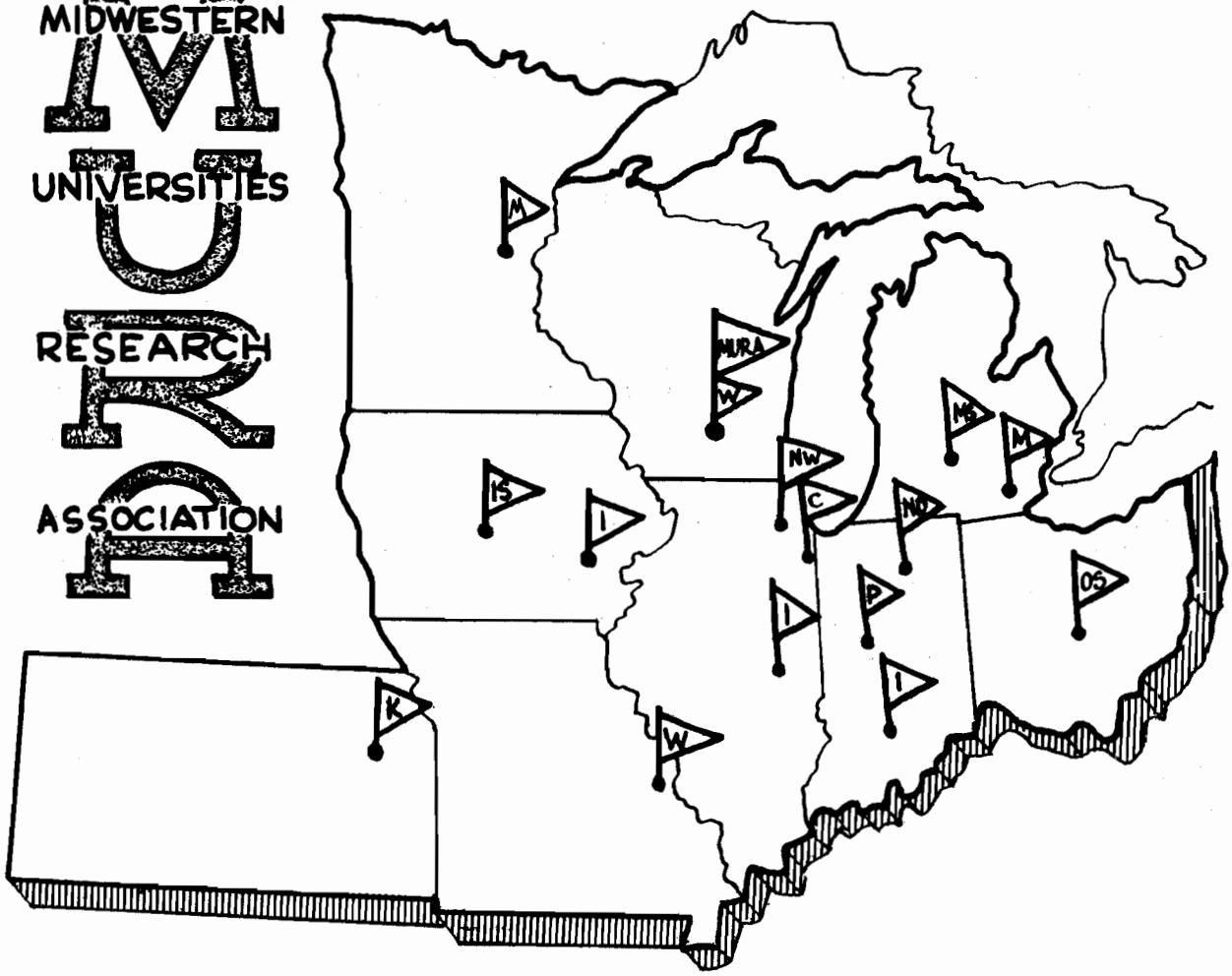
567

Fermilab Library
0 1160 0038853 2

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



SCHWARZ- CHRISTOFFEL TRANSFORMATIONS PERTAINING TO MAGNET EDGES OR PEELERS

L. Jackson Laslett

REPORT **NUMBER** 567

MURA-567
Internal

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

SCHWARZ - CHRISTOFFEL TRANSFORMATIONS PERTAINING TO MAGNET EDGES OR PEELERS

L. Jackson Laslett**

April 5, 1960

ABSTRACT

Schwarz-Christoffel transformations pertaining to the following two-dimensional electrostatic or magnetostatic situations are presented and simple formulas given for the consequent field in the median plane:

1a. Thin, parallel electrodes
b. Thick, parallel poles
c. Step poles
d. Neighboring, opposed magnet poles
2. Field between parallel poles in the presence of a magnetic shunt.

In Figs. 1-4 graphs are shown to illustrate, in representative cases, the spatial variation of the median-plane field in these various situations.

*AEC Research and Development Report. Supported by the Atomic Energy Commission, Contract No. AEC AT(11-1)-384.

**Department of Physics and Institute for Atomic Research, Iowa State University, Ames, Iowa.

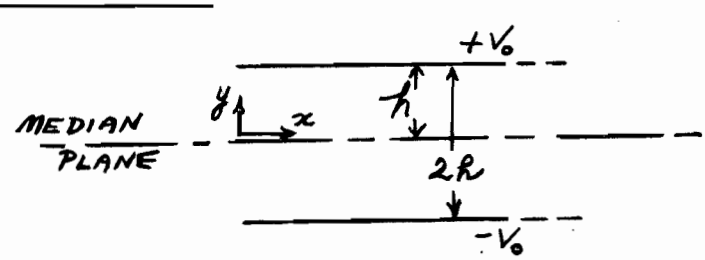
A. INTRODUCTION

It has proven of interest on occasion in the past to write down Schwarz-Christoffel transformations pertaining to two-dimensional electrostatic or magnetostatic situations descriptive of MURA technical problems. Among the possible applications may be mentioned the evaluation of the fringe field at a magnet edge or of the shielding provided by a magnetic shunt ("peeler") of assumed high permeability. It is the purpose of the present report to record the transformations which have been obtained and the specific relations for the median-plane field which follow from these transformations.

