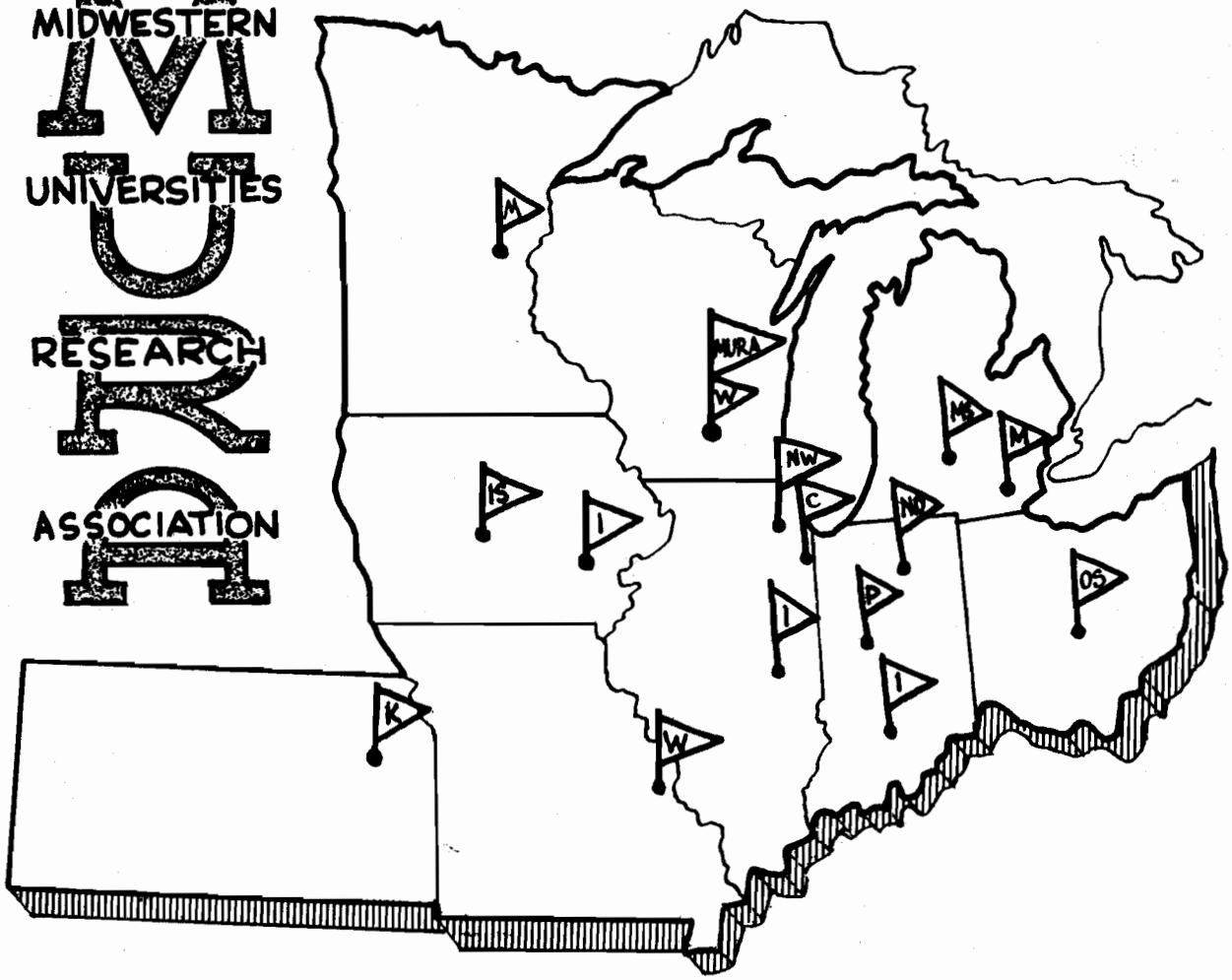
569

Fermilab Library
0 1160 0038855 7

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



THE PRODUCTION BY SURFACE CURRENTS OF MAGNETIC FIELDS SUITABLE FOR SPIRAL RIDGE ACCELERATORS

Edward S. Akeley

REPORT **NUMBER** 569

TABLE OF CONTENTS

I.	Introduction	Page 2
II.	Elementary Magnets	Page 4
III.	Subregions of Type (r)	Page 8
IV.	Subregions of Type (θ)	Page 12
V.	Subregions of Type (φ)	Page 18
VI.	Second and Third Order Regions	Page 22
VII.	Concluding Remarks	Page 25
VIII.	References	Page 26

MURA-569
UC-28, Particle Accelerators and
High Voltage Machines
TID-4500 (14th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

THE PRODUCTION BY SURFACE CURRENTS OF MAGNETIC FIELDS SUITABLE FOR SPIRAL RIDGE ACCELERATORS.

Edward S. Akeley**

Purdue University

April 15, 1960

ABSTRACT

The problem is treated of finding surface current distributions required to produce magnetic fields suitable for spiral ridge accelerators, where the field is required to be zero outside the region of the current distributions and where media of two different permeabilities are used. The problem is reduced to the design of an "elementary magnet". The final accelerator magnet can be formed of a single such magnet or built up using a number of them.

*AEC Research and Development Report. Research supported by the U.S. Atomic Energy Commission, Contract No. AT(11-1)-384.

**Purdue University, Lafayette, Indiana

INTRODUCTION

Methods have been developed for finding current distributions which would produce magnetic fields required for radial sector FFAG accelerators. This was accomplished first by using only surface current distributions[1] and then by using only volume current distributions[2]. In this report, these methods are used to determine surface currents to produce the magnetic fields for spiral ridge accelerators.* A somewhat more general treatment will be used than that of the previous reports. Media of two permeabilities will be used, so that the results could be applied to other than air core magnets.

We shall consider the magnetic field represented by the complex magnetic scalar potential:

$$V_{(r,\gamma,\phi)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(o)}{L}{}^{(mN)}_{\alpha+1}(\gamma)\, e^{-imN\phi} \tag{1.1}$$

where r_0 and H_0 are real constants, $\alpha = k + imK$ with k and K real constants. m is a positive or negative integer. N is a positive integer. (r, θ, ϕ) or (r, γ, ϕ) represents a spherical coordinate system, $\gamma = \frac{\pi}{2} - \theta$. $\underset{(o)}{L}{}^{(mN)}_{\alpha+1}(\gamma)$ and $\underset{(e)}{L}{}^{(mN)}_{\alpha+1}(\gamma)$ are respectively odd and even functions of γ, which are also solutions of the Legendre equation:

$$\frac{d^2}{d\gamma^2} L^{(mN)}_{\alpha+1} - \tan\gamma \frac{d}{d\gamma} L^{(mN)}_{\alpha+1} + \left[(\alpha+1)(\alpha+2) - \frac{m^2 N^2}{\cos^2\gamma}\right] L^{(mN)}_{\alpha+1} \tag{1.2}$$

For small values of γ and large values of $|\omega|\gamma$, the following expansions[3] represent rapidly convergent series.

*A report on solume current distributions for spiral ridge accelerators will appear shortly.

$$\underset{(o)}{\overset{(mN)}{L}}{}^{(y)}_{\alpha+1} = \sin\omega y + \frac{1}{4}\left[\frac{y}{\omega}\cos\omega y + y^2\sin\omega y\right] + \ldots$$

$$\underset{(e)}{\overset{(mN)}{L}}{}^{(y)}_{\alpha+1} = \cos\omega y + \ldots \tag{1.3}$$

where $\omega^2 = (\alpha+1)(\alpha+2) - m^2N^2$

We shall throughout this report assume the following approximations:

$$\underset{(o)}{\overset{(mN)}{L}}{}^{(y)}_{\alpha+1} = \sin\omega y \quad ; \quad \underset{(e)}{\overset{(mN)}{L}}{}^{(y)}_{\alpha+1} = \cos\omega y \tag{1.4}$$

which are relatively good approximations for large FFAG accelerators. For small accelerators, it might be necessary to use more terms of the series (1.3). The real and pure imaginary parts of V in (1.1) represent separately harmonic components of the spiral ridge field. This follows because any spiral ridge field (even of accelerators with radial straight sections) may be analyzed into components of this type. Since the equations for the magnetic fields and currents are linear, the sum of any number of solutions is also a solution. Hereafter in this report, unless otherwise stated, we shall deal with the complex field (1.1) and shall determine the complex current distributions that produce it. Then the real and pure imaginary parts of this complex field will be produced by the real and pure imaginary parts respectively of the complex current distributions.

As soon as any spiral ridge field has been resolved into harmonic components, the current distributions required to produce it, can be found by adding the current distributions required for its harmonic components.

The field (1.1) is to be produced in a certain region of space K_o surrounding part of the median plane $\psi = 0$. This corresponds to the region between the magnet pole faces of the accelerator. It is in this region that the vacuum chamber is located. The surface current distributions producing the magnetic field lie in and on a region K_1 surrounding K_o. Region K_2 is all space surrounding K_o and K_1. The magnetic field is required to vanish everywhere in K_2. Such a field can be produced by one or more magnets of a specific type, which we shall call elementary magnets and which are described in the neat section.

