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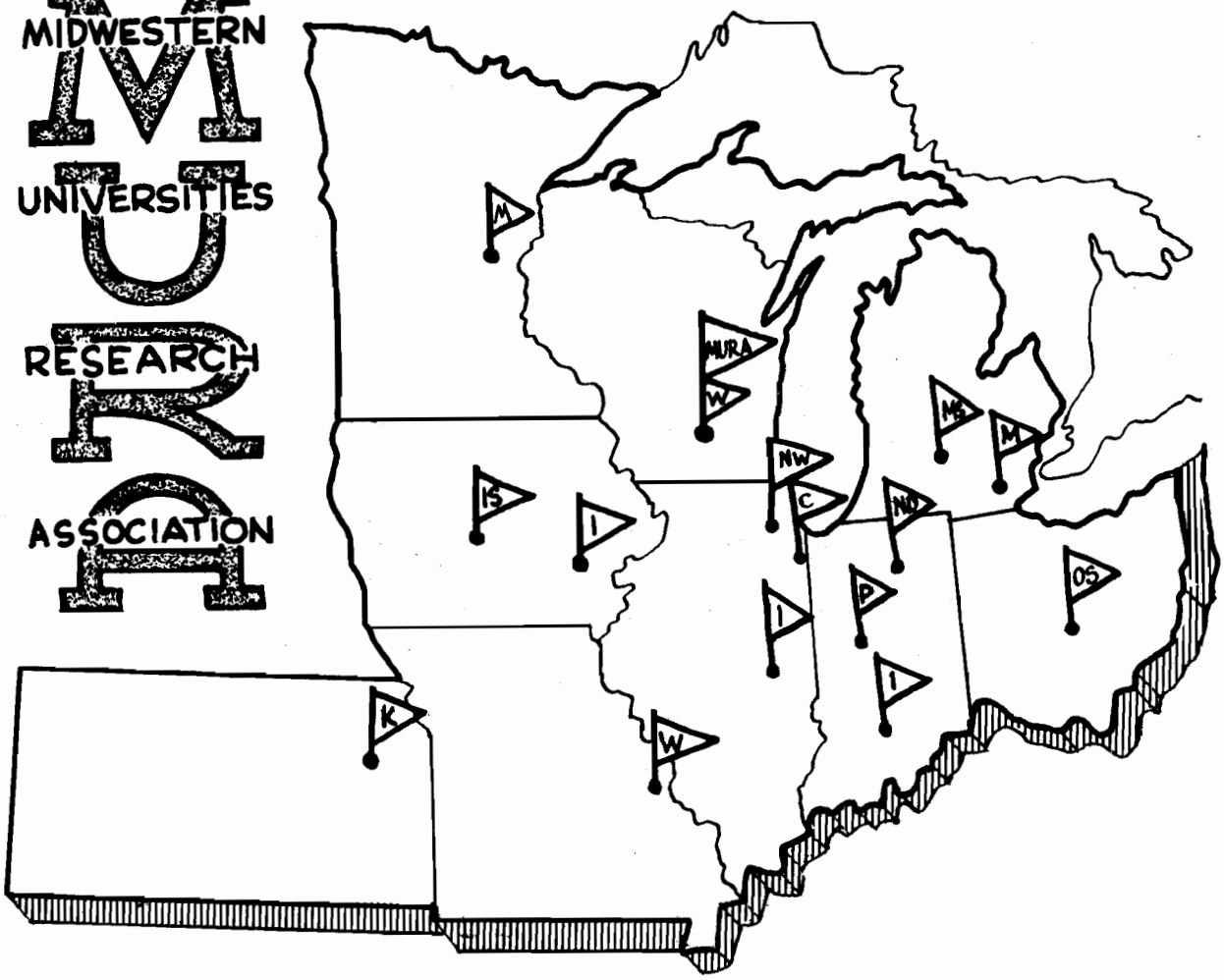

ACCELERATION ACROSS TRANSITION ENERGY

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ABSTRACT

Study of the phase motion across transition energy shows that the non-linear term has an important effect. To avoid increase in amplitude of synchrotron oscillation, the shift of the stable point should occur when the energy is very close to the transition energy and with a program keeping the linear approximation valid as long as possible. Computer study agrees with these results. When the above requirements are fulfilled, the final bucket area increases less than 20% for an initial area $A_i = .3A_c$, A_c being the maximum bucket area near transition.

