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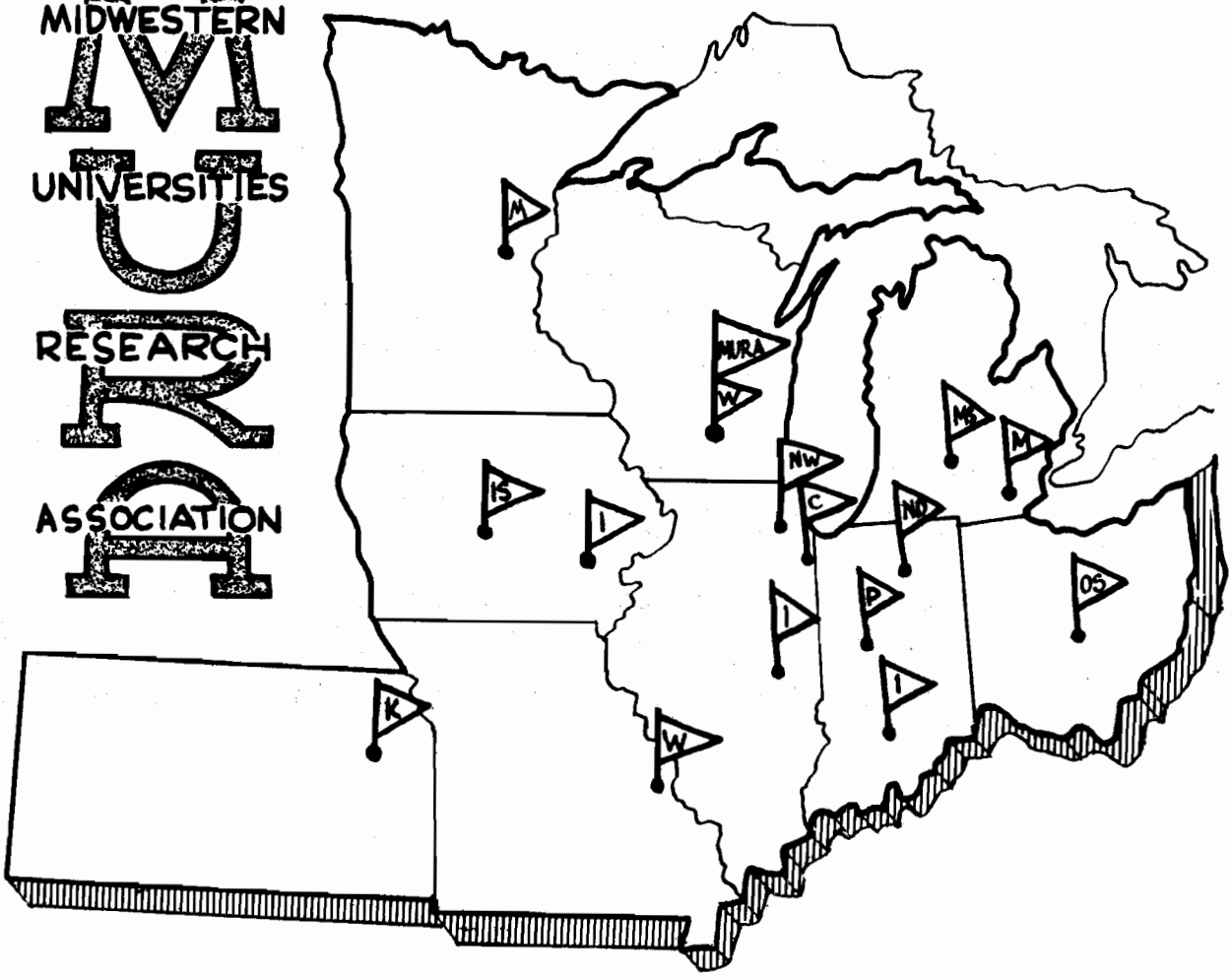

SOME NOTES ON THE KINEMATICS OF HIGH ENERGY NUCLEON-NUCLEON COLLISIONS

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SOME NOTES ON THE KINEMATICS OF HIGH ENERGY NUCLEON-NUCLEON COLLISIONS

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ABSTRACT

The kinematics of high energy nucleon-nucleon collisions are examined in the laboratory system and the rest-frame of the centre-of-mass, and the appropriate Lorentz transformation is described in detail. Some of the results are applied to the production of pions and anti-nucleons.