ELEMENTARY MAGNETS

An accelerator magnet could be built out of elementary magnets using them as building blocks. We start with an (r, ψ, ϕ) spherical coordinate system. All space is divided by the elementary magnet into three regions K_o, K_1, and K_2. K_o is a curvilinear parallelopiped bounded by spherical surfaces $r = r_1$ and $r = r_{-1}$, by the cones $\psi = \psi_1$ and $\psi = \psi_{-1}$ and by the azimuthal planes $\phi = \phi_1$ and $\phi = \phi_{-1}$ The region K_1 surrounds K_o. It is bounded on the inside by K_o and on the outside by the spherical surfaces $r = r_2$ and $r = r_{-2}$, the cones $\psi = \psi_2$ and $\psi = \psi_{-2}$ and the azumuthal planes $\phi = \phi_2$ and $\phi = \phi_{-2}$.

Here

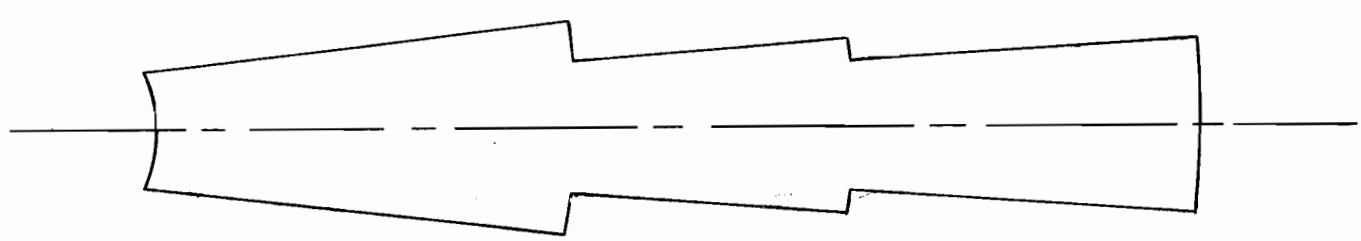
$$\begin{aligned} &\gamma_{-2} < \gamma_{-1} < \gamma_1 < \gamma_2 \\ &\phi_{-2} < \phi_{-1} < \phi_1 < \phi_2 \\ &\Gamma_{-2} < \Gamma_{-1} < \Gamma_1 < \Gamma_2 \end{aligned} \tag{1.5}$$

where the above quantities are given constants and the ϕ's and Γ's are positive. The region K_0 is the region of the magnetic field that we are interested in producing. The region K_1 contains the surface current distributions that are required to produce this field. The region K_2 represents all space outside of K_0 and K_1. The permeability of the regions K_0 and K_2 will be assumed to be the unity while that of K_1 will have a constant value μ. When an accelerator magnet is made up of a number of elementary magnets, the regions K_0 and K_1 of the elementary magnets will combine to form the regions K_0 and K_1 of the accelerator magnet.

The region K_1 of the elementary magnet will be subdivided into various subregions which can be identified by a three dimensional integral coordinate system. This is illustrated in figures 1-8. Figure 1a is a three dimensional diagram of an elementary magnet. Figure 1b represents the same view of the elementary magnet as figure 1a, but where the various regions of 1a have been pulled apart as if by an explosion. Figures 2,3, and 4 represent three azimuthal cross sections of the elementary magnet. Figures 5 and 6 represent projections on the median plane by cones γ = const. Figures 7 and 8 represent projections on azimuthal planes of cross sections of an elementary magnet by spheres. In these figures, each region is represented by a triple of numbers. Consider the

azimuthal cross sections at ϕ_0 where $\phi_{-1} < \phi_0 < \phi_1$ (figure 3). This azimuthal cross section cuts through the region K_0 which is denoted by (0,0,0). A point moving along a radial line (ψ = const ϕ = const) and starting from a point in (0,0,0) will enter into the region (1,0,0) and then into K_2 as r increases. Again starting in (0,0,0) and moving in the opposite direction along the same radial line this point will pass into (-1,0,0) and then into K_2. As r assumes the values r_2, r_1, r_{-1} and r_{-2} the point will cross surface current distributions. If a point starts in (0,0,0) and moves along a line (r = const., ψ = const.), it passed at $\phi = \phi_1$ into the region (0,0,1) and at $\phi = \phi_{-1}$ into the region (0,0,-1). On moving further in either direction it passes into K_2.

If $\phi_+ = \phi_{-1} + 2\pi$, then the region K_0 is a toroidal ring shaped region with the orgin of the spherical coordinate system at its center. This elementary magnet could be built out of elementary magnets, using them as building blocks. Using them in this way, an azimuthal cross section of the region K_0 of the completed magnet could be made in the figure



The azimuthal cross section could also be made to vary in azimuth.

When the various elemental magnets are fitted together, this must be done so that the region K_0 of the complete accelerators magnet is free of surface currents. This will be accompolished if azimuthal currents for one elemental magnet coincide with those of the adjoining magnet. While the same need not be true for the surface distribution on the γ and Γ surfaces, nevertheless, if this were done, the resulting current distributions would be greatly simplified. Each of these types or regions has its own characteristic form for the magnetic scalar potential. Furthermore, as soon as the magnetic scalar potential is known for the first order sub regions (a,0,0,) (0,b,0) and (0,0,c), the scalar potentials can at once be written down for the (0,b,c) (a,0,c) (a,b,0) and the (a.b.c) sub regions. Thus if the scalar potential is known for all the first order sub regions of the magnet, it can at once be written down for all the sub regions of the magnet and all the fields and surface current densities can be computed.

In order to simplify the notation, the following abreviations will be used in the following sections:

$$\begin{array}{lll} L_{(o)}^{(mN)}{}^{(\gamma)}{}_{\alpha+1} & \text{will be replaced by} & L_{(o)} \\ L_{(e)}^{(mN)}{}^{(\gamma)}{}_{\alpha+1} & \text{will be replaced by} & L_{(e)} \end{array} \qquad (1.6)$$

$$\frac{d}{dy} L_{(\theta)}^{(mN)}(y)_{\alpha+1} \quad \text{will be replaced by} \quad L'_{(\theta)}$$
$$\frac{d}{dy} L_{(\theta)}^{(mN)}(y)_{\alpha+1} \quad \text{will be replaced by} \quad L'_{(\theta)} \qquad (1.6)$$

The various subregions into which the K_1 region of an elementary magnet is divided may be classified into types as follows:

First order regions (a,0,0) are of type (r)

First order regions (0,b,c) are of type (y)

First order regions (0,0,c) are of type (ϕ)

Second order regions (a,b,c) are of type (r, y)

Second order regions (0,b,c) are of type (y, ϕ)

Second order regions (a,0,c) are of type (r, ϕ)

Third order regions (a,b,c) are of type (r, y, ϕ)

where the quantities a,b,c take in values +1 or -1.

SUBREGIONS OF TYPE (1)

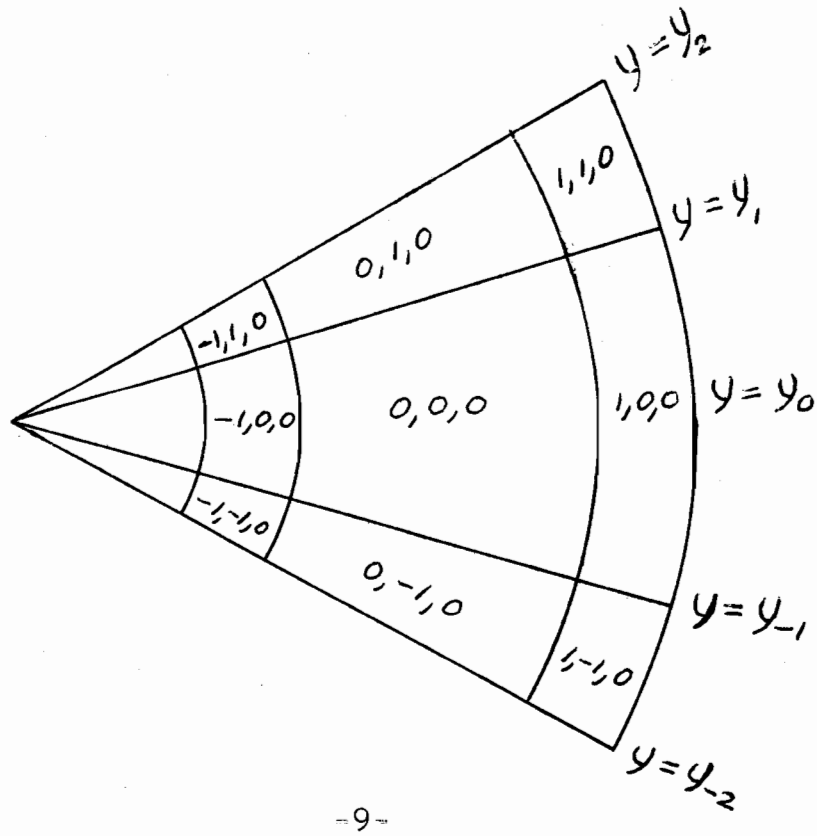
There are two subregions of this type, namely (1,0,0) and (-1,0,0). In order to find the fields in these regions, it is only necessary to know the fields in (0,0,0) and K_2. We start with the equation:

$$V^{(0,0,0)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} \; e^{-im N_\phi}$$

$$V^{(1,0,0)} = -r_0 H_0 R^{(+1)} \underset{(0)}{L} \; e^{-im N_\phi} \qquad (2.1)$$

$$V^{(-1,0,0)} = -r_0 H_0 R^{(-1)} \underset{(0)}{L} \; e^{-im N_\phi}$$

$$V^{(K_2)} = 0$$



where $$R^{(+1)} = v_{(+1)}^{(1,0,0)} \left(\frac{r}{r_0}\right)^{\alpha+1} + v_{(-1)}^{(1,0,0)} \left(\frac{r}{r_0}\right)^{-(\alpha+2)}$$

$$R^{(-1)} = v_{(+1)}^{(-1,0,0)} \left(\frac{r}{r_0}\right)^{\alpha+1} + v_{(-1)}^{(-1,0,0)} \left(\frac{r}{r_0}\right)^{-(\alpha+2)} \tag{2.2}$$

$$\bar{R}^{(+1)} = \frac{d\,R^{(+1)}}{d\left(\frac{r}{r_0}\right)} \quad ; \quad \bar{R}^{(-1)} = \frac{d\,R^{(-1)}}{d\left(\frac{r}{r_0}\right)}$$

where the v's are constants to be determines by boundary conditions then,

$$H_r^{(0,0,0)} = H_0 (\alpha+1) \left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L} e^{-imN\phi}$$

$$H_y^{(0,0,0)} = H_0 \left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'} e^{-imN\phi} \tag{2.3}$$

$$H_\phi^{(0,0,0)} \cos y = -imN H_0 \left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L} e^{-imN\phi}$$

$$H_r^{(1,0,0)} = H_0 \bar{R}^{(+1)} \underset{(0)}{L} e^{-imN\phi}$$

$$H_y^{(1,0,0)} = H_0 \left(\frac{r_0}{r}\right) R^{(+1)} \underset{(0)}{L'} e^{-imN\phi} \tag{2.4}$$

$$H_\phi^{(1,0,0)} = -imN H_0 \left(\frac{r_0}{r}\right) R^{(+1)} \underset{(0)}{L} e^{-imN\phi}$$

$$H_r^{(-1,0,0)} = H_0 \bar{R}^{(-1)} \underset{(0)}{L} e^{-imN\phi}$$

$$H_y^{(-1,0,0)} = H_0 \left(\frac{r_0}{r}\right) R^{(-1)} \underset{(0)}{L'} e^{-imN\phi} \tag{2.5}$$

$$H_\phi^{(-1,0,0)} \cos y = -imN \left(\frac{r_0}{r}\right) R^{(-1)} \underset{(0)}{L} e^{-imN\phi}$$

We shall as ume permeability 1 in K_o and permeability μ in K_1 In order that B_r be continuous at $r = r_1$ and $r = r_2$, we require

$$(\alpha+1)\, v_{(+1)} \left(\frac{r_1}{r_0}\right)^{\alpha} - (\alpha+2)\, v_{(-1)} \left(\frac{r_1}{r_0}\right)^{-(\alpha+3)} = \frac{1}{\mu}\left(\frac{r_1}{r_0}\right)^{\alpha+1}$$

$$(\alpha+1)\, v_{(+1)} \left(\frac{r_2}{r_0}\right)^{\alpha} - (\alpha+2)\, v_{(-1)} \left(\frac{r_2}{r_0}\right)^{-(\alpha+3)} = 0 \tag{2.6}$$

or

$$v_{(+1)} - \frac{\alpha+2}{\alpha+1}\left(\frac{r_1}{r_0}\right)^{-(2\alpha+3)} v_{(-1)} = \frac{1}{\mu} \tag{2.7}$$

$$v_{(+1)} - \frac{\alpha+2}{\alpha+1}\left(\frac{r_2}{r_0}\right)^{-(2\alpha+3)} v_{(-1)} = 0$$

then

$$v_{(+1)} = -\frac{1}{\mu}\,\frac{1}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1} \quad ; \quad v_{(-1)} = -\frac{1}{\mu}\,\frac{\alpha+1}{\alpha+2}\,\frac{\left(\frac{r_2}{r_0}\right)^{2\alpha+3}}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1} \tag{2.8}$$

Substituting (2.8) in (2.2), one finds

$$R^{(+1)} = -\frac{1}{\mu}\,\frac{\left(\frac{r}{r_0}\right)^{\alpha+1}}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1}\left\{1 + \frac{\alpha+1}{\alpha+2}\left(\frac{r_2}{r}\right)^{2\alpha+3}\right\} \tag{2.9}$$

and

$$\bar{R}^{(+1)} = -\frac{\alpha+1}{\mu}\,\frac{\left(\frac{r}{r_0}\right)^{\alpha}}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1}\left\{1 - \left(\frac{r_2}{r}\right)^{2\alpha+3}\right\} \tag{2.10}$$

In the same way, one can find

$$R^{(-1)} = -\frac{1}{\mu}\,\frac{\left(\frac{r}{r_0}\right)^{\alpha+1}}{\left(\frac{r_{-2}}{r_{-1}}\right)^{2\alpha+3}}\left\{1 + \frac{\alpha+1}{\alpha+2}\left(\frac{r_{-2}}{r}\right)^{2\alpha+3}\right\} \tag{2.11}$$

$$\bar{R}^{(-1)} = -\frac{\alpha+1}{\mu}\,\frac{\left(\frac{r}{r_0}\right)^{\alpha+1}}{\left(\frac{r_{-2}}{r_{-1}}\right)^{2\alpha+3}}\left\{1 - \left(\frac{r_{-2}}{r}\right)^{2\alpha+3}\right\} \tag{2.12}$$

Also
$$V^{(1,0,0)} = \frac{r_0 H_0}{\mu} \frac{\left(\frac{r}{r_0}\right)^{\alpha+1}}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1} \left\{ 1 + \frac{\alpha+1}{\alpha+2} \left(\frac{r_2}{r}\right)^{2\alpha+3} \right\} \Big|_{(0)} e^{-imN\phi} \tag{2.13}$$

$$V^{(-1,0,0)} = \frac{r_0 H_0}{\mu} \frac{\left(\frac{r}{r_0}\right)^{\alpha+1}}{\left(\frac{r_{-2}}{r_{-1}}\right)^{2\alpha+3} - 1} \left\{ 1 + \frac{\alpha+1}{\alpha+2} \left(\frac{r_{-2}}{r}\right)^{2\alpha+3} \right\} \Big|_{(0)} e^{-imN\phi} \tag{2.14}$$

The following quantities are useful for computing surface current densities,

$$R^{(+1)}_{(r_1)} = -\frac{1}{\mu} \frac{\left(\frac{r_1}{r_0}\right)^{\alpha+1}}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1} \left\{ 1 + \frac{\alpha+1}{\alpha+2} \left(\frac{r_2}{r_1}\right)^{2\alpha+3} \right\}$$

$$R^{(+1)}_{(r_2)} = -\frac{1}{\mu} \frac{\left(\frac{r_2}{r_1}\right)^{\alpha+1}}{\left(\frac{r_2}{r_1}\right)^{2\alpha+3} - 1} \left\{ 1 + \frac{\alpha+1}{\alpha+2} \left(\frac{r_2}{r_1}\right)^{2\alpha+3} \right\} \tag{2.15}$$

$$\bar{R}^{(+1)}_{(r_1)} = \frac{\alpha+1}{\mu} \left(\frac{r_{-1}}{r}\right)^{\alpha}$$

$$\bar{R}^{(+1)}_{(r_2)} = 0$$

The surface current densities on the two spherical surfaces $r = r_1$ and $r = r_2$ where $\gamma_{-1} \leq \gamma \leq \gamma_{+1}$ are given by

$$4\pi i y(r_1) = H_\phi^{(1,0,0)} - H_\phi^{(0,0,0)} = -imN H_0 \left[\left(\frac{r_0}{r}\right) R^{(+1)} - \left(\frac{r}{r_0}\right)^{\alpha} \right]_{(r=r_1)}^{(0)} e^{-imN\phi}$$

$$4\pi i_\phi(r_1) = H_y^{(0,0,0)} - H_y^{(1,0,0)} = H_0\left[\left(\frac{r}{r_0}\right)^{\alpha} - R^{(+1)}\left(\frac{r_0}{r}\right)\right]\underset{(0)}{L}\, e^{-imN\phi}\Big|_{(r=r_1)} \tag{2.16}$$

$$4\pi i_y(r_2) = -H_\phi^{(1,0,0)} = imNH_0\frac{r_0}{r_2}R^{(+1)}(r_2)\underset{(0)}{L}\, e^{-imN\phi}$$

$$4\pi i_\phi(r_2) = H_y^{(1,0,0)} = H_0\left(\frac{r_0}{r_2}\right)R^{(+1)}(r_2)\underset{(0)}{L'}\, e^{-imN\phi}$$