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I. INTRODUCTION

The phase of the reference particle around which particles describe phase oscillations must be shifted from the first to the second quadrant while crossing the transition energy. If this is done without particular care, the amplitude of synchrotron oscillations will increase and particles, being outside the bucket, will be lost. Usually the study of the phase motion is limited, with good accuracy, to the linear approximation. When crossing transition energy, non-linear terms can become very important. The program of the voltage and the frequency of the cavity should be such that in the linear approximation the amplitude of the phase oscillation far above transition energy is the same as far below, with the supplementary requirement that the non-linear term remains small during nearly all the time the particles need to cross transition energy. Programs are compared both analytically and using the computer

II. EQUATIONS OF THE SYNCHROTRON OSCILLATION

The equations of the phase motion are derived in MURA-106[1]

$$\frac{dW}{d\tau} = V(\tau)\sin\phi, \qquad \frac{d\phi}{d\tau} = 2\pi\left[f_c(\tau) - h\,f(W)\right] \tag{1}$$

where $f_c(\tau)$ is the frequency of the cavity, $V(\tau)$ the accelerating voltage, h the harmonic number, ϕ the phase of the rf voltage when the particle goes through the accelerating gap, E and f the energy and frequency of the particle and W is defined by

$$W = \int \frac{dE}{f(E)} \,. \tag{2}$$

We shall use the subscript s for the reference particle whose motion is represented by the stable fixed point far below and above the transition energy. For particles close to the reference particle, it is convenient to use the relative coordinates:

$$W^* = W - W_s \,, \qquad \phi^* = \phi - \phi_s \,. \tag{3}$$

Then the equations of motion become

$$\begin{aligned} \dot{W}^* &= V(\tau) \sin(\phi_s + \phi^*) - V(\tau) \sin \phi_s \,, \\ \dot{\phi}^* &= -2\pi h \left[f(W) - f_s(W) \right] . \end{aligned} \tag{4}$$

If W^* and ϕ^* are small, we can expand and get

$$\dot{W}^* = V \cos \phi_s \cdot \phi^* - \tfrac{1}{2} V \sin \phi_s \cdot \phi^{*2} + \cdots \,, \tag{5}$$

$$\dot{\phi}^* = -2\pi h \left(\frac{df}{dW} \right)_s \cdot W^* - \pi h \left(\frac{d^2 f}{dW^2} \right)_s \cdot W^{*2} + \cdots \,. \tag{6}$$

These equations are derivable from the Hamiltonian

$$H^* = -\pi h f_s' W^{*2} - \tfrac{1}{2} V \cos \phi_s \cdot \phi^2 - \tfrac{1}{3} \pi h f_s'' W^{*3} + \tfrac{1}{6} V \sin \phi_s \cdot \phi^{*3} + \cdots \tag{7}$$

III. LINEAR APPROXIMATION

A. Adiabatic Motion

For small oscillations and not too close to transition energy, only the two first terms are important in the Hamiltonian and the equations of motion reduce to

$$\dot{W}^* = V\cos\phi_s \cdot \phi^*$$
$$\dot{\phi}^* = -2\pi h f_s' W^*. \tag{8}$$

Far from the transition energy, the limit of this region being given a previous report,[2] it is usual to make $V(\tau)$ equal a constant V_0 and to choose $f_c(\tau)$ such that ϕ_s remains constant. As f_s' varies slowly with time, the solution of (8) is the well known adiabatic motion.