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I. INTRODUCTION

In this report we consider the collisions of high energy protons with stationary target nucleons, and the subsequent production of various secondary particles, e.g. pions, anti-nucleons and strange particles. The kinematics of these processes are of particular importance to the design of accelerators if the targets are placed inside the machine, since we are then faced with the problem of extracting the secondaries.

It is convenient to consider the kinematics in both the laboratory system of coordinates, LS, and also the rest-frame of the centre-of-mass of the colliding nucleons. When the energy of the bombarding proton is relativistic, these two coordinate systems are related by a Lorentz transformation, the properties of which are described below. Throughout the discussion we neglect the neutron-proton mass difference.

Suppose that (E_i, p_i) , (E_i', p_i') are the energy-momentum four-vectors, in LS and CMS respectively, of a particle with rest-mass m_i ; the units are chosen so that the velocity of light is unity. The Lorentz transformation between them is[1]

$$E_i^2 - p_i^2 = m_i^2 = E_i'^2 - p_i'^2$$

$$\underline{p}_i = \underline{p}_i' + \underline{\beta}_c \left[\frac{(\underline{\beta}_c \cdot \underline{p}_i')(1-\sqrt{1-\beta_c^2}) + \beta_c^2 E_i'}{\beta_c^2 \sqrt{1-\beta_c^2}} \right] \qquad (1.1)$$

$$E_i = \frac{E_i' + (\underline{\beta}_c \cdot \underline{p}_i')}{\sqrt{1-\beta_c^2}}$$

Where $_c$ is the velocity of the centre-of-mass relative to LS. Let (E_{LS}, P_{LS}) be the total energy and momentum of the initial particles in LS and (E'_{CMS}, P'_{CMS}) those in CMS:

$$\left.\begin{aligned} E_{LS} &= \sum_i E_i \quad ; \quad E'_{CMS} = \sum_i E'_i \\ \underline{P}_{LS} &= \sum_i \underline{p}_i \quad ; \quad \underline{P}'_{CMS} = \sum_i \underline{p}'_i \end{aligned}\right\} \qquad (1.2)$$

From (1.1), (1.2) and the fact that $\underline{P}'_{CMS}$ must, by definition, be zero, it follows that

$$\left.\begin{aligned} E'_{CMS} &= \sqrt{1-\beta_c^2}\, E_{LS} \\ \beta_c &= P_{LS}/E_{LS} \end{aligned}\right\} \qquad (1.3)$$

Suppose now that the direction of motion of the bombarding proton is chosen as z-axis for both LS and CMS and that the momenta P_i, P'_i are inclined at angles θ_i, θ'_i to it. (see figure (i)). Denote the longitudinal and

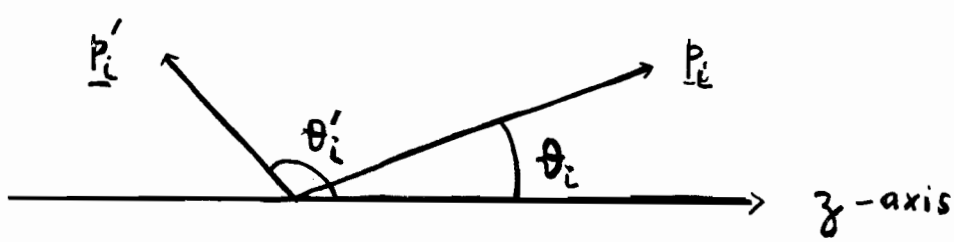

Fig. (i)

transverse components of momentum in LS, CMS by (l_i, t_i) and (l_i', t_i') respectively. The transformation (1.1) may be written