The corresponding densities on the spherical surfaces $r = r_{-1}$ and $r = r_{-2}$ where $y_{-1} \leq y \leq y_{+1}$ are given by:

$$4\pi i_y(r_{-1}) = H_\phi^{(0,0,0)} - H_\phi^{(-1,0,0)}$$

$$4\pi i_\phi(r_{-1}) = H_y^{(-1,0,0)} - H_y^{(0,0,0)} \tag{2.17}$$

$$4\pi i_y(r_{-2}) = H_\phi^{(-1,0,0)}$$

$$4\pi i_\phi(r_{-2}) = H_y^{(-1,0,0)}$$

SUBREGIONS OF TYPE (y)

There are two subregions of this type, namely (0,1,0) and (0,-1,0) To find the fields in these regions it is only necessary to know the fields in (0,0,0) and in K_2.

We therefore start with the equations:

$$V^{(0,0,0)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(o)}{L} e^{-imN\phi}$$

$$V^{(0,1,0)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \Gamma^{(1)} e^{-imN\phi} \tag{3.1}$$

$$V^{(0,-1,0)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \Gamma^{(-1)} e^{-imN\phi}$$

where

$$\Gamma^{(+1)} = v_o^{(0,1,0)} \underset{(o)}{L} + v_e^{(0,1,0)} \underset{(e)}{L}$$

$$\Gamma^{(-1)} = v_o^{(0,-1,0)} \underset{(o)}{L} + v_e^{(0,-1,0)} \underset{(e)}{L} \tag{3.2}$$

and

$$\bar{\Gamma}^{(+1)} = v_o^{(0,1,0)} \underset{(o)}{L'} + v_e^{(0,1,0)} \underset{(e)}{L'} \tag{3.3}$$

$$\bar{\Gamma}^{(-1)} = v_o^{(0,-1,0)} \underset{(o)}{L'} + v_e^{(0,-1,0)} \underset{(e)}{L'}$$

where the v's are constants that are determined by the boundary conditions.

Then

$$H_r^{(0,0,0)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} L_{(0)} e^{-imN\phi}$$

$$H_y^{(0,0,0)} = H_0\left(\frac{r}{r_0}\right)^{\alpha} L'_{(0)} e^{-imN\phi} \tag{3.4}$$

$$H_\phi^{(0,0,0)} \cos y = -imNH_0\left(\frac{r}{r_0}\right)^{\alpha} L_{(0)} e^{-imN\phi}$$

$$H_r^{(0,1,0)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \Gamma^{(+1)} L_{(0)} e^{-imN\phi}$$

$$H_y^{(0,1,0)} = H_0\left(\frac{r}{r_0}\right)^{\alpha} \Gamma^{(+1)} L_{(0)} e^{-imN\phi} \tag{3.5}$$

$$H_\phi^{(0,1,0)} \cos y = -imNH_0\left(\frac{r}{r_0}\right)^{\alpha} \Gamma^{(+1)} L_{(0)} e^{-imN\phi}$$

$$H_r^{(0,-1,0)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \Gamma^{(-1)} L_{(0)} e^{-imN\phi}$$

$$H_y^{(0,-1,0)} = H_0\left(\frac{r}{r_0}\right)^{\alpha} \Gamma^{(-1)} L_{(0)} e^{-imN\phi} \tag{3.6}$$

$$H_\phi^{(0,-1,0)} \cos y = -imNH_0\left(\frac{r}{r_0}\right)^{\alpha} \Gamma^{(-1)} L_{(0)} e^{-imN\phi}$$

$$H_r^{(R_2)} = H_y^{(R_2)} = H_\phi^{(R_2)} = 0$$

We are assuming permeability 1 in K_0 and μ in K_1. In order to preserve the continuity of B_y at $y = y_1$ and $y = y_2$, we set

$$v_0 L'_{(0)}(y_1) + v_{(e)} L'_{(e)}(y_1) = \frac{1}{\mu} L'_{(0)}(y_1) \tag{3.7}$$

$$v_0 L'_{(0)}(y_2) + v_{(e)} L'_{(e)}(y_2) = 0$$

Then
$$v_o = \frac{1}{\mu D} L'_{(o)}(\gamma_1)\, L'_{(e)}(\gamma_2) \tag{3.8}$$

$$v_e = -\frac{1}{\mu D} L'_{(o)}(\gamma_1)\, L'_{(o)}(\gamma_2)$$

where

$$D = \begin{vmatrix} L'_{(o)}(\gamma_1) & L'_{(e)}(\gamma_1) \\ L'_{(o)}(\gamma_2) & L'_{(e)}(\gamma_2) \end{vmatrix}$$

Then

$$\Gamma^{(0)} = \frac{1}{\mu D} L'_{(o)}(\gamma_1)\left[L'_{(e)}(\gamma_2)\, L_{(o)}(\gamma) - L'_{(o)}(\gamma_2)\, L_{(e)}(\gamma)\right] \tag{3.9}$$

In the same way we find

$$\Gamma^{(-1)} = \frac{1}{\mu D} L'_{(o)}(\gamma_{-1})\left[L'_{(e)}(\gamma_{-2})\, L_{(e)}(\gamma) - L'_{(o)}(\gamma_{-2})\, L_{(e)}(\gamma)\right] \tag{3.10}$$

Also the surface current densities on the conical surfaces $\gamma = \gamma_1$ and $\gamma = \gamma_2$ when $r_{-1} \leq r \leq r_{+1}$ are given by

$$4\pi i\, r(\gamma_1) = H^{(0,0,0)}_{\phi}(\gamma_1) - H^{(0,1,0)}_{\phi}(\gamma_1) = -\frac{imNH_o}{\cos\gamma}\left(\frac{r}{r_o}\right)^{\alpha}\left[L_{(o)}(\gamma_1) \quad (\gamma_1)\right] e^{-imN\phi} \tag{3.11}$$

$$4\pi i\, \phi(\gamma_1) = H^{(0,1,0)}_{r}(\gamma_1) - H^{(0,0,0)}_{r}(\gamma_1) = H_o\left(\frac{r}{r_o}\right)^{\alpha}\left[\Gamma^{(+1)}_{(\gamma_1)} - L'_{(o)}(\gamma_1)\right] e^{-imN\phi}$$

$$4\pi i\, r(\gamma_2) = H^{(0,-1,0)}_{\phi}(\gamma_2) = -\frac{imNH_o}{\cos\gamma}\left(\frac{r}{r_o}\right)^{\alpha}\Gamma^{(+1)}_{(\gamma_2)}\, e^{-imN\phi} \tag{3.12}$$

$$4\pi i\, \phi(\gamma_2) = -H^{(0,-1,0)}_{r}(\gamma_2) = H_o(\alpha+1)\left(\frac{r}{r_o}\right)^{\alpha}\Gamma^{(+1)}_{(\gamma_2)}\, e^{-imN\phi}$$

The corresponding surface current densities on the conical surfaces $y = y_{-1}$ and $y = y_{-2}$ where $r_{-1} \leq r \leq r_{+1}$ are given by

$$4\pi i_r(y_{-1}) = H_\phi^{(0,-1,0)}(y_{-1}) - H_\phi^{(0,0,0)}(y_{-1}) = -\frac{imNH_0}{\cos y}\left(\frac{r}{r_0}\right)^\alpha \left[\Gamma^{(-1)}_{(y_{-1})} - L_{(0)}(y_{-1})\right] e^{-imN\phi}$$

$$4\pi i_\phi(y_{-1}) = H_r^{(0,0,0)}(y_{-1}) - H_r^{(0,-1,0)}(y_{-1}) = H_0\left(\frac{r}{r_0}\right)^\alpha \left[L_{(0)}(y_{-1}) - \Gamma^{(-1)}_{(y_{-1})}\right] \tag{3.13}$$

$$4\pi i_r(y_{-2}) = -H_\phi^{(0,-1,0)}(y_{-2}) = \frac{imNH_0}{\cos y}\left(\frac{r}{r_0}\right)^\alpha \Gamma^{(-1)}_{(y_{-2})}\, e^{-imN\phi}$$

$$4\pi i_\phi(y_{-2}) = H_r^{(0,-1,0)}(y_{-2}) = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^\alpha \bar{\Gamma}^{(-1)}_{(y_{-2})}\, e^{-imN\phi} \tag{3.14}$$