B. Non-Adiabatic Motion

When the condition for adiabatic motion is no longer fulfilled, the linearized equations become, if no assumption is made about $V(\tau)$ and $\dot{\phi}_s(\tau)$:

$$\ddot{\phi}^* - \frac{1}{f_s'}\frac{d f_s'}{d\tau}\cdot\dot{\phi}^* + 2\pi h f_s' V\cos\phi_s \cdot \phi^* = 0 \tag{9}$$

$$\ddot{W}^* - \frac{1}{V\cos\phi_s}\frac{d(V\cos\phi_s)}{d\tau}\dot{W}^* + 2\pi h f_s' V\cos\phi_s \cdot W^* = 0. \tag{10}$$

We shall assume that near transition energy f_s' varies with time according to

$$f_s' = -b\tau, \tag{11}$$

where b is positive and t = 0 at transition energy. This is true only if $V \sin \phi_s$ is constant.

In the case ϕ_s and V are not functions of time, but ϕ_s has the value ϕ_{s_0} below and $\pi - \phi_{s_0}$ above transition energy, equation (9) becomes

$$\ddot{\phi}^* - \frac{1}{\tau}\dot{\phi}^* \pm 2\pi h b V_0 \cos\phi_{s_0} \cdot \tau \phi^* = 0 \tag{12}$$

In (12) as in the following equation (14), the upper sign applies above and the lower sign below transition energy. The solution is given by means of the Bessel function of order 2/3:

$$\phi^* = A_1 u^{2/3} J_{2/3}(u) + A_2 u^{2/3} J_{-2/3}(u), \tag{13}$$

with

$$u = \frac{2}{3}\left(\mp 2\pi h b V_0 \cos\phi_{s_0}\right)^{1/2} (\pm\tau)^{3/2} \tag{14}$$

and A_1, A_2 represent the initial conditions.

The ϕ^* solution given by (12) and the W^* expression we get from (10) are both even functions of t. This means that, in the hypothetical case that at t = 0 we suddenly shift the stable phase ϕ_s from ϕ_{s_0} to $\pi - \phi_{s_0}$, ϕ^* and W^* will have at $\tau = \tau_f$ the same values they had at $t = t_i$ if $\tau_f = -t_i$.

If the program $V(\tau)$ and $f_s(\tau)$ is such that the stable phase is shifted during the interval $t = t_0$ to $t = t_1$ according to some relation $\phi_s(\tau)$, then the solution of equation (9) for $t_0 < \tau < t_1$ depends on the functions $V \cos\phi_s$ and $f'_s(\tau)$.

The program must be chosen such that the solutions ϕ^* and ω^* are even functions of time. But we may observe that if t_o and t_1 are very small, the synchrotron oscillation will be only slightly larger far above transition energy when ϕ^* and ω^* are not even functions.

A program based on the linear approximation has been proposed by N. Christofilos, K. R. Symon and K. M. Terwilliger.[3] V(t) and $fc^{(t)}$ are such that the following relations are satisfied.

$$V \sin \phi_s = \text{constant} \tag{15}$$

$$V \cos \phi_s = V_o \cos \phi_{so} \cdot \frac{\tau}{\tau_o} \tag{16}$$

$$\operatorname{Cotg} \phi_s = \operatorname{Cotg} \phi_{so} \cdot \frac{\tau}{\tau_o} . \tag{17}$$

Then the solutions of the linear equations are

$$\phi^* = \phi_m^* \sin(\alpha \tau^2 + \theta) \tag{18}$$

$$\omega^* = \omega_m^* \cos(\alpha \tau + \theta), \tag{19}$$

where ϕ_m^*, ω_m^* and θ represent the initial conditions (and must be consistent with (7)),and α can be calculated from (16), (10) and (9). The solutions $\phi^*(t)$ and $\omega^*(t)$ are even functions as required, and give constant amplitude oscillation during the shifting time.

Note that it is not necessary, if $t_1 = -t_o$, to start to shift the stable point when the motion is still adiabatic. In that case the solution for the non-adiabatic motion is the combination of (13) and (18).

IV. VALIDITY OF THE LINEAR APPROXIMATION

In the linear approximation, we made two different assumptions.