$$\left.\begin{aligned} l_i &= p_i \cos\theta_i = \gamma_c p_i' (\cos\theta_i' + \delta_i') \\ t_i &= p_i \sin\theta_i = t_i' = p_i' \sin\theta_i' \\ E_i &= \gamma_c (E_i' + \beta_c l_i') \end{aligned}\right\} \tag{1.4}$$

where

$$\gamma_c \equiv 1/\sqrt{1-\beta_c^2} \tag{1.5}$$

and

$$\delta_i' \equiv \beta_c \Big/ \left(p_i'/E_i'\right) \equiv \beta_c \Big/ \left(\beta_i' \text{ for } m_i \text{ in CMS}\right) \tag{1.6}$$

As will be shown later the value of δ_i' determines the character of the transformation.

If M is the nucleon mass, and T the kinetic energy* of the bombarding proton, then

* T will denote kinetic, and E total, energy throughout this discussion: $T_i = E_i - m_i$.

$$\left.\begin{aligned} E_{LS} &= T + 2M \\ P_{LS} &= \sqrt{T(T+2M)} \end{aligned}\right\} \tag{1.7}$$

and from (1.3)

$$\left.\begin{aligned} \beta_c &= \sqrt{\frac{T}{T+2M}} \\ E_{CMS}' &= \sqrt{2M(T+2M)} \end{aligned}\right\} \tag{1.8}$$

The above formulae provide the basis for our discussion of the kinematics of nucleon-nucleon collisions, and in the course of it we shall give a detailed discussion of the Lorentz transformation in (1.4)

2. KINEMATICS

Consider a nucleon-nucleon collision

$$N_1 + N_2 \rightarrow m_1 + m_2 + m_3 + \ldots \tag{2.1}$$

in which N_2 is at rest in LS and N_1 has kinetic energy T. The secondary particles, and also their masses, are denoted by m_1, m_2, m_3 ... . It will be assumed that there are no less than three secondaries, and by baryon conservation, at least two of them are baryons: therefore

$$2M < \sum_i m_i \tag{2.2}$$

and

$$M < \left(\sum_i m_i\right) - m_j = \sum_{i \neq j} m_i \tag{2.3}$$

for any m_j.

From the conservation of energy and momentum in CMS, we have

$$\left.\begin{aligned} E'_{CMS} &= \sum_i \left(m_i^2 + \ell_i'^2 + t_i^2\right)^{1/2} \\ \sum_i \ell_i' &= \sum_i t_i' = 0 \end{aligned}\right\} \tag{2.4}$$

or

$$E'_{CMS} - \left(m_j^2 + \ell_j'^2 + t_j'^2\right)^{1/2} = \sum_{i \neq j} \left(m_i^2 + \ell_i'^2 + t_i'^2\right)^{1/2} \tag{2.5}$$

The jth secondary will have maximum energy when the right-hand side of (2.5) is a minimum, subject to the conditions

$$\sum_{i \neq j} \ell'_i = -\ell'_j \quad ; \quad \sum_{i \neq j} t'_i = -t'_j \tag{2.6}$$

It can be shown in this way (see Appendix for details) that the maximum energy, in the CMS, of the jth secondary is

$$E'_{j\,max} = \left(m_j^2 + p'^2_{j\,max}\right)^{1/2} = \frac{E'^2_{CMS} + m_j^2 - \left(\sum_{i \neq j} m_i\right)^2}{2\,E'_{CMS}} \tag{2.7}$$

2.1 Secondaries in the backward LS Cone

The jth secondary travels backwards in the LS when

$$\ell_j < 0 \tag{2.8}$$

From (1.4) a <u>necessary</u> condition for l_j to be negative is

$$\delta'_j < 1 \tag{2.9}$$

or, from (1.5) , (1.6) and (1.8),

$$E'_j > m_j \gamma_c = \left(\frac{m_j}{2M}\right) E'_{CMS} \tag{2.10}$$

It is easy to show that:

<u>(a) δ'_j cannot be less than unity for all secondaries, at the same time.</u>

for if (a) were not true, the sum of the energies of the secondaries would exceed E'_{CMS} (see 2.10, 2.2). It can also be shown that, as

a consequence of baryon conservation:

> (b) δ'_j is never less than unity for secondaries with rest-mass greater than, or equal to, the nucleon rest-mass, i.e. no such particle can appear in the backwards LS cone.