In case we assume the approximations,

$$L_{(0)}(y) = \sin \omega y \qquad L_{(e)}(y) = \cos \omega y$$

$$L'_{(0)}(y) = \omega \cos \omega y \qquad L'_{(e)}(y) = -\omega \sin y \tag{3.15}$$

then

$$D = \frac{1}{\omega^2 \sin \omega(y_1 - y_2)} \tag{3.16}$$

$$\Gamma^{(-1)} \cong \frac{1}{\mu}\,\frac{\cos \omega y_1}{\sin \omega(y_2 - y_1)} \cos \omega(y_2 - y)$$

$$\bar{\Gamma}^{(-1)} \cong -\frac{\omega}{\mu}\,\frac{\cos \omega y_1}{\sin \omega(y_2 - y_1)} \sin \omega(y_2 - y) \tag{3.17}$$

and
$$\Gamma^{(-1)} \cong \frac{i}{\mu} \frac{\cos \omega \gamma_{-1}}{\sin \omega(\gamma_{-2} - \gamma_{-1})} \cos \omega(\gamma_{-2} - \gamma)$$
$$\Gamma^{(-1)} \cong \frac{i}{\mu} \frac{\cos \omega \gamma_{-1}}{\sin \omega(\gamma_{-2} - \gamma_{-1})} \sin \omega(\gamma_{-2} - \gamma) \tag{3.18}$$

The latter two can be obtained from the former simply be replacing (+1, +2) by (-1, -2), then

$$V^{(0,1,0)} \cong -\frac{r_0 H_0}{\mu} \left(\frac{r}{r_0}\right)^{\alpha+1} \frac{\cos \omega \gamma_1}{\sin \omega(\gamma_2 - \gamma_1)} \cos \omega(\gamma_2 - \gamma)\, e^{-imN\phi}$$
$$V^{(0,-1,0)} \cong -\frac{r_0 H_0}{\mu} \left(\frac{r}{r_0}\right)^{\alpha+1} \frac{\cos \omega \gamma_{-1}}{\sin \omega(\gamma_{-2} - \gamma_{-1})} \sin \omega(\gamma_{-2} - \gamma) \tag{3.19}$$

Likewise the magnetic field components and the surface current densities may easily be obtained by substituting (3.17) and (3.18) into (3.4), (3.5), (3.6), (3.11), (3.12), (3.13), and (3.14).

It should be noted than in the case[2] of a radial sector machine that a harmonic component of the field could be produced in (0,0,0) with zero field in K_2 by using a surface distribution only on the cones $\gamma = \gamma_1$ and $\gamma = \gamma_{-1}$, the current density in the cones $\gamma = \gamma_2$ and $\gamma = \gamma_{-2}$ being zeros, if the angles $\gamma = \gamma_1$ and $\gamma = \gamma_{-1}$ were chosen properly. It will now be shown that this is not possible in the case of the spiral ridge accelerator. In this case $V^{(0,1,0)} = 0$ and in order that H_γ be continuous at $\gamma = \gamma_1$, it follows that,

$$\left(\frac{r}{r_0}\right)^{\alpha+1} L'_{\alpha}(\gamma_1)\, e^{-imN\phi} = 0 \qquad \left(\frac{r}{r_0}, \phi\right) \tag{3.20}$$

Now as $\frac{r_0}{r}$ and ϕ change, the phase angle of $\left(\frac{r}{r_0}\right)^{\alpha+1} e^{-imN\phi}$ will change. Therefore $L'_{(0)}(y) = 0$, or using the approximation (1.4), we find

$$\cos \omega y_1 = 0 \tag{3.21}$$

If ω is real, this equation has real solutions for y_1 and one can then produce the field by a surface current distribution in the case $y = y_1$ where $\cos \omega y_1 = 0$. This is the case in a radial sector machine. But in a spiral ridge machine ω is not real but complex. In this case $\omega = a_1 + i a_2$ where a_1 and a_2 are real and $a_2 \neq 0$

Then

$$\cos a_1 y_1 \cosh a_2 y_1 = 0$$

$$\sin a_1 y_1 \sinh a_2 y_1 = 0 \tag{3.22}$$

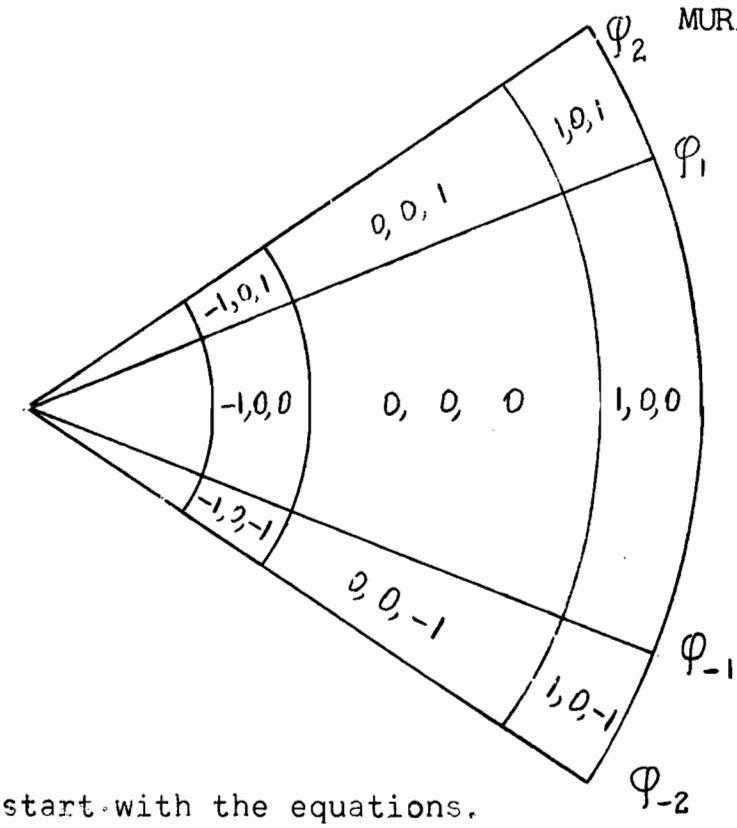
or $\cosh 2 a_2 y_1 = 0$

or $e^{2a_2 y_1} = -e^{-2a_2 y_1}$ or $e^{4a_2 y_1} = -1$

which has no real solution for y_1 Q.E.D.

SUBREGIONS OF TYPE (ϕ)

There are two subregions of this type, namely (0,0,1) and (0,0,-1). The fields in these regions are determined by the fields in (0,0,0) and K_2.



We therefore start with the equations.

$$V^{(0,0,0)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} e^{-im N_\phi}$$

$$V^{(0,0,1)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} \Phi^{(+1)} \tag{4.1}$$

$$V^{(0,0,-1)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} \Phi^{(-1)}$$

where $\Phi^{(+1)} = v_c^{(+1)} \cos m N_\phi + v_s^{(+1)} \sin m N_\phi$

$$\Phi^{(-1)} = v_c^{(-1)} \cos m N_\phi + v_s^{(-1)} \sin m N_\phi \tag{4.2}$$

where these v's are constants to be determined to satisfy certain boundary conditions.

Also set $\bar{\Phi}^{(+1)} = \dfrac{d\,\Phi^{(+1)}}{d\phi}$, $\bar{\Phi}^{(-1)} = \dfrac{d\,\Phi^{(-1)}}{d\phi}$ (4.3)

Then

$$H_r^{(0,0,0)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L} \left[\cos mN\phi - i \sin mN\phi\right]$$

$$H_y^{(0,0,0)} = H_0\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'} \left[\cos mN\phi - i \sin mN\phi\right] \tag{4.4}$$

$$H_\phi^{(0,0,0)} = -H_0 mN\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L} \left[\sin mN\phi + i \cos mN\phi\right]$$

$$H_r^{(0,0,1)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L}$$

$$H_y^{(0,0,1)} = H_0\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'}\, \Phi^{(+1)} \tag{4.5}$$

$$H_\phi^{(0,0,1)} \cos y = H_0\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L}\, \bar{\Phi}^{(+1)}$$

$$H_r^{(0,0,-1)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L}\, \Phi^{(-1)}$$

$$H_y^{(0,0,-1)} = H_0\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'}\, \Phi^{(-1)} \tag{4.6}$$

$$H_\phi^{(0,0,-1)} \cos y = H_0\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'}\, \bar{\Phi}^{(-1)}$$

The continuity of B_ϕ at $\phi = \phi_1$ and $\phi = \phi_2$ gives:

$$-mN\xi_c^{(+1)} \sin mN\phi_1 + mN\xi_s^{(+1)} \cos mN\phi = -\frac{mN}{\mu} \sin mN\phi_1 - \frac{imN}{\mu} \cos mN\phi_1 \tag{4.7}$$

$$-mN\xi_c^{(+1)} \sin mN\phi_1 + mN\xi_s^{(+1)} \cos mN\phi_2 = 0$$

From which one finds

$$v_c^{(+1)} = -\frac{i}{\mu} e^{-imN\phi_1} \frac{\cos mN\phi_2}{\sin mN(\phi_2-\phi_1)} \tag{4.8}$$

$$v_s^{(+1)} = -\frac{i}{\mu} e^{-imN\phi_1} \frac{\sin mN\phi_2}{\sin mN(\phi_2-\phi_1)}$$