A. We neglected $\pi h f_s'' \omega^{*2}$ compared to $2\pi h f_s' \omega^*$:

If $V \sin \phi_s$ is approximately constant, equation (11) holds and this assumption can be written

$$\tau \gg -\frac{f_s''}{b} W. \tag{20}$$

Replacing f_s'' by its value, (20) becomes

$$\tau \gg \frac{\omega}{V \sin \phi_s}. \tag{21}$$

Practically, it turns out that the time limit for the validity of (21) is very small, of the order of 10 to 100 revolutions.

The effect of this term is to change ϕ by

$$\Delta\phi = \frac{K \pi h b \overline{W}^3}{V^2 \sin^2 \phi_s}, \tag{22}$$

where K is a coefficient specifying how long the non-linear term is important. The value of K is probably less than 10, and $\Delta\phi$ calculated by (22) is very small and negligible.

Note that this effect does not depend on the program.

B. We neglected $\frac{1}{2} V \sin \phi_s \cdot \phi^{*2}$ compared to $V \cos \phi_s \cdot \phi^*$:

As long as V and ϕ_s are constant, the assumption gives the usual approximation for small oscillations which is right except near the separatrix. The smaller ϕ_s , the better is the approximation.

During the time the stable point is shifted, $V \cos \phi_s$ becomes very small and the term $\frac{1}{2} V \sin \phi_s \cdot \phi^{*2}$ is larger than $V \cos \phi_s \cdot \phi^*$. In this case the solutions of (5) and (6) can increase very rapidly, depending on how $V \cos \phi_s$ varies with time. If we start to shift the stable point early, this effect lasts a long time and becomes

very important when the program is such that the solutions of equations (5) and (6) are rapidly increasing functions during the phase shifting time.

V. GENERAL METHOD TO ACCELERATE ACROSS TRANSITION ENERGY

From the above considerations, we see, as the non-linear term in equation (5) has a very important effect, that the main requirement on the choice of the program $V(\tau)$ and $f_c(\tau)$ is that $V\cos\phi_s$ remains larger than $V\sin\phi_s$ (linear approximation valid) as long as possible, even if the motion is not adiabatic during a long interval of time.

A possible program for shifting phase is

$$V\sin\phi_s = \text{constant}, \tag{23}$$

$$V\cos\phi_s = a\tau^{1/9}, \tag{24}$$

where a is a constant $(a < 0)$,

$$a = \frac{V_0\cos\phi_{s0}}{\tau_0^{1/9}}. \tag{25}$$

The time τ_0 at which ϕ_s is shifted must be small.

The meaning of " τ_0 small" will appear later but we can remark immediately that the smaller τ_0 is, the smaller is the effect of the non-linear term. With this program, the differential equations (9) and (10) for the linear approximations become

$$\ddot{\phi}^* - \frac{1}{\tau}\dot{\phi}^* - 2\pi h b a\,\tau^{\frac{10}{9}}\phi^* = 0, \tag{26}$$

$$\ddot{w}^* - \frac{1}{9\tau}\dot{w}^* - 2\pi h b a\,\tau^{10/9} w^* = 0. \tag{27}$$

To solve equation (26), we use the new variables

$$u = \frac{9}{7}(-2\pi h b a)^{1/2}\, \tau^{14/9} \tag{28}$$

and

$$Z = u^{-\frac{1}{7}}\, \phi^{*}. \tag{29}$$

Then equation (26) becomes:

$$\frac{d^2 Z}{du^2} + \left(\frac{1}{4} - \frac{8}{49 u^2}\right) Z = 0. \tag{30}$$

An approximate solution of (30) is known in the case

$$u^2 \ll \frac{32}{49} \tag{31}$$

or

$$\tau^{28/9} \ll -\frac{32}{81}\, \frac{1}{2\pi h b a}. \tag{32}$$

It is

$$Z = C_1 u^{16/14} + C_2 u^{-\frac{2}{14}}. \tag{33}$$

Then

$$\phi^{*} = C_1' \tau^2 + C_2 \tag{34}$$

where C_1' and C_2 are given by the initial conditions. The approximation (31) means that for small time we neglect the last term in equation (26). Note that with this program the non-linear term is also small compared to the two first terms of equation (26), even when it becomes larger than the linear term.

A similar calculation can be made for W^* and equation (27).