Suppose $m_j \gg M$: δ'_j will be less than unity if

$$E'_{j\max} \gg E'_j \geqslant \left(\frac{m_j}{2M}\right) E_{CMS}$$

or, from (2.7), if

$$-(m_j - M)E'^2_{CMS} + M\left(\sum_{i\neq j} m_i + m_j\right)\left(m_j - \sum_{i\neq j} m_i\right) > 0 \qquad (2.11)$$

When* $m_j \leq \sum_{i\neq j} m_i$, both terms of the left-hand side are negative; when $m_j > \sum_{i\neq j} m_i$, then from (2.3)

$$m_j - M > m_j - \sum_{i\neq j} m_i$$

and since $E'^2_{CMS} > M\left(\sum_{i\neq j} m_i + m_j\right)$ the left-hand side of (2.11) is again negative. Therefore (2.11) is never satisfied when $m_j \gg M$, and δ'_j must always be greater than unity.

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* From the mass spectrum of elementary particles[2], the inequality $m_j < \sum_{i\neq j} m_i$ is seen to hold for all processes that are allowed in the Gell-Mann-Nishijima strangeness scheme. However, it may not be true for forbidden ones, e.g. $n + n \rightarrow \Lambda + n + \pi$, where $m_n + m_\pi < m_\Lambda$

If the rest-masses of the secondaries are given, then E'_{CMS} (and therefore T) must satisfy (2.11) before any secondary can appear in the backward LS cone.

A necessary <u>and</u> sufficient condition for (2.8) to hold is

$$\ell'_j < -\delta'_j p'_j = -\beta_c E'_j \tag{2.12}$$

Therefore

$$\tan^2 \theta'_j = \left(\frac{t'_j}{\ell'_j}\right)^2 < \frac{(1-\beta_c^2)}{\beta_c^2} - \frac{m_j^2}{\beta_c^2 E'^2_j} < \frac{1}{\gamma_c^2 \beta_c^2} \tag{2.12}$$

and the jth secondary will appear in the backward LS cone if and only if

i) its CMS energy satisfies (2.10)

ii) it appears in a backward CMS cone of small half-angle, the tangent of which is given in (2.13).

2.2 Maximum and Minimum Energies in LS

The LS energy of the jth secondary is greatest when it has maximum CMS energy and travels parallel to the centre-of-mass:

$$E_{j\,max} = \gamma_c \left[E'_{j\,max} + \beta_c p'_{j\,max} \right] \tag{2.14}$$

The minimum LS energy is merely the rest mass if $E'_{jmax} \geqslant m_j \gamma_c$; otherwise it occurs when the secondary travels anti-parallel to the centre-of-mass, with its maximum CMS energy:

$$E_{j\,min} = \begin{cases} m_j & \text{if } E'_{j\,max} \geqslant m_j \gamma_c \\ \gamma_c \left[E'_{j\,max} - \beta_c p'_{j\,max} \right] & \text{otherwise} \end{cases} \tag{2.15}$$

The value of E_{jmax} has also been obtained by Meyer and Holladay (reference 3) in a different form.

3. THE LORENTZ TRANSFORMATION

The direction of motion of the jth secondary in LS is related to that in the CMS by

$$\tan\vartheta = \frac{\sin\vartheta'}{\gamma_c(\cos\vartheta' + \delta')} \tag{3.1}$$

(see 1.4; for convenience the suffix j is omitted.) Since $\tan\vartheta$ depends on CMS energy (i.e. δ') as well as on ϑ', we shall treat δ' as a parameter and see how its value affects the relation between ϑ and ϑ'.