B. THE TRANSFORMATIONS

1. Field Between Magnet Poles (or Electrodes) of Various Shapes

(a) Thin, parallel electrodes:



In this case, which has also been treated by Jeans[1], the appropriate transformation for the depicted configuration is

$$z = \frac{h}{\pi}\left[\frac{\pi W}{V_o} - (1 + e^{-\pi W/V_o})\right],$$

where $z = x + iy$ and $W = U + iV$;

hence

$$x = \frac{h}{\pi} \left[\frac{\pi U}{V_o} - 1 - e^{-\pi U/V_o} \cos \frac{\pi V}{V_o} \right]$$

$$y = \frac{h}{\pi} \left[\frac{\pi V}{V_o} + e^{-\pi U/V_o} \sin \frac{\pi V}{V_o} \right].$$

The magnitude of the field strength is

$$|H| = \left| {}^{dW}/_{dz} \right| = \left| {}^{dz}/_{dW} \right|^{-1}$$

$$= \frac{V_o}{h} \frac{1}{\left| 1+e^{-\pi W/V_o} \right|} = \frac{H_o}{\left| 1+e^{-\pi W/V_o} \right|} ,$$

where H_o denotes the limiting value of the field as attained at points well to the right between the plates. The field strength in the median plane is, specifically,

$$\frac{H}{H_o} = \frac{1}{\left| 1+e^{-\pi U/V_o} \right|}$$

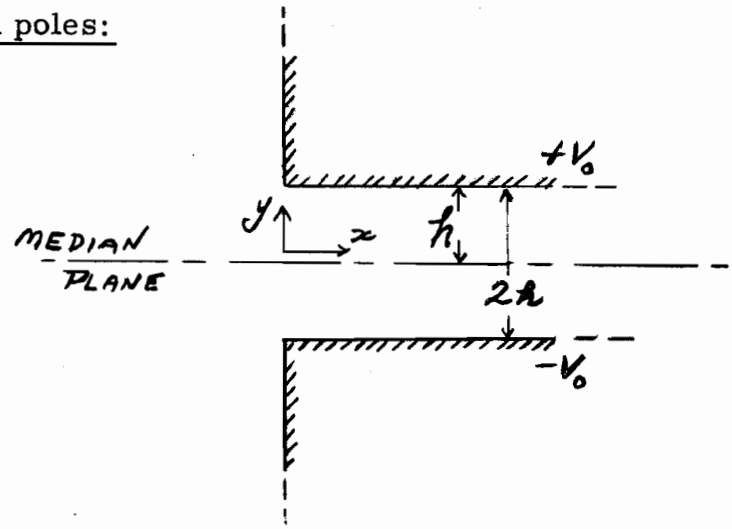
and

$$\frac{x}{h} = \frac{1}{\pi} \left[\frac{\pi U}{V_o} - (1+e^{-\pi U/V_o}) \right]$$

$$= -\frac{1}{\pi} \left[\ln \left(\frac{1 - H/H_o}{H/H_o} \right) + \frac{1}{H/H_o} \right] .$$

A graph of this relation, showing H/H_o vs. x/h, is given in Fig. 1 [curve (a)] .

(b) Thick, parallel poles:



In this case we make a transformation to an intermediate, z_1 - plane

$$z = -\frac{h}{\pi}\left[2\sqrt{1-z_1} + \ln \frac{\sqrt{1-z_1} - 1}{\sqrt{1-z_1} + 1} \right],$$

for which the connection with the potential function (W) is

$$W = -\frac{V_o}{\pi}\left(\ln z_1 - i\pi \right).$$

The field strength, then, is

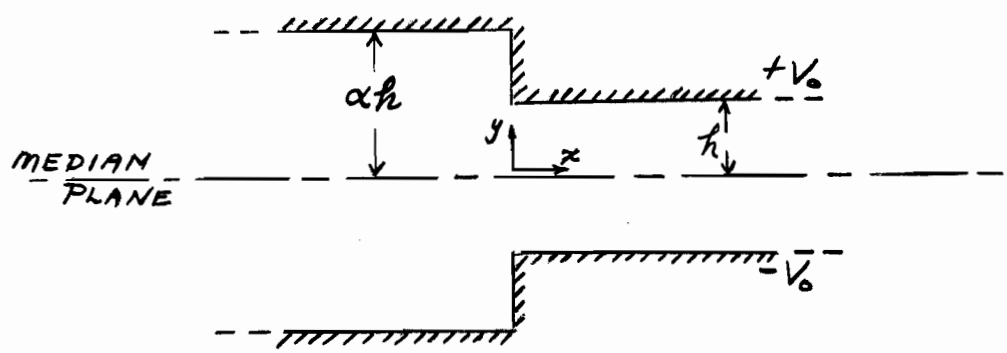
$$|H| = \frac{|dW/dz_1|}{|dz/dz_1|} = \frac{V_o}{h}\frac{1}{|\sqrt{1-z_1}|} = \frac{H_o}{|\sqrt{1-z_1}|} .$$

Accordingly the relation between the field and the location in the median plane may be expressed in the form

$$\frac{x}{h} = -\frac{1}{\pi}\left[\ln\left(\frac{1-H/H_o}{1+H/H_o}\right) + \frac{2}{H/H_o} \right] .$$

This relation in plotted in Fig. 1 [curve (b)] . The result may be helpful in providing information concerning the fringing field at the edge of magnet poles which can be treated as individually at constant magnetostatic potential.