It turns out that

$$W^* = C_3 \tau^{10/9} + C_4, \tag{35}$$

with the condition

$$u^2 \ll \frac{74}{49}. \tag{36}$$

The constants C_3 and C_4 are related to C_1 and C_2 so that there are only two independent initial conditions. Both functions $\phi^*(\tau)$ and $W^*(\tau)$ given by (34) and (35) are even functions of time.

It should be noted that the condition (36) implies a short time and any program to shift the stable point in such a short time would not have an important effect. But with the program (23) and (24), the only increase in amplitude of the phase oscillations comes from the non-linear term $f_s'' W^{*2}$. From equations (23), (24), and (1), we get the relation $\phi_s(\tau)$ and the parameters of the cavity $V(\tau)$ and $f_c(\tau)$:

$$\cot g\, \phi_s = \frac{\cot g\, \phi_{s0}}{\tau_0^{1/9}}\, \tau^{1/9} \tag{37}$$

$$V(\tau) = V_0 \left[\sin^2 \phi_{s0} + \cos^2 \phi_{s0} \left(\frac{\tau}{\tau_0}\right)^{2/9} \right]^{1/2} \tag{38}$$

$$f_c(\tau) = h f_\tau + \frac{h b^2}{2 f_{s0}''} \tau^2 - \frac{1}{18\pi} \frac{\cot g\, \phi_{s0} \cdot \frac{\tau^{-8/9}}{\tau_0^{1/9}}}{1 + \cot g^2 \phi_{s0} \left(\frac{\tau}{\tau_0}\right)^{2/9}} . \tag{39}$$

VI. COMPUTER RESULTS

Calculations were made using the TTT program. We started to study the motion far below transition energy in the adiabatic region. Particles represented by points on a curve of area A_i were accelerated and far above transition energy each point described a curve of area A_p. In Fig. 1 are represented the averages of the areas A_p, corresponding to the same A_i, as a function of the initial area A_i for two different programs. In Fig. 2 the root mean square

of the distributions is plotted under the same conditions.

The areas are measured in units of the maximum bucket area A_c when near transition energy the parameter η_c has the value

$$A_c = \frac{1}{\beta_\tau}\left(\frac{k E_\tau^3 V}{4\pi h}\right)^{1/3} \alpha(\Gamma), \tag{40}$$

where $\alpha(\Gamma)$ is given by computer study, $\alpha = 2.7$ for $\Gamma = \frac{\sqrt{2}}{2}$ and $\alpha = 4.9$ for $\Gamma = .5$ (cf. MURA-106 and 423). In the numerical calculation $\phi_{s_0} = \frac{\pi}{4}$ and the maximum initial bucket area A_{imax} is $\frac{1}{2} A_c$.

The solid curves correspond to a program where at small τ_0 a sudden constant jump is made in the cavity frequency, while the voltage is constant, in such a way that the stable point is shifted to $\pi - \phi_{s_0}$. It is slightly different from the proposed program but the requirement that the phase shifting time is small is fulfilled.

The broken curves correspond to the program of reference 3 in which we started to shift the stable point when the motion was adiabatic (τ_0 large). From equation (5) we see that for some initial conditions W^* is zero for large negative values of time, then $W^*(\tau)$ is not an even function and $|W^*|$ increases rapidly. In Fig. 3 a typical case of such an effect is represented. The broken curve corresponds to the computer result while the crosses are the values given by the linear approximation. The solid curve corresponds, for the smae initial conditions, to the second program which keeps the amplitude almost constant.

The analytical solutions for the case of the jump in the cavity frequency are not known and are probably not even functions of time.

It is believed that relations (23) and (24) should give a better result. Note that by a calculation similar to that of equations (26)-(36), the program of reference 3, but with a short shifting time, will also give even functions for the solutions $\phi^*(\tau)$ and $\omega^*(\tau)$, when τ is small. Therefore this program gives no increase in the amplitude of the synchrotron oscillations.

From the computer results we see that we can accelerate across transition energy at least one-third of the maximum bucket area, with very little increase in the synchrotron oscillations.