First suppose that $\delta' > 1$: $\tan\vartheta$ vanishes when $\vartheta = 0, \pi$, and is positive elsewhere. Its maximum value,

$$\tan\vartheta \leq \tan\vartheta_{max}(E') = \sqrt{\frac{1-\beta_c^2}{\delta'^2-1}} \qquad (\delta' > 1) \tag{3.2}$$

occurs when the derivative

$$\frac{d}{d\vartheta'}\tan\vartheta = \frac{(1+\delta'\cos\vartheta')}{\gamma_c(\cos\vartheta' + \delta')^2} \tag{3.3}$$

vanishes, and is always less than $\pi/2$.

If $\beta_c \leq \delta' \leq 1$ (by definition $\delta' \geq \beta_c$, see 1.6) $\tan\vartheta$ is discontinuous at $\cos\vartheta' = -\delta'$ and $d\vartheta/d\vartheta'$ is always positive. As ϑ' increases from zero to $\cos^{-1}(-\delta')$, $\tan\vartheta$ increases from zero to infinity, and as ϑ' increases from $\cos^{-1}(-\delta')$, $\tan\vartheta$ increases from minus infinity to zero.

Equation (3.1) may be solved for $\cos\theta'$:

$$\cos\theta'_{\pm} = \frac{-\delta'\gamma_c^2 \tan\theta \pm \sqrt{\gamma_c^2(1-\delta'^2)\tan^2\theta + 1}}{\gamma_c^2\left(\sec^2\theta - \beta_c^2\right)} \tag{3.4}$$

the square root is always real (by virtue of 3.2 when $\delta' > 1$) and its sign is chosen in the following way:

$\underline{\delta' > 1}$

$$0 \leqslant \theta'_+ \leqslant \cos^{-1}(-1/\delta') \qquad \text{for} \quad 0 \leqslant \tan\theta \leqslant \tan\theta_{max}(E')$$
$$\cos^{-1}(-1/\delta') \leqslant \theta'_- \leqslant \pi \qquad \tan\theta_{max}(E') \geqslant \tan\theta \geqslant 0 \tag{3.5a}$$

$\underline{\delta' < 1}$

$$0 \leqslant \theta'_+ \leqslant \cos^{-1}(-\delta') \qquad 0 \leqslant \theta \leqslant \pi/2$$
$$\cos^{-1}(-\delta') \leqslant \theta'_- \leqslant \pi \qquad \pi/2 \leqslant \theta \leqslant \pi \tag{3.5b}$$

The relation between θ and θ' is illustrated in figures (ii), (iii). Comparing these, we see that angles with single arrows in (ii) correspond to θ'_+ in (iii) and those with double arrows, to θ'_- . Notice that when $\delta' > 1$, both θ'_+ and θ'_- correspond to forward angles in LS ($\theta < \pi/2$); when $\delta' < 1$, θ'_+ corresponds to forward LS angles and θ'_- to backwards LS angles ($\theta > \pi/2$).

3.1 LS Energies and Angles Corresponding to a Given CMS Energy

If a secondary particle has a given CMS energy E', its LS energy E, and angle θ, must satisfy

$$E' = \gamma_c \left[E - \beta_c p \cos\theta \right] \tag{3.6}$$

(see 1.4) To study the behaviour of (3.6), we shall again treat E' as a parameter.

The solutions of (3.6) for E in terms of E' and .

$$E_{\pm}(\theta, E') = \frac{E' \pm \sqrt{\beta_c^2 \cos^2\theta \left[E'^2 - m^2\gamma_c^2 (1 - \beta_c^2 \cos^2\theta) \right]}}{\gamma_c \left(1 - \beta_c^2 \cos^2\theta\right)} \tag{3.7}$$

are real provided

$$E'^2 > m^2 \gamma_c^2 \left(1 - \beta_c^2 \cos^2\theta\right) \tag{3.8}$$

When $E' < m\gamma_c$ (i.e. $\delta > 1$, see 2.9, 2.10). this condition is only satisfied if $\theta \leq \theta_{max}(E')$ (see 3.2); when $E' > m\gamma_c$, it is satisfied by all values of θ.

