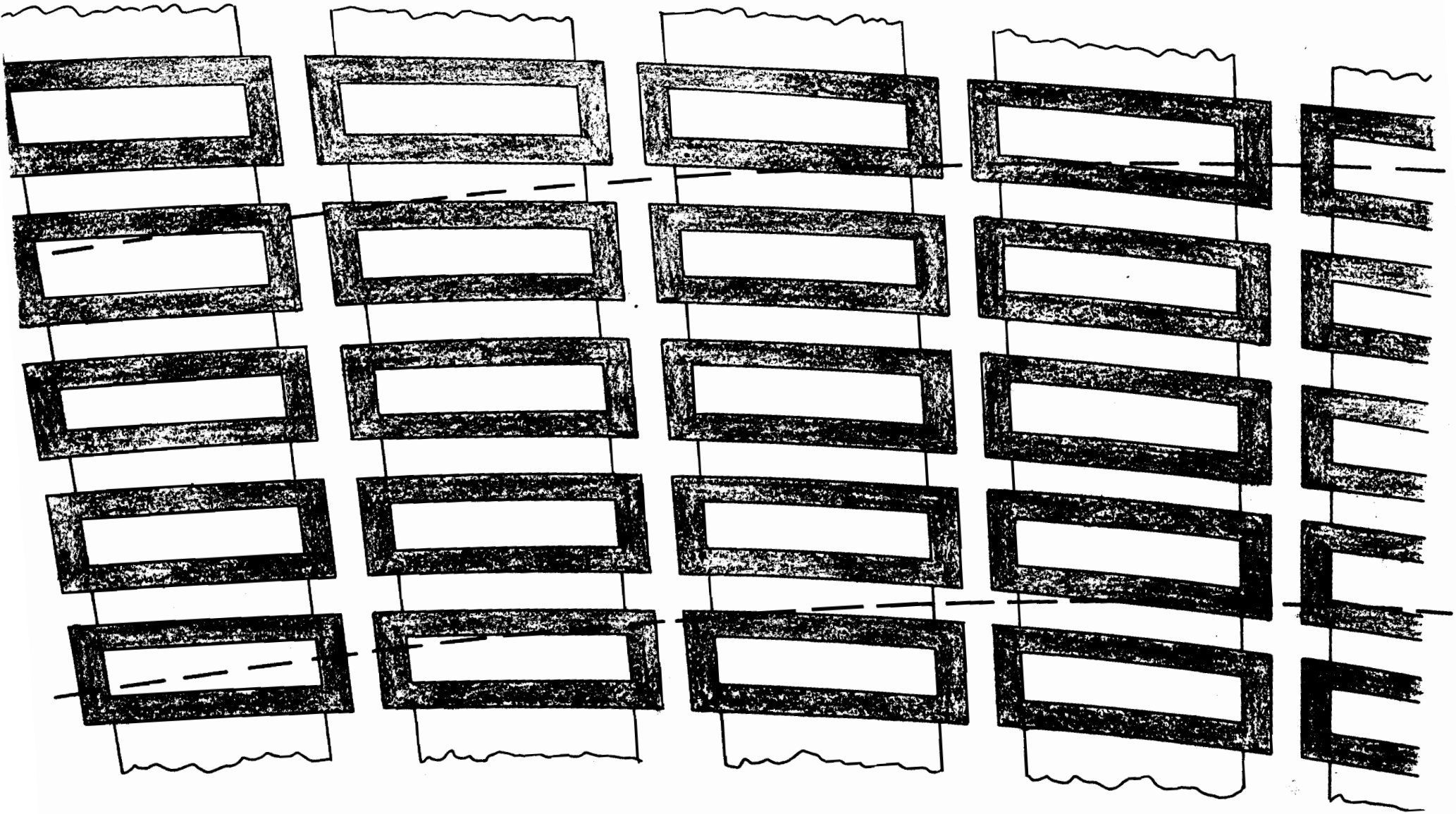

FIG. 1 -INTEGRAL SCALING POLE FACE