(c) Step - poles:



This problem first arose in the MURA work in connection with tests of a Hall-effect field-gradient meter.[2] The same pole configuration has been treated by Ramo and Whinnery.[3] We introduce a transformation to an intermediate z_1 - plane

$$z = \frac{h}{\pi}\left[\cosh^{-1}\frac{2z_1 + \alpha + 1/\alpha}{\alpha - 1/\alpha} - \alpha\cosh^{-1}\frac{2/z_1 + \alpha + 1/\alpha}{\alpha - 1/\alpha}\right]$$

and relate z_1 to the potential function by

$$W = \frac{V_o}{\pi}\ln z_1 .$$

The field strength is then found as

$$|H| = \frac{|dW/dz_1|}{|dz/dz_1|} = \frac{V_o}{h}\sqrt{\frac{z_1 + 1/\alpha}{z_1 + \alpha}} = H_o\sqrt{\frac{z_1 + 1/\alpha}{z_1 + \alpha}} ,$$

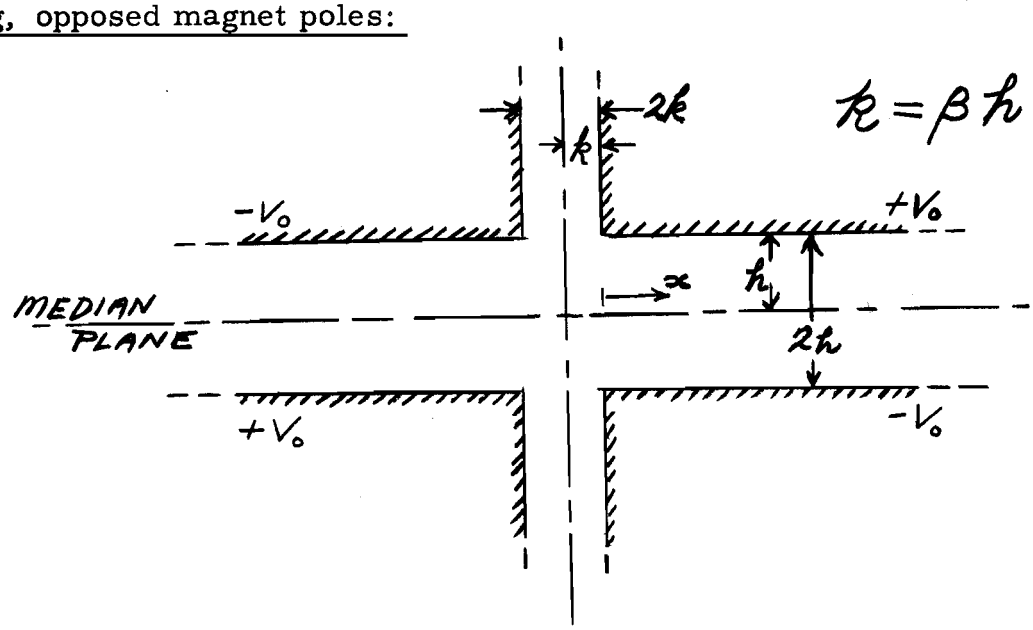
where H_o denotes the field strength prevailing well into the gap of half-aperture h. By elimination of z_1, a direct relation is obtained between the position and field in the median plane:

$$\frac{x}{h} = \frac{1}{\pi}\left[\cosh^{-1}\frac{1+(H/H_o)^2}{1-(H/H_o)^2} - \alpha\cosh^{-1}\frac{(H/H_o)^2+1/\alpha^2}{(H/H_o)^2-1/\alpha^2}\right]$$

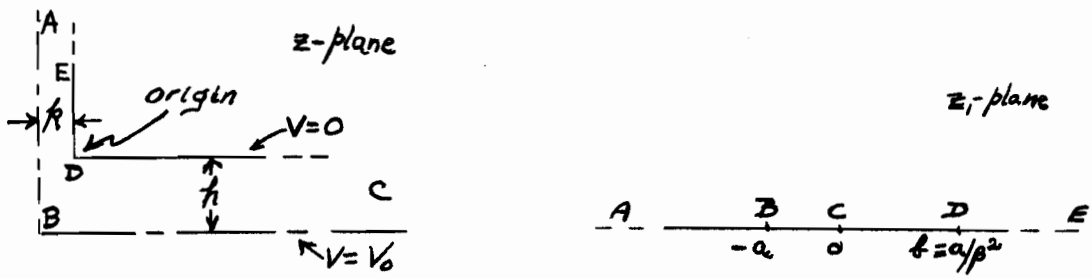
$$= -\frac{1}{\pi}\left[\ln\frac{1-H/H_o}{1+H/H_o} + \alpha\ln\frac{H/H_o+1}{H/H_o-1}\right] .$$

For $\alpha \to \infty$, this relation approaches, as it should, that found in (b) for thick poles. Results are plotted in Fig. 1 [curves (c)] for $\alpha = 2, 4,$ and 10.

(d) Neighboring, opposed magnet poles:



This configuration may be considered as representative of adjacent magnet blocks in a reversed-field structure. It is convenient to transform one quadrant of the z-plane into the (upper half) of the z_1 - plane,[3] and to shift the origin of y and V to avoid inconsequential additive constants.



We then find

$$z = i\frac{h}{\pi}\left[\beta \ln\frac{\sqrt{z_1/a+1}+\sqrt{z_1/a-1/\beta^2}}{\sqrt{z_1/a+1}-\sqrt{z_1/a-1/\beta^2}} - 2\tan^{-1}\beta\sqrt{\frac{z_1/a-1/\beta^2}{z_1/a+1}}\right]$$

with

$$W = \frac{V_o}{\pi}\ln\frac{z_1}{a} .$$

For the region of interest, in the median plane between B and C, it is convenient to introduce $t = -z_1/a$, with $0 < t \leq 1$.