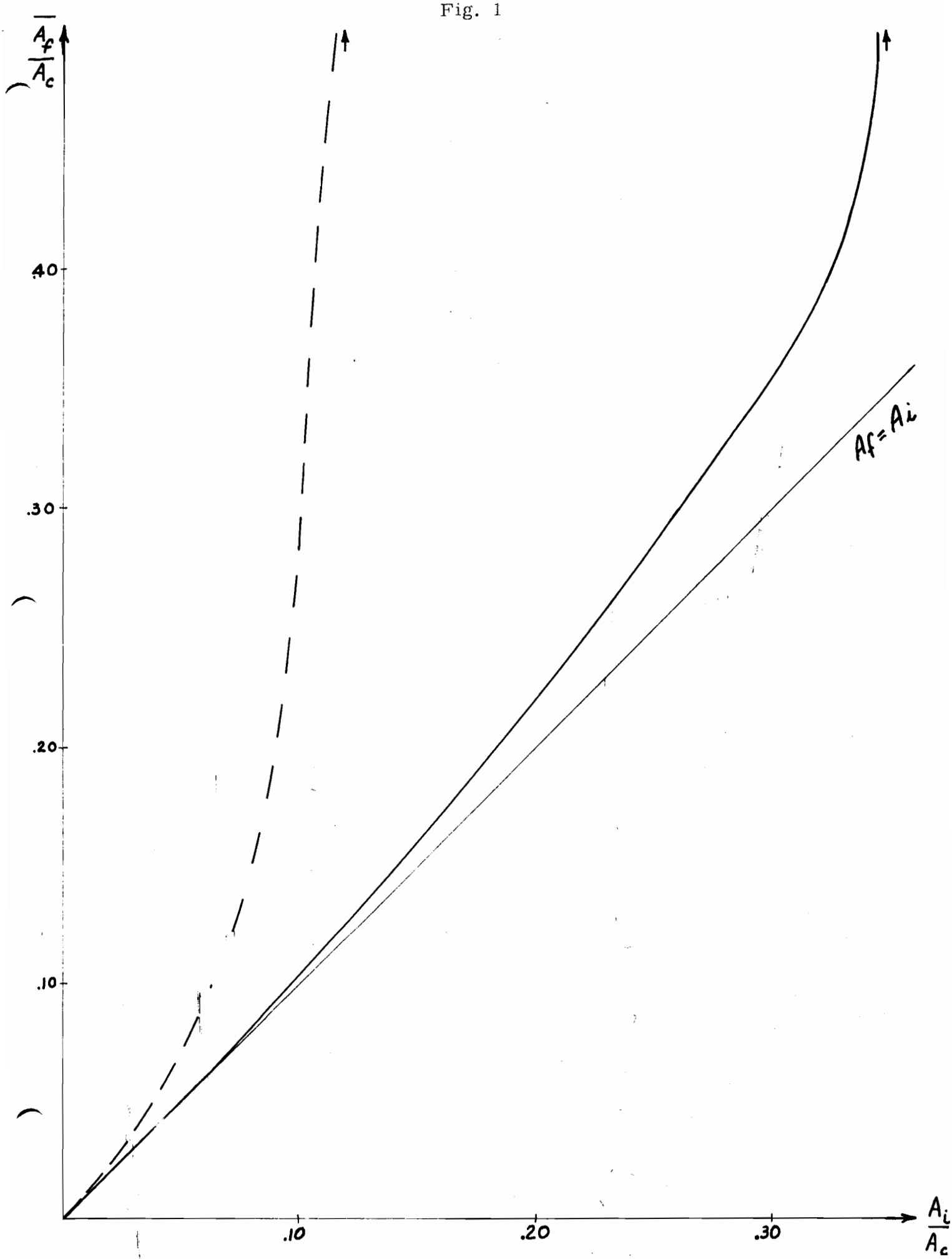
REFERENCES

1) K. R. Symon and A. M. Sessler, "Methods for Radio Frequency Acceleration in Fixed Field Accelerators", MURA-106 or Proceedings of CERN symposium on High Energy Accelerators, Vol. I, page 44, (1956).