<u>$E' < m\gamma_c$:</u> there are two real solutions, $E_{\pm}(\theta, E')$ for every $\theta \leq \theta_{max}(E')$ and none when $\theta > \theta_{max}(E')$. $E_+(\theta, E')$ is greater than $E_-(\theta, E')$ for all θ except $\theta = \theta_{max}$, at which point the solutions are smoothly connected. As E' approaches $m\gamma_c$, $E_+(\theta, E')$ and $\theta_{max}(E')$ increase, while $E_-(\theta, E')$ decreases.

<u>$E' > m\gamma_c$:</u> in this case there is one real solution for each θ between 0 and π. The LS energy in the forward cone($\theta \leq \pi/2$) is always greater than that in the backward cone*:

$$E(\phi) > E(\pi-\phi) \qquad \phi < \pi/2 \tag{3.9}$$

since
$$\left.\begin{array}{l} E_\pm(\vartheta,E') = E_\pm(\pi-\vartheta,E') \\ E_+(\vartheta,E') \geqslant E_-(\vartheta,E') \end{array}\right\} \text{ for } 0 \leq \vartheta \leq \pi$$

the only solution of (3.6) which satisfies (3.9) is

*Suppose $E = E_1$ when $\vartheta = \phi < \pi/2$ and $E = E_2$ when $\vartheta = \pi - \phi$: then from (3.6)

$$E_1 - \beta_c p_1 \cos\phi = E_2 + \beta_c p_2 \cos\phi = E'/\gamma_c$$

Since p_1, p_2, $\cos\phi$ are all positive, $E_1 > E_2$

$$E(\vartheta,E') = \begin{cases} E_+(\vartheta,E') & 0 \leq \vartheta \leq \pi/2 \\ E_-(\vartheta,E') & \pi/2 \leq \vartheta \leq \pi \end{cases}$$

or, since $\cos\vartheta$ changes sign at $\vartheta = \pi/2$

$$E(\vartheta,E') = \frac{E' + \beta_c \cos\vartheta \sqrt{E'^2 - m^2\gamma_c^2(1-\beta_c^2\cos^2\vartheta)}}{\gamma_c(1-\beta_c^2\cos^2\vartheta)} \tag{3.10}$$

Notice that as E' increases, $E(\vartheta,E')$ increases for all ϑ.

The curves af E versus ϑ for several values of E' are shown in figure (iv); no two curves intersect one another.

3.2 Maximum Energy and Angel in LS

For a given LS angle ϑ, the LS energy of the secondary particle varies between certain limits, corresponding to the variation of its CMS energy between m and E'_{max}.

$E'_{max} < m\gamma_c$: ϑ must always be less than $\vartheta_{max}(E'_{max})$ and the limits of E (see 3.2, 3.7),

$$E_+(\vartheta,E'_{max}) \geqslant E \geqslant E_-(\vartheta,E'_{max}) \tag{3.11}$$

are readily apparent from figure (iv).

$\underline{E'_{max} > m\gamma_c}$: since $E(\vartheta, E')$ increases with E' (see 3.10), the limits of E are

$$E(\vartheta, E'_{max}) \geqslant E \geqslant m \tag{3.12}$$

For a given LS energy, the largest possible value of ϑ may be obtained from (3.6):

$$0 \leqslant \vartheta \leqslant \Theta$$
$$\cos\Theta = \left[E - \frac{1}{\gamma_c} E'_{max}\right] \Big/ \sqrt{E^2 - m^2} \tag{3.13}$$

3.3 Transverse Momentum

The transverse momentum of a given secondary is the same in both LS and CMS (2.9) and is therefore a useful quantity to measure. If the transverse momenta of all secondaries are bounded, then their LS energies and angles are restricted to a certain band in the (E, ϑ) plane.