Then

$$|H| = \frac{|dW/dz_1|}{|dz/dz_1|} = \frac{V_o}{h}\left|\sqrt{\frac{z_1+a}{\beta^2 z_1 - a}}\right|$$

$$= \frac{V_o}{h}\sqrt{\frac{1-t}{1+\beta^2 t}} = H_o\sqrt{\frac{1-t}{1+\beta^2 t}}$$

and

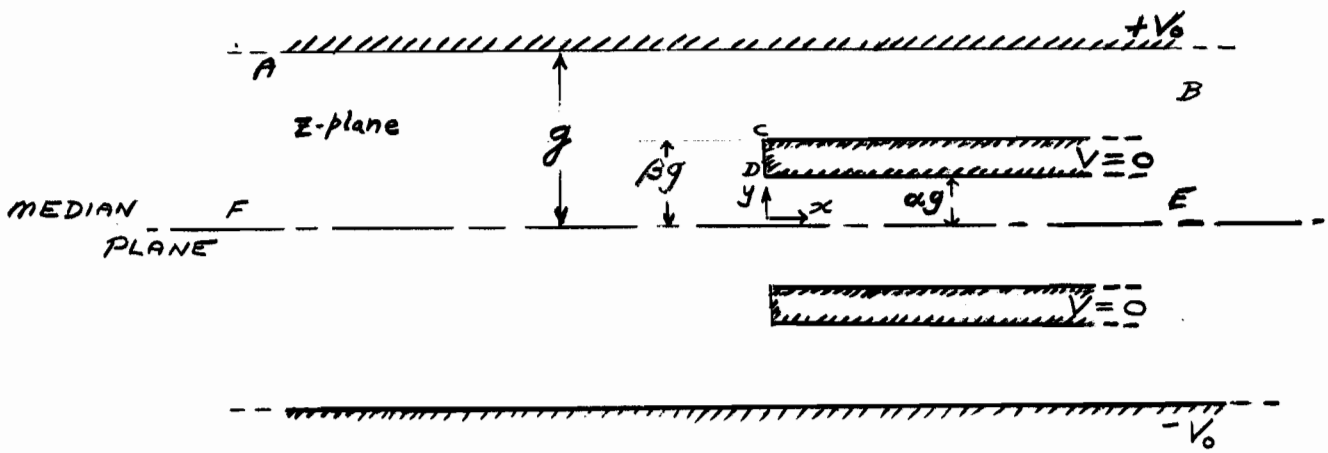
$$\frac{x}{h} = \frac{1}{\pi}\left[\ln\frac{\sqrt{1+\beta^2 t}+\sqrt{1-t}}{\sqrt{1+\beta^2 t}-\sqrt{1-t}} - 2\beta\tan^{-1}\frac{1}{\beta}\sqrt{\frac{1+\beta^2 t}{1-t}}\right]$$

$$= -\frac{1}{\pi}\left[\ln\frac{1-H/H_o}{1+H/H_o} + 2\beta\tan^{-1}\frac{1}{\beta(H/H_o)}\right] .$$

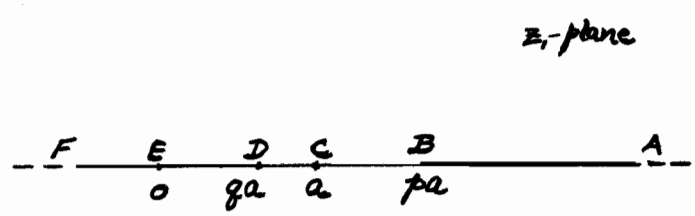
It is seen that with $\beta \to \infty$, this result correctly approaches that found in (b) for thick poles. Results are plotted in Fig. 2 for $\beta = 0, 1, 2, 4,$ and 10.

2. Field Between Parallel Poles in the Presence of a Magnetic Shunt

(a) General case:



In this situation the material at V = 0 may typify a magnetic shunt or peeler, into which a beam may be driven to implement extraction from a circular accelerator.[4]



It is convenient to transform the upper half of the region of interest into the upper portion of an intermediate, z_1 plane. A suitable form for the transformation appears to be

$$z = \frac{2g}{\pi}\left\{\alpha \cosh^{-1}\sqrt{\frac{z_1 - qa}{(1-q)\,z_1}} + (1-\beta)\cosh^{-1}\sqrt{\frac{p-q}{1-q}\cdot\frac{z_1 - a}{z_1 - pa}} - \cosh^{-1}\sqrt{\frac{z_1 - qa}{(1-q)\,a}}\right\} + ig$$

with

$$W = -\frac{V_o}{\pi}\ln\left(\frac{z_1 - pa}{a}\right) + iV_o,$$

where p and q are parameters such that

$$\frac{\sqrt{q}}{p} = \alpha \qquad \text{and} \qquad \frac{\sqrt{p-1}\,\sqrt{p-q}}{p} = 1-\beta .$$

Then

$$|H| = \frac{|dW/dz_1|}{|dz/dz_1|} = \frac{V_o}{g}\left|\frac{z_1}{\sqrt{z_1-a}\,\sqrt{z_1-qa}}\right| = H_o\left|\frac{z_1}{\sqrt{z_1-a}\,\sqrt{z_1-qa}}\right|$$

Again, since the median plane is the region of primary interest, it is convenient to introduce the real, positive variable $t = -z_1/a$; we thus have the following relationships, connecting parametrically the field strength and location in the median plane:

$$\frac{H}{H_o} = \frac{t}{\sqrt{(t+1)\,(t+q)}}$$

$$\frac{x}{g} = \frac{2}{\pi}\left\{ \alpha \cosh^{-1}\sqrt{\frac{t+q}{(1-q)\,t}} + (1-\beta)\cosh^{-1}\sqrt{\frac{p-q}{1-q}\cdot\frac{t+1}{t+p}} - \cosh^{-1}\sqrt{\frac{t+1}{1-q}}\right\}.$$

Graphs of this relationship are presented in Figs. 3 and 4, for $\beta-\alpha = 0.05$ and $\beta = 0.50$ or 0.10, respectively. Values of q and p for some representative values of α and β are given in Table I.

(b) Special cases:

The result just obtained may be checked in certain special cases.