In the same way, one can obtain

$$v_c^{(-1)} = -\frac{i}{\mu} e^{-imN\phi_{-1}} \frac{\cos mN\phi_{-2}}{\sin mN(\phi_{-2}-\phi_{-1})} \tag{4.9}$$

$$v_s^{(-1)} = -\frac{i}{\mu} e^{-imN\phi_1} \frac{\sin mN\phi_{-1}}{\sin mN(\phi_{-2}-\phi_{-1})}$$

$$\Phi^{(+1)} = -\frac{i}{\mu} \frac{e^{-imN\phi_1}}{\sin mN(\phi_2-\phi_1)} \cos mN(\phi_2-\phi)$$

$$\Phi^{(-1)} = -\frac{i}{\mu} \frac{e^{-imN\phi_{-1}}}{\sin mN(\phi_{-2}-\phi_{-1})} \cos mN(\phi_{-2}-\phi) \tag{4.10}$$

$$V^{(0,0,1)} = r_0 H_0 \frac{i}{\mu} \frac{e^{-imN\phi_1}}{\sin mN(\phi_2-\phi_1)} \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} \cos mN(\phi_2-\phi)$$

$$V^{(0,0,-1)} = r_0 H_0 \frac{i}{\mu} \frac{e^{-imN\phi_{-1}}}{\sin mN(\phi_{-2}-\phi_{-1})} \left(\frac{r}{r_0}\right)^{\alpha+1} \cos mN(\phi_{-2}-\phi) \tag{4.11}$$

The surface current densities in the azimuthal planes $\phi = \phi_1$ and $\phi = \phi_2$ where $r_{-1} \leq r \leq r_{+1}$, $y_{-1} \leq y \leq y_{+1}$ are given by

$$4\pi i_r(\phi_1) = H_y^{(0,0,1)} - H_y^{(0,0,0)} = H_0 \left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'} \left[\Phi^{(+1)}(\phi_1) - e^{-imN\phi_1}\right]$$

$$4\pi i_y(\phi_1) = H_r^{(0,0,0)} - H_r^{(0,0,1)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L} \left[-\Phi^{(+1)}(\phi_1) + e^{-imN\phi_1}\right] \tag{4.12}$$

$$4\pi i_r(\phi_2) = -H_y^{(0,0,1)} = -H_0 \left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L'} \Phi^{(+1)}(\phi_2)$$

$$4\pi i_y(\phi_2) = H_r^{(0,0,1)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^{\alpha} \underset{(0)}{L}$$

Likewise the surface current densities in the azimuthal planes $\phi = \phi_{-1}$ and $\phi = \phi_{-2}$ where $r_{-1} \leq r \leq r_{+1}$, $y_{-1} \leq y \leq y_{+1}$ are given by

$$
\begin{aligned}
4\pi i_r(\phi_{-1}) &= H_y^{(0,0,0)} - H_y^{(0,0,-1)} \\
4\pi i_y(\phi_{-1}) &= H_r^{(0,0,-1)} - H_r^{(0,0,0)} \\
4\pi i_r(\phi_{-2}) &= H_y^{(0,0,-1)} \\
4\pi i_y(\phi_{-2}) &= -H_r^{(0,0,-1)}
\end{aligned}
\tag{4.13}
$$

In the case of a radial sector machine, it was possible to produce a single harmonic component by one azimuthal surface distribution on each side of K_0 if the angles for these distributions were chosed properly. This is no longer possible for the spiral ridge accelerator, where a pair of azimuthal surface distributions are required on each side of K_0. The proof is very similar to the case of the (y) region which has been treated in the previous section. The proof follows. Assume that the surface distribution at $\phi = \phi_2$ is missing. Then

$$H_\phi^{(0,0,0)}(r, y, \phi_1) = 0 = -imNH_0\left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} e^{-imN\phi_1} \left(\frac{r}{r_0}, y\right)$$

therefore $e^{-imN\phi_1} = 0$

which has no real solution ϕ_1. Q.E.D.

SECOND AND THIRD ORDER REGIONS

The first order regions can be designated by (a,0,0) (0,b,0) and (0,0,c) where a,b, and c can take on the values +1 or -1. Likewise (0,b,c) (a,0,c) and (a,b,0) will designate second order and (a,b,c) third order regions. The potentials in the first order regions can be written.

$$V^{(a,0,0)} = -r_0 H_0 R^{(a)} \underset{(0)}{L} e^{-im N\phi}$$

$$V^{(0,b,0)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \Gamma^{(b)} e^{-im N\phi} \tag{5.1}$$

$$V^{(0,0,c)} = -r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \underset{(0)}{L} \Phi^{(c)}$$

where $R^{(\alpha)}$ are given by (2.9) and (2.11), $\Gamma^{(\alpha)}$ by (3.2) and $\Phi^{(\alpha)}$ by (4.10)

Since the normal components of B are continuous, it follows that

$$V^{(a,b,0)} = -\mu r_0 H_0 R^{(a)} \Gamma^{(b)} e^{-im N\phi}$$

$$V^{(0,b,c)} = -\mu r_0 H_0 \left(\frac{r}{r_0}\right)^{\alpha+1} \Gamma^{(b)} \Phi^{(c)}$$

$$V^{(a,0,c)} = -\mu r_0 H_0 R^{(a)} \underset{(0)}{L} \Phi^{(c)} \tag{5.2}$$

$$V^{(a,b,c)} = -\mu^2 r_0 H_0 R^{(a)} \Gamma^{(b)} \Phi^{(c)}$$

To illustrate:

$$V^{(1,0,-1)} = -\mu r_0 H_0 R^{(+1)} \underset{(0)}{L} \Phi^{(-1)}$$

$$V^{(-1,1,-1)} = -\mu^2 r_0 H_0 R^{(-1)} \Gamma^{(+1)} \Phi^{(-1)} \tag{5.3}$$

From these potentials, the fields in all the regions may be computed. Thus we find,

$$H_r^{(a,0,0)} = H_0 \bar{R}^{(a)} \underset{(0)}{L} e^{-imN\phi}$$

$$H_y^{(a,0,0)} = H_0 \left(\frac{r_0}{r}\right) R^{(a)} \underset{(0)}{L'} e^{-imN\phi} \qquad (5.4)$$

$$H_\phi^{(a,0,0)} \cos\gamma = -imNH_0 \left(\frac{r_0}{r}\right) R^{(a)} e^{-imN\phi}$$

$$H_r^{(0,b,0)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^\alpha \Gamma^{(b)} e^{-imN\phi}$$

$$H_y^{(0,b,0)} = H_0 \left(\frac{r}{r_0}\right)^\alpha \bar{\Gamma}^{(b)} e^{-imN\phi} \qquad (5.5)$$

$$H_\phi^{(0,b,0)} \cos\gamma = -imNH_0 \left(\frac{r}{r_0}\right)^\alpha \Gamma^{(b)} e^{-imN\phi}$$

$$H_r^{(0,0,c)} = H_0(\alpha+1)\left(\frac{r}{r_0}\right)^\alpha \underset{(0)}{L} \Phi^{(c)}$$

$$H_y^{(0,0,c)} = H_0 \left(\frac{r}{r_0}\right)^\alpha \underset{(0)}{L'} \Phi^{(c)} \qquad (5.6)$$

$$H_\phi^{(0,0,c)} \cos\gamma = H_0 \left(\frac{r}{r_0}\right)^\alpha \underset{(0)}{L} \bar{\Phi}^{(c)}$$

$$H_r^{(a,b,0)} = \mu H_0 \bar{R}^{(a)} \Gamma^{(b)} e^{-imN\phi}$$

$$H_y^{(a,b,0)} = \mu H_0 \left(\frac{r_0}{r}\right) R^{(a)} \bar{\Gamma}^{(b)} e^{-imN\phi} \qquad (5.7)$$

$$H_\phi^{(a,b,0)} \cos\gamma = -imN\mu H_0 \left(\frac{r_0}{r}\right) R^{(a)} \Gamma^{(b)} e^{-imN\phi}$$

$$H_r^{(0,b,c)} = \mu H_0(\alpha+1)\left(\frac{r}{r_0}\right)^\alpha \Gamma^{(b)} \Phi^{(c)}$$

$$H_y^{(0,b,c)} = \mu H_0 \left(\frac{r}{r_0}\right)^\alpha \bar{\Gamma}^{(b)} \Phi^{(c)} \qquad (5.8)$$

$$H_\phi^{(0,b,c)} \cos\gamma = \mu H_0 \left(\frac{r}{r_0}\right)^\alpha \Gamma^{(b)} \bar{\Phi}^{(c)}$$

$$H_r^{(a,0,c)} = \mu H_0 \bar{R}^{(a)} \underset{(0)}{L} \Phi^{(c)}$$

$$H_y^{(a,0,c)} = \mu H_0 \left(\frac{r_0}{r}\right) R^{(a)} \underset{(0)}{L'} \Phi^{(c)} \qquad (5.9)$$

$$H_\phi^{(a,0,c)} \cos\gamma = \mu H_0 \left(\frac{r_0}{r}\right) R^{(a)} \underset{(0)}{L} \bar{\Phi}^{(c)}$$

$$H_r^{(a,b,c)} = \mu^2 H_0 \bar{R}^{(a)} \Gamma^{(b)} \Phi^{(c)}$$

$$H_y^{(a,b,c)} = \mu^2 H_0 \left(\frac{r_0}{r}\right) R^{(a)} \bar{\Gamma}^{(b)} \Phi^{(c)} \qquad (5.10)$$

$$H_\phi^{(a,b,c)} \cos\gamma = \mu^2 H_0 \left(\frac{r_0}{r}\right) R^{(a)} \Gamma^{(b)} \bar{\Phi}^{(c)}$$

From the magnetic field strengths, the surface current densities at the boundaries between the various regions may be computed.