Suppose the secondary has CMS energy E', and non-zero transverse momentum t ($E' > \sqrt{m^2 + t^2}$): then the corresponding LS energy E and angle ϑ must satisfy two equations (3.6).

$$E' = \gamma_c [E - \beta_c p \cos\vartheta]$$
$$t = p \sin\vartheta \tag{3.14}$$

which define two curves in the (E, ϑ) plane (see Figure (v)). These curves intersect in two points

$$\cot\vartheta_\pm = \gamma_c \left(\beta_c E' \pm \sqrt{E'^2 - m^2 - t^2}\right) \Big/ t$$
$$E_\pm = \gamma_c \left(E' \pm \beta_c \sqrt{E'^2 - m^2 - t^2}\right) \tag{3.15}$$

such that

$$\vartheta_+ < \pi/2$$
$$\vartheta_- \lessgtr \pi/2 \quad \text{according as } E' \lessgtr \gamma_c \sqrt{m^2 + t^2} \tag{3.16}$$

As E' varies between m and E'_{max}, ϑ lies in the range

$$\Theta_{min} \leq \vartheta \leq \Theta_{max}$$

$$\cot \Theta_{\substack{min \\ max}} = \frac{\gamma_c}{t}\left(\beta_c E'_{max} \pm \sqrt{E'^2_{max} - m^2 - t^2}\right) \tag{3.17}$$

and E lies between

$$E_{max} \geq E > E_{min}$$

$$E_{max} = \gamma_c\left[E'_{max} + \beta_c\sqrt{E'^2_{max} - m^2 - t^2}\right], \quad E_{min} = \begin{cases} \gamma_c\left[E'_{max} - \beta_c\sqrt{E'^2_{max} - m^2 - t^2}\right] \\ \sqrt{m^2 + t^2}, \quad \text{if } E'_{max} > \gamma_c\sqrt{m^2+t^2} \end{cases} \tag{3.18}$$

Suppose the transverse momentum is bounded, $T_1 \leq t \leq T_2$: the points of the (E, ϑ) plane which represent the possible LS energies and angles of the secondary will be contained in the area between the curves:

$$\left.\begin{aligned} T_1 &= p\sin\vartheta\,; \quad T_2 = p\sin\vartheta \\ E'_{max} &= \gamma_c\left(E - \beta_c\, p\cos\vartheta\right) \end{aligned}\right\} \tag{3.19}$$

This area is shown in Figure (v).

4. TRANSFORMATION OF DIFFERENTIAL CROSS-SECTIONS AND ANGULAR DISTRIBUTIONS

The relation between LS and CMS differential cross-sections is*

$$2\pi\sigma(\vartheta,E)\,d(\cos\vartheta)\,dE = 2\pi\sigma'(\vartheta',E')\,d(\cos\vartheta')\,dE' \qquad (4.1)$$

Where primes denote quantities measured in the CMS. The dependence of

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* The azimuthal angles Ø, Ø' are always identical in the coordinate systems used here, and therefore need not be considered

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E', ϑ' on E, ϑ is obtained from (1.4),

$$\vartheta'(\vartheta,E) \Longrightarrow \quad \tan\vartheta' = \frac{\sin\vartheta}{\gamma_c(\cos\vartheta-\delta)} \quad ; \quad \delta = \beta_c E/p$$

$$E'(\vartheta,E) \Longrightarrow \quad E' = \gamma_c\left(E - \beta_c p\cos\vartheta\right) \qquad (4.2)$$

and hence the Jacobian for the transformation is

$$J(\vartheta,E) = \left|\frac{\partial(\cos\vartheta',E')}{\partial(\cos\vartheta,E)}\right| = \left[\gamma_c^2(\cos\vartheta-\delta)^2 + \sin^2\vartheta\right]^{-1/2} \qquad (4.3)$$

The expression under the square root is positive and therefore J exists for all values of ϑ and δ ; from (4.1), (4.2) and (4.3),

$$\sigma(\vartheta,E)\,d(\cos\vartheta)\,dE = \sigma'\left[\vartheta'(\vartheta,E),E'(\vartheta,E)\right]J(\vartheta,E)\,d(\cos\vartheta)\,dE \qquad (4.4)$$