(i) $\alpha = 0$:

$$q = 0, \qquad p = \frac{1}{1-(1-\beta)^2};$$

$$\frac{x}{g} = \frac{2}{\pi}\left\{(1-\beta)\cosh^{-1}\sqrt{\frac{t+1}{[1-(1-\beta)^2]t+1}} - \cosh^{-1}\sqrt{t+1}\right\}$$

$$= \frac{2}{\pi}\left\{(1-\beta)\cosh^{-1}\sqrt{\frac{1}{1-(1-\beta)^2\,(H/H_o)^2}} - \cosh^{-1}\sqrt{\frac{1}{1-(H/H_o)^2}}\right\}$$

$$= \frac{2}{\pi}\left\{(1-\beta)\tanh^{-1}\left[(1-\beta)\,(H/H_o)\right] - \tanh^{-1}(H/H_o)\right\}.$$

A graph illustrating this special case, for $\beta = 0.10$, is included on Fig. 4 (dashed curve).

(ii) $\alpha = 0$ and $\beta = 1$:

From the above, by setting $\beta = 1$,

$$\frac{x}{g} = -\frac{2}{\pi}\tanh^{-1}(H/H_o).$$

These special results may be checked by an independent direct calculation.

(iii) $\alpha = \beta$ (i.e., shield is of zero thickness):

$$q = 1, \qquad p = 1/\alpha = 1/\beta;$$

$$\frac{x}{g} = \frac{1}{\pi}\left[(1-\alpha)\ln\frac{1-\alpha}{1+\alpha t} - \alpha\ln t\right]$$

$$= \frac{1}{\pi}\left[(1-\alpha)\ln\frac{1-H/H_o}{\left(\frac{1}{1-\alpha}\right)-H/H_o} - \alpha\frac{H/H_o}{1-H/H_o}\right].$$

(iv) $\alpha = \beta$, with $\alpha \to 0$ and $\alpha g \to h$

(corresponding to thin sheets serving as shields at distances $\pm$ h from the median plane):

$$\frac{x}{h} = -\frac{1}{\pi}\left[\frac{1}{1 - H/H_o} + \ln \frac{H/H_o}{1 - H/H_o}\right].$$

[It may be noted that if we take a uniform field, of strength H_o directed downward, and substract from it the fields of the last special example (iv), we obtain a situation of the type described earlier (in Sect. la) under the designation "Thin, parallel electrodes". Mathematically, we may make this correspondence by replacing H/H_o in the above by $1 - H/H_o$, to obtain

$$\frac{x}{h} = -\frac{1}{\pi}\left(\frac{1}{H/H_o} + \ln \frac{1 - H/H_o}{H/H_o}\right),$$ as found in Sect. la.]

(v) α and $\beta \to 0$, with $\beta g = H$ and $\alpha g = \gamma H$:

$$\gamma = \alpha/\beta = 2\,\frac{\sqrt{q}}{q+1}, \qquad p \to \infty$$

$$\frac{x}{H} = -\frac{2}{\pi}\left[\frac{\sqrt{t+1}\cdot\sqrt{t+q}}{q+1} + \cosh^{-1}\sqrt{\frac{t+1}{t-q}} - \gamma\cosh^{-1}\sqrt{\frac{t+1}{(1-q)\,t}}\right],$$

as may be derived directly by an independent calculation. [In the limit that $\gamma \to 1$ ($\alpha \to \beta$ and $q \to 1$), this last result reduces to

$$\frac{x}{H} = -\frac{1}{\pi}\,(t + 1 + \ln t)$$

$$= -\frac{1}{\pi}\left(\frac{1}{1-H/H_o} + \ln \frac{H/H_o}{1 - H/H_o}\right),$$ in agreement with example (iv).]

TABLE I.

Values of q and p for Representative α and β.

β	$\beta-\alpha$	α	q	p
1	0	1	1	1
	0.05	0.95	0.9025	1
	0.5	0.5	0.25	1
	1	0	0	1
0.5	0	0.5	1	2
	0.05	0.45	0.506691915	1.58182878
	0.1	0.4	0.354463995	1.48842197
	0.25	0.25	0.118426251	1.37652462
	0.5	0	0	4/3
0.25	0	0.25	1	4
	0.05	0.20	0.293547024	2.70899900
	0.10	0.15	0.137611531	2.47306854
	0.25	0	0	16/7
0.1	0	0.1	1	10
	0.05	0.05	0.078468102	5.60243170
	0.10	0	0	100/19

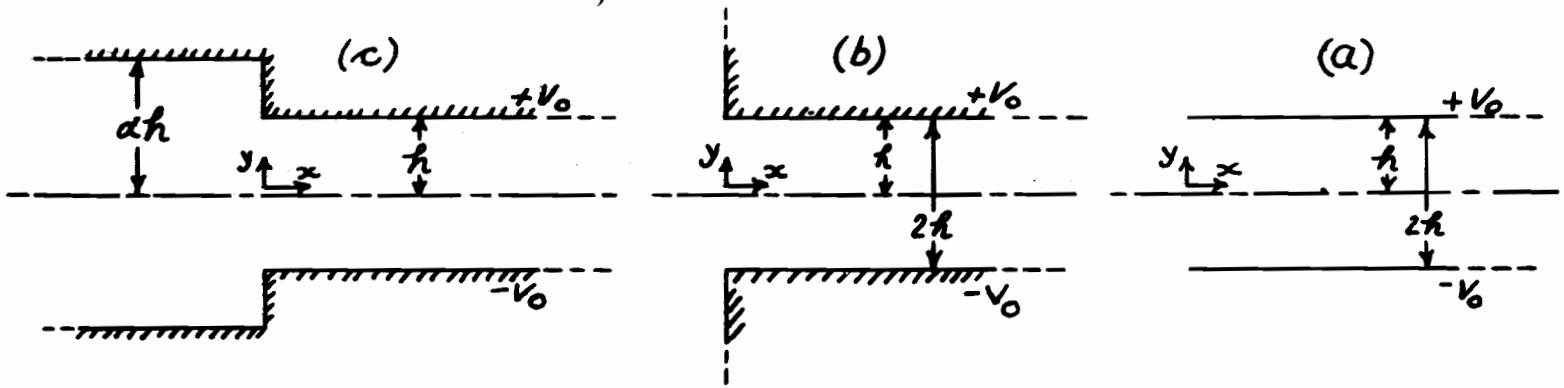
C. ACKNOWLEDGEMENT

The writer acknowledges the stimulation afforded by conversations with Drs. D. W. Kerst, K. R. Symon, F. T. Cole, N. Vogt-Nilsen, and C. L. Hammer at various times, in the interval from 1956 to the present, when problems have been encountered to which the foregoing transformations relate.