2) E. Bronca, "Adiabatic Behavior near Transition Energy", MURA-560.

3) N. Christofilos, K. R. Symon, K. M. Terwilliger, "Acceleration across the Transition Energy without increase in Amplitude of Synchrotron Oscillations". MURA-491.

Fig. 1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$\frac{A_i}{A_c}$
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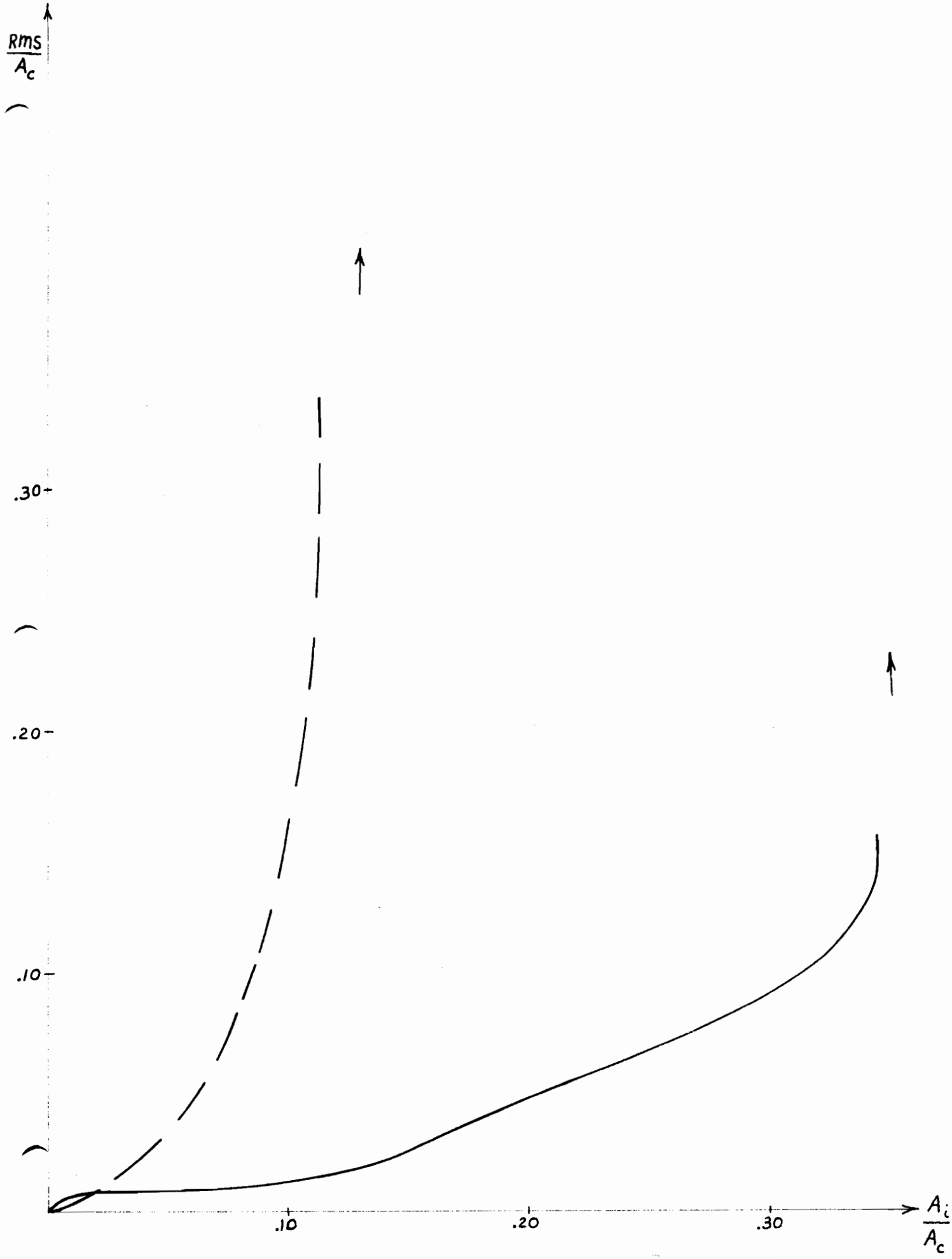