CONCLUDING REMARKS

1. Magnetic fields are always produced in practice by volume currents. Surface currents distributions may be of interest as a first approximation to volume currents. This approximation would only be expected to be good if the thickness of the volume current distribution is sufficiently small. For most accelerators, such thin volume current distributions would require impractically high current densities. nevertheless, the methods and results of this paper could be useful in the following ways:

(a) for the design of cryogenic magnets where sufficiently high volume current densities might be achieved so that the surface current would form a valid approximation.

(b) As a very rough zeroth approximation which might be useful in rough computations.

(c) This paper should help clarify certain parts of another paper to appear shortly on the production of spiral ridge fields by volume current distributions. In particular the figures corresponding to (1a) and (1b) for the volume current distributions are very difficult to draw in such a way as to make them inteligible. Figures 1a and 1b will be useful in this case.

2. It should be remembered that the analytical expressions for the fields and surface currents are complex. The real and pure imaginary parts of these fields appear much more complicated than the complex expressions.

3. No attempt has been made in this report to represent graphically the surface current densities required, or to try to construct more practical surface current distributions by adding several harmonic components of the field represented by (1.1).

4. It has been assumed that the permeability is μ everywhere in K_1. This can be easily modified. For instance, one could assume that in K_1 the permeability is everywhere μ, except in the regions (a,b,c) where it is unity. It is only necessary to modify the expression for $V^{(a,b,c)}$ accordingly.

REFERENCES

(1) Minutes of the MURA Technical Group staff meeting, April 29,1957 Pages 2-6.

(2) The production by volume current distributions of magnetic fields which are represented by spherical harmonics in a current free region, which encloses part of the median plane. MURA Report No. 339, August, 1957.

(3) The magnetic field and the equations of motion in spherical coordinates for the Mark V FFAG accelerator. Page 11, 12, 13. MURA/ESA. - 4. May, 1956.

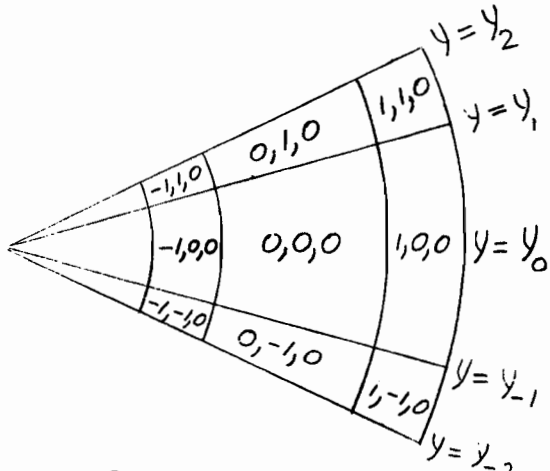


Figure 2. CROSS SECTION AT AZIMUTHAL PLANE φ_0 WHERE $\varphi_{-1} < \varphi_0 < \varphi_{+1}$

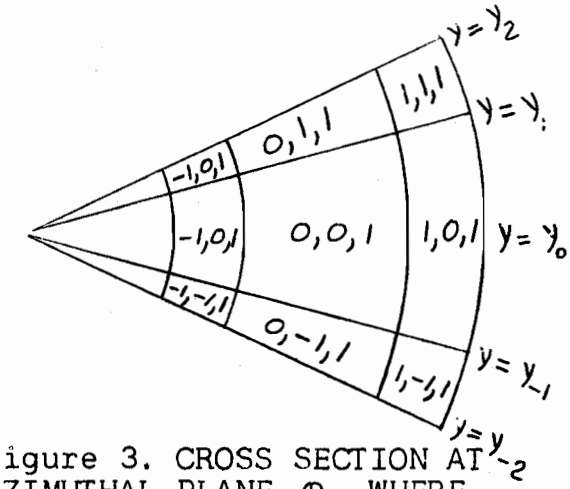


Figure 3. CROSS SECTION AT AZIMUTHAL PLANE φ_0 WHERE $\varphi_{+1} < \varphi_0 < \varphi_{+2}$

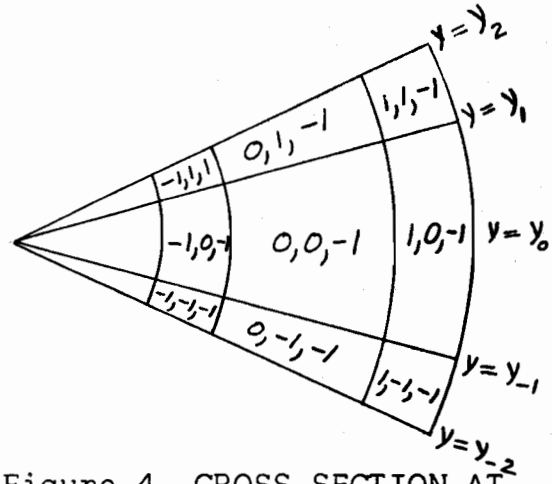


Figure 4. CROSS SECTION AT AZIMUTHAL PLANE φ_0 WHERE $\varphi_{-2} < \varphi_0 < \varphi_{-1}$

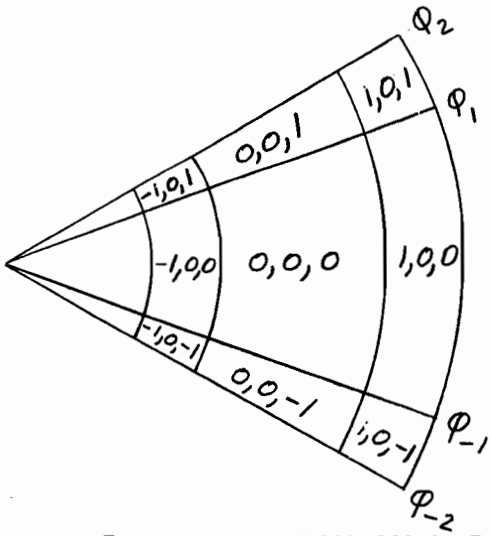


Figure 5. PROJECTION ON MEDIAN PLANE OF CROSS SECTION OF ELEMENTARY MAGNET BY CONE $y = y_0$ WHERE $y_{-1} < y_0 < y_{+1}$

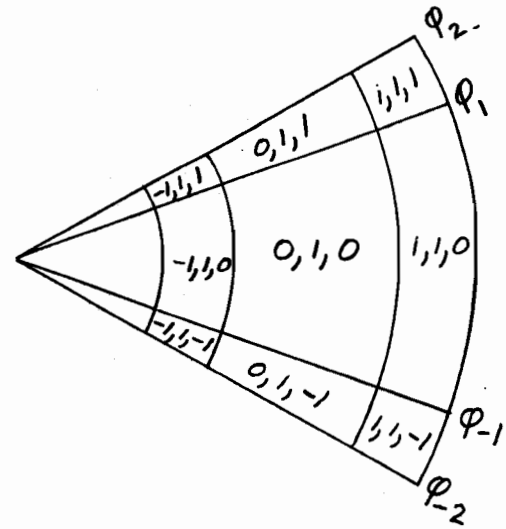


Figure 6. PROJECTION ON MEDIAN PLANE OF CROSS SECTION OF ELEMENTARY MAGNET BY CONE $y = y_0$ WHERE $y_1 < y_0 < y_2$

0,1,-1	0,1,0	0,1,1	y_2
0,0,-1	0,0,0	0,0,1	y_1
0,-1,-1	0,-1,0	0,-1,1	y_{-1}
φ_{-2}	φ_{-1}	φ_1	φ_2 y_{-2}