D. REFERENCES

1. Sir James Jeans, The Mathematical Theory of Electricity and Magnetism (Cambridge University Press, 1948), Sect. 328.
2. R. D. Redin and G. C. Danielson, Iowa State College Journal of Science, 32, 463-469 (February 15, 1958).
3. S. Ramo and J. R. Whinnery, Fields and Waves in Modern Radio (John Wiley and Sons, Inc., N.Y., 1944), Ed. 1, Table 3.10 (p. 113).
4. L. S. Skaggs, G. M. Almy, D. W. Kerst, and L. H. Lanzl, Phys. Rev. 70, 95 (L) (1946).

FIG. 1.


(c)
(b)
(a)
αh
h
$2h$
$+V_0$
$-V_0$
y
x

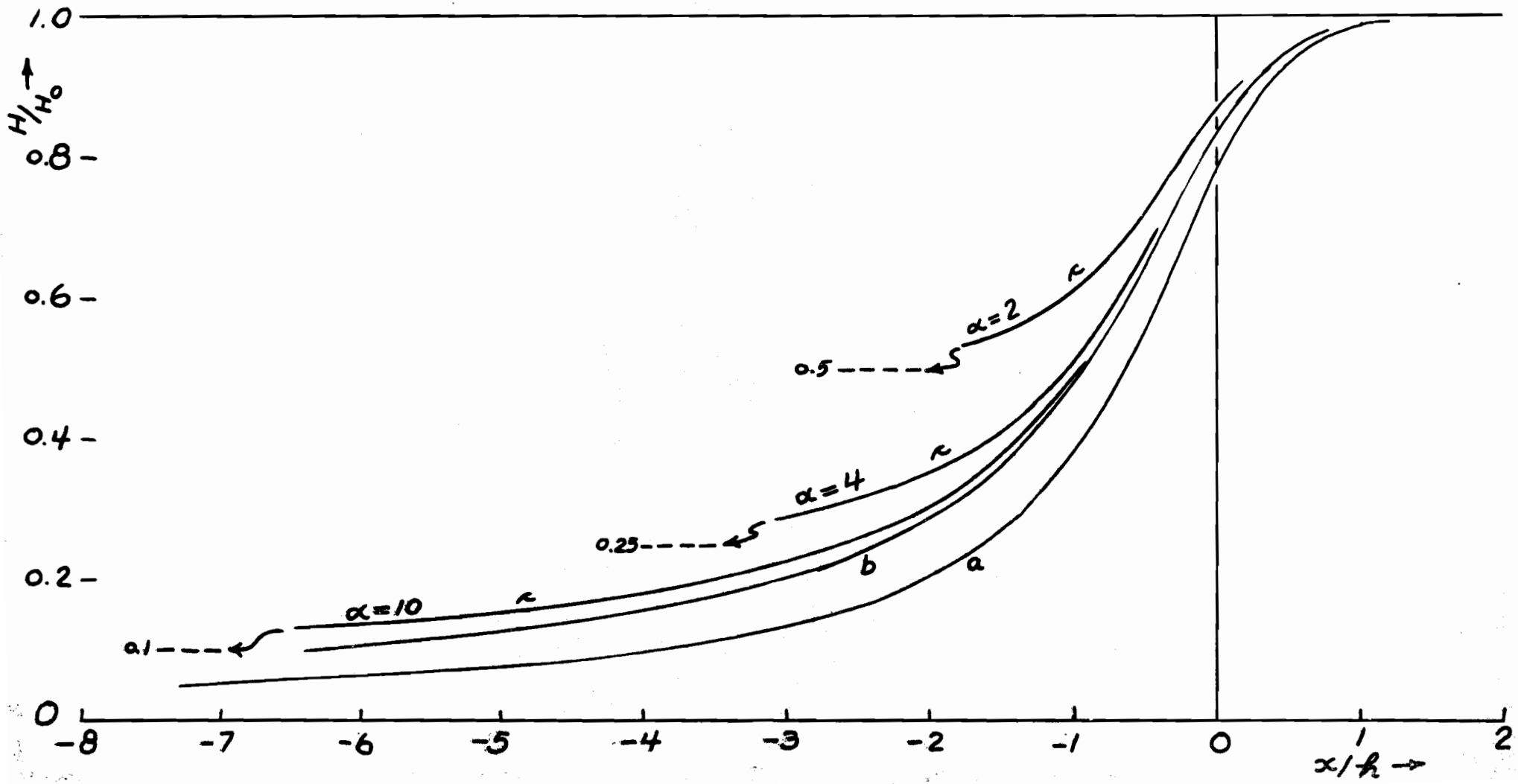

H/H_0
x/h
1.0
0.8
0.6
0.4
0.2
0
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
$\alpha=2$
$\alpha=4$
$\alpha=10$
0.5
0.25
0.1
a
b
c

FIG. 2.