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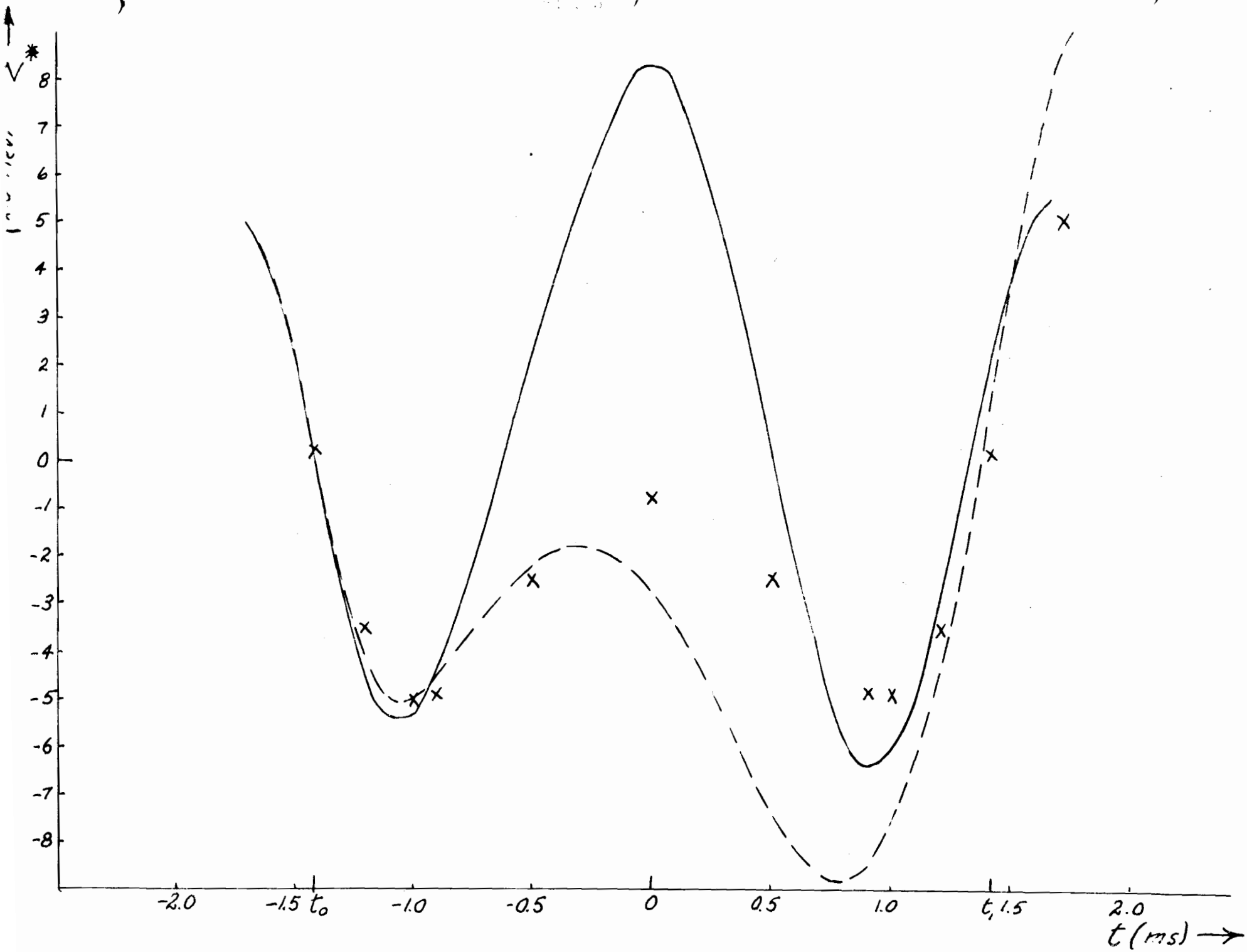
V*
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5
4
3
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1
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-1.5
t_0
-1.0
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0
0.5
1.0
t_1
1.5
2.0
t (ms)