Fig. 1 S.C. Snowdon MURA-583

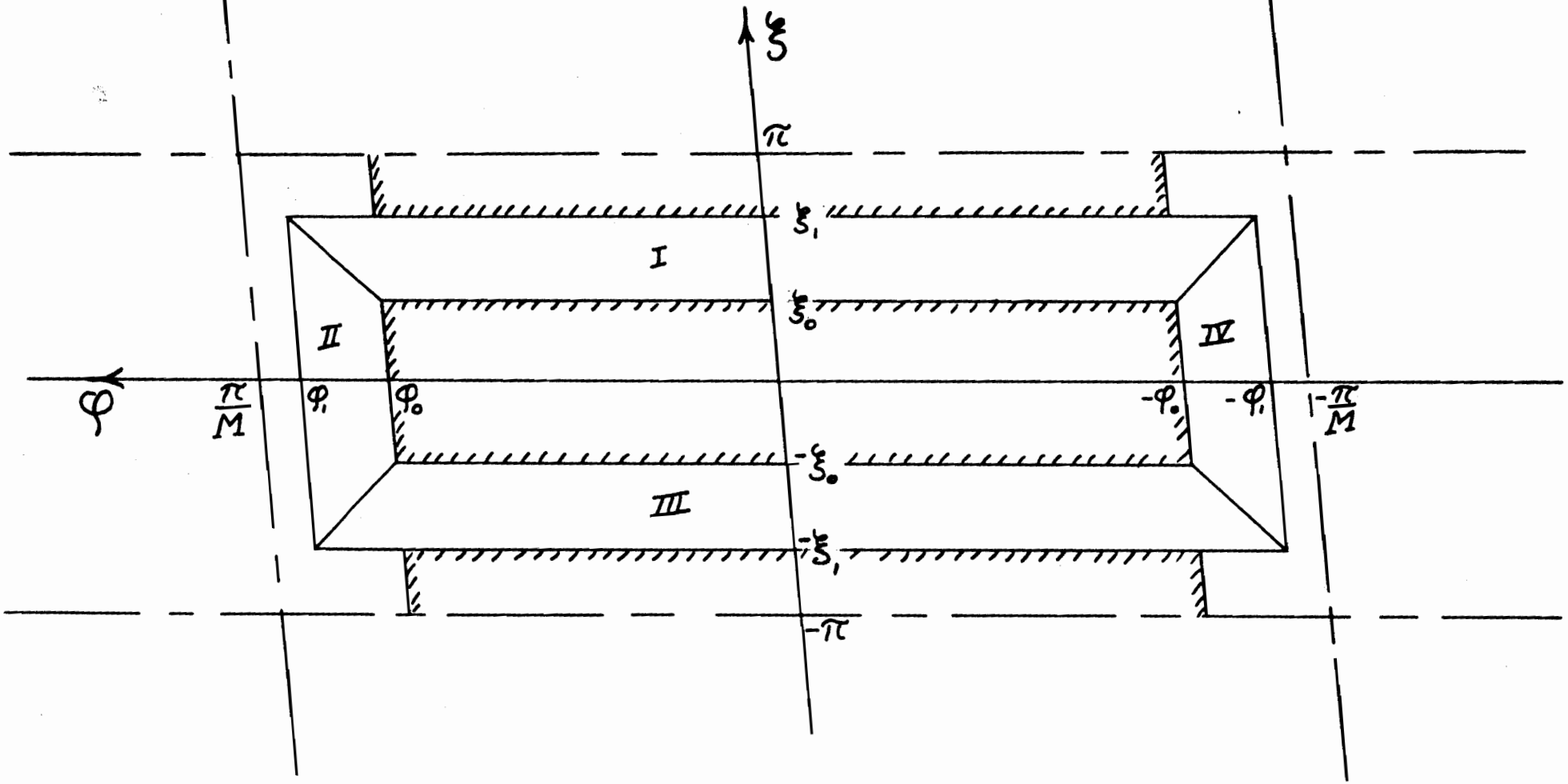

FIG. 2 -POLE FACE UNIT CELL