Figure 7. PROJECTION ON AN AZIMUTHAL PLANE OF THE CROSS SECTION OF ELEMENTARY MAGNET BY SPHERE $r = r_0$ WHERE $r_{-1} < r_0 < r_{+1}$

1,1,-1	1,1,0	1,1,1	y_2
1,0,-1	1,0,0	1,0,1	y_1
1,-1,-1	1,-1,0	1,-1,1	y_{-1}
φ_{-2}	φ_{-1}	φ_1	φ_2 y_{-2}

Figure 8. PROJECTION ON AN AZIMUTHAL PLANE OF THE CROSS SECTION OF ELEMENTARY MAGNET BY SPHERE $r = r_0$ WHERE $r_{+1} < r_0 < r_{+2}$

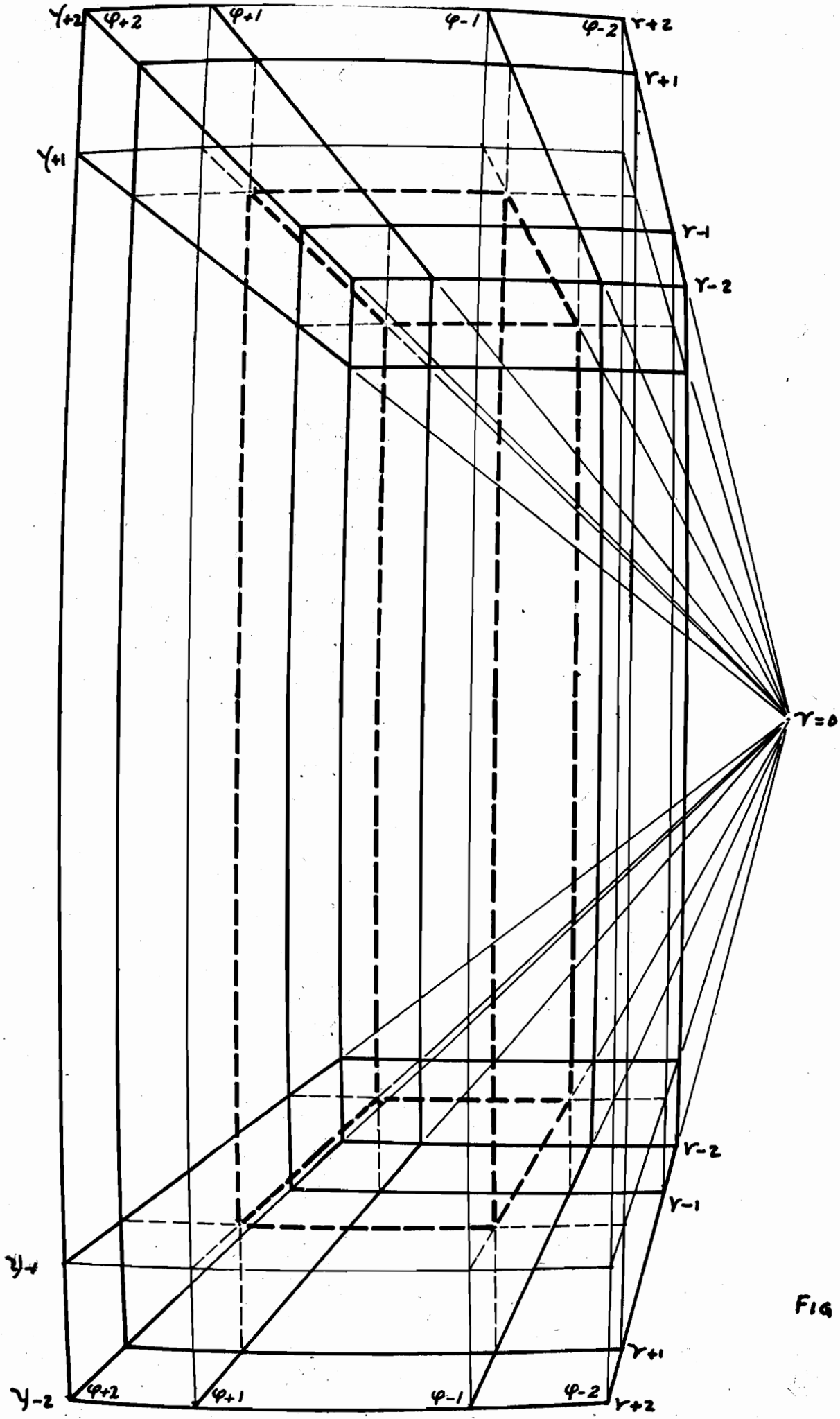


FIG 11

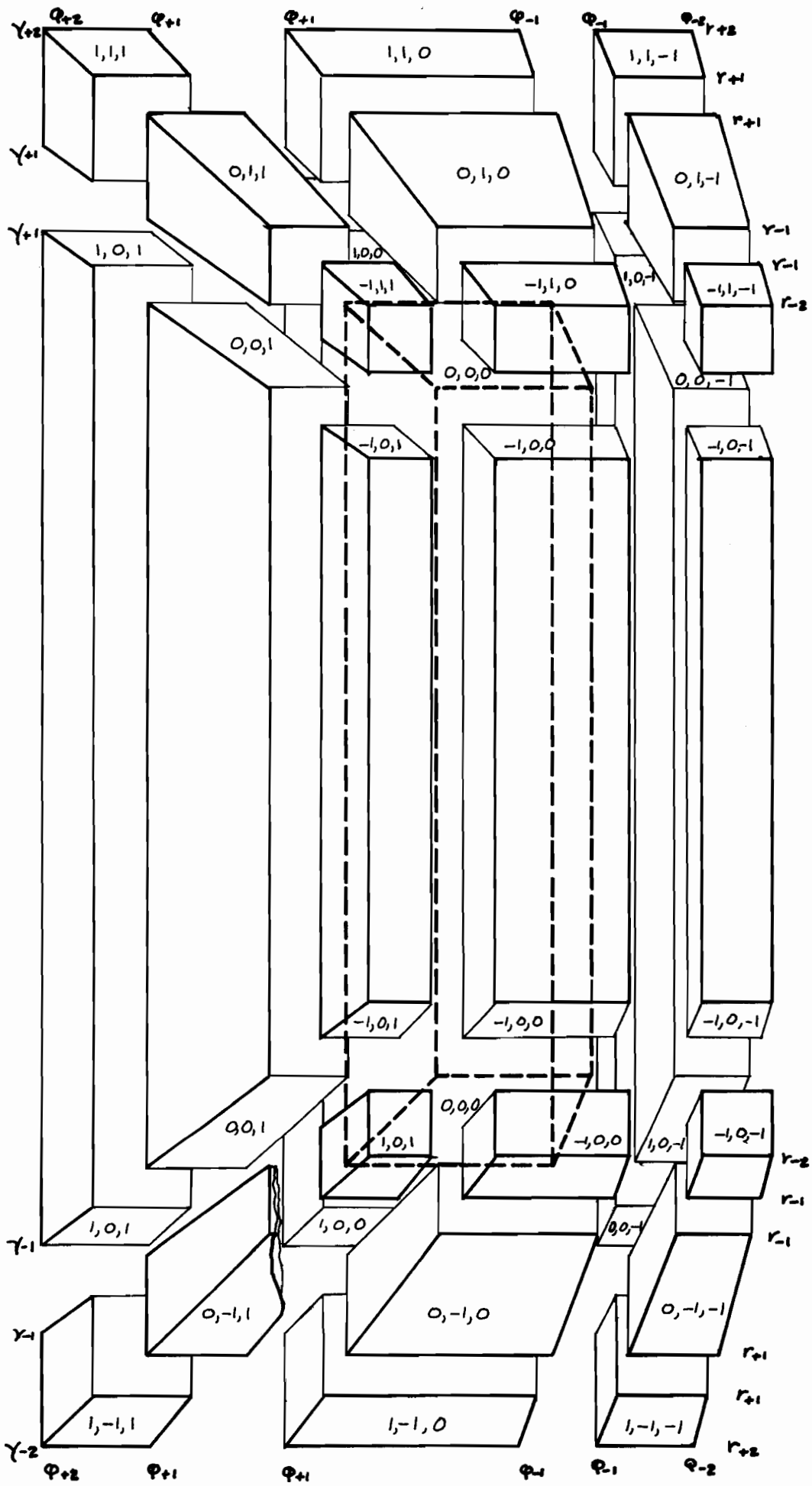

1,1,1
1,1,0
1,1,-1
0,1,1
0,1,0
0,1,-1
1,0,1
1,0,0
1,0,-1
-1,1,1
-1,1,0
-1,1,-1
0,0,1
0,0,0
0,0,-1
-1,0,1
-1,0,0
-1,0,-1
0,0,1
1,0,1
-1,0,0
-1,0,-1
1,0,1
1,0,0
0,0,-1
0,-1,1
0,-1,0
0,-1,-1
1,-1,1
1,-1,0
1,-1,-1

Fig 1B