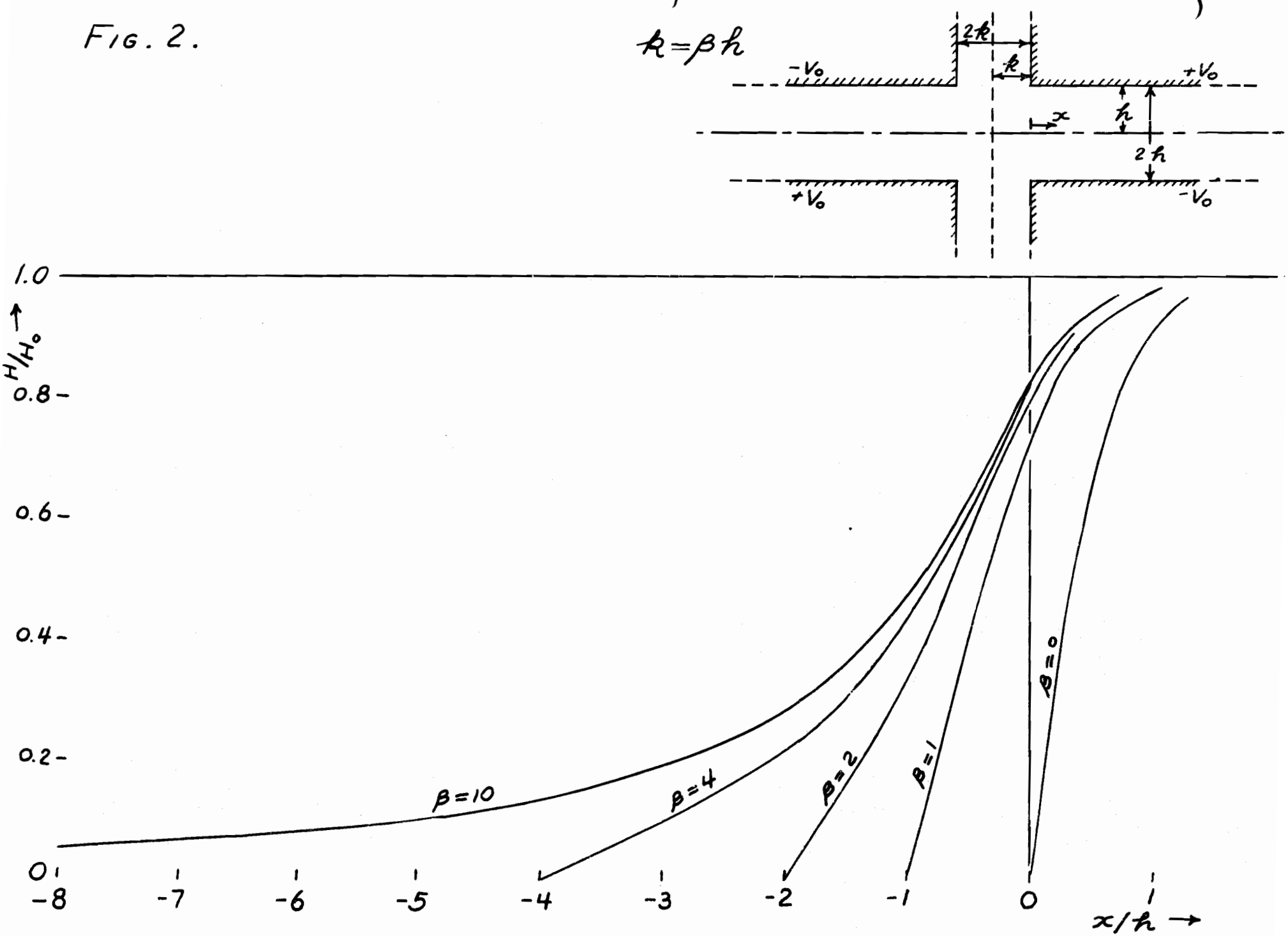

k = βh
2k
k
-V₀
+V₀
x
h
2h
+V₀
-V₀
1.0
0.8
0.6
0.4
0.2
0
H/H₀
β=10
β=4
β=2
β=1
β=0
-8
-7
-6
-5
-4
-3
-2
-1
0
1
x/h

FIG. 3. MEDIAN-PLANE FIELD IN PRESENCE OF PEELER
with $\alpha = 0.45$ & $\beta = 0.50$
[Checked by Laplace Phase of Forcyl Run 15060049041, with $r = 0.5$]

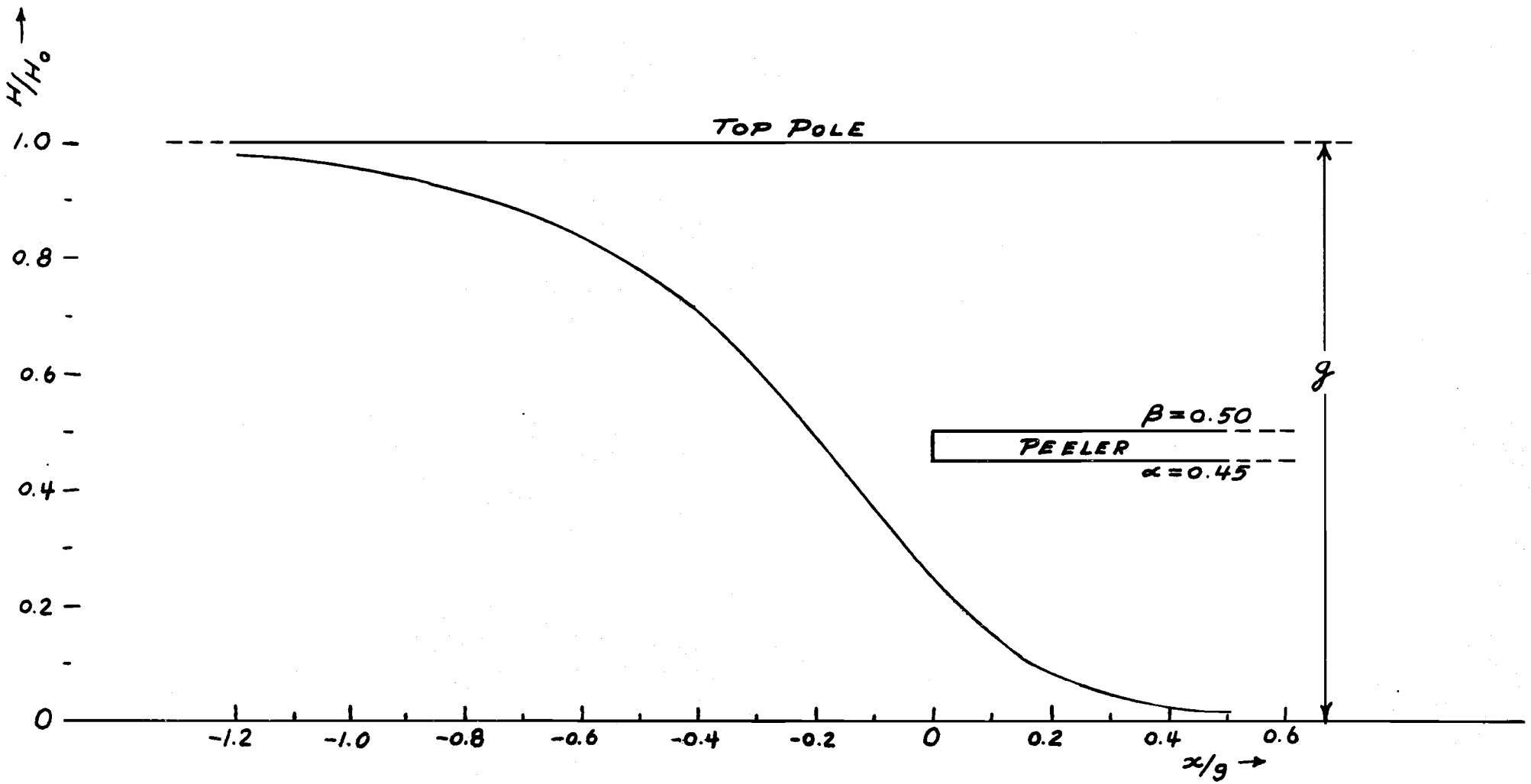
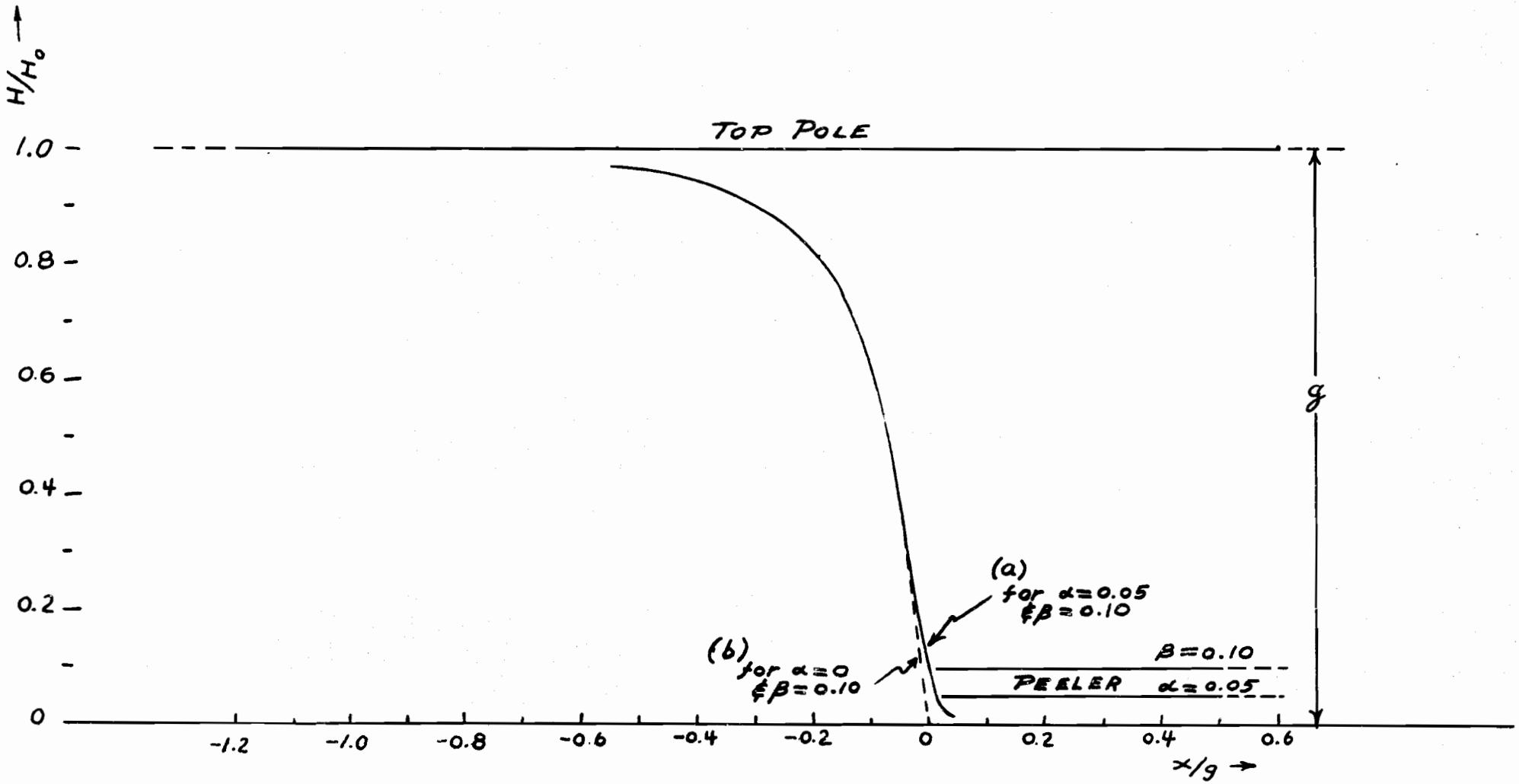


FIG. 4. MEDIAN-PLANE FIELD IN PRESENCE OF PEELER

(a) For $\alpha = 0.05$ & $\beta = 0.10$ (solid curve) and

(b) Limiting case of $\alpha = 0$ & $\beta = 0.10$ (dashed curve).