If the CMS differential cross-section is known, three interesting LS quantities can be obtained from (4.4):

a) the angular distribution for a fixed LS energy E; must always lie in the range $0 \leq \vartheta \leq \Theta$ (3.13).

b) the energy distribution for a fixed angle ; the range of values of E is given by(3.11) or (3.12) depending on whether $E'_{max} \lesssim m\gamma_c$.

c) the angular distribution per unit solid angle summed over all energies

$$A(\vartheta)\,d(\cos\vartheta) = \left\{ \int \sigma(\vartheta,E)\,dE \right\} d(\cos\vartheta) \tag{4.5}$$

$\underline{E'_{max} < m\gamma_c}$: from (3.11)

$$A(\vartheta) = \int_{E_-(\vartheta,E'_{max})}^{E_+(\vartheta,E'_{max})} \sigma'\left[\vartheta'(\vartheta,E),E'(\vartheta,E)\right] J(\vartheta,E)\,dE \tag{4.6}$$

$$(0 \leq \vartheta \leq \vartheta_{max}(E'_{max}) < \pi/2)$$

$\underline{E'_{max} > m\gamma_c}$: from (3.12)

$$A(\vartheta) = \int_{m}^{E(\vartheta,E'_{max})} \sigma'\left[\vartheta'(\vartheta,E),E'(\vartheta,E)\right] J(\vartheta,E)\,dE \tag{4.7}$$

$$(0 \leq \vartheta \leq \pi)$$

Equations (4.6), (4.7) are an agreement with the results of Kaplon and Yamanouchi (reference 4).

An alternative way of calculating $A(\vartheta)$ is to express ϑ' as a function of ϑ and E' (see 3.4) and then integrate over CMS energy:

$$A(\vartheta) = \int \sigma'\left[\vartheta'(\vartheta,E'),E'\right] \left| \frac{d(\cos\vartheta')}{d(\cos\vartheta)} \right| dE' \tag{4.8}$$

Now from (3.4):

$$\vartheta'(\vartheta,E') \Longrightarrow \cos\vartheta'_{\pm} = \frac{-\gamma_c^2 \beta_c E' \tan\vartheta \pm \sqrt{L}}{\gamma_c^2 \sqrt{E'^2 - m^2}\,(\sec^2\vartheta - \beta_c^2)}$$

$$\left| \frac{d(\cos\vartheta'_{\pm})}{d(\cos\vartheta)} \right| = \left| \frac{\left[\beta_c E' \pm \sqrt{L}\right]^2}{\gamma_c \sqrt{E'^2 - m^2}\,\cos^3\vartheta\,(\sec^2\vartheta - \beta_c^2)\,\sqrt{L}} \right| \tag{4.9}$$

$$\sqrt{L} = \sqrt{E'^2 \sec^2\vartheta - m^2\gamma_c^2(\sec^2\vartheta - \beta_c^2)}$$

the square root must be real, and therefore

$$E'_{max} \geqslant E' \geqslant m\gamma_c \sqrt{\frac{\sec^2\vartheta - \beta_c^2}{\sec^2\vartheta}} = E'_o \tag{4.10}$$

<u>$E'_{max} < m\gamma_c$:</u> both ϑ'_+ , ϑ'_- correspond to the forward LS cone $0 \leqslant \vartheta \leqslant \vartheta_{max}(E'_{max}) < \pi/2$ (see § 3), and so

$$A(\vartheta) = \int_{E'_o}^{E'_{max}} \left\{ \sigma'[\vartheta'_+, E'] \left| \frac{d(\cos\vartheta'_+)}{d(\cos\vartheta)} \right| + \sigma'[\vartheta'_-, E'] \left| \frac{d(\cos\vartheta'_-)}{d(\cos\vartheta)} \right| \right\} dE'$$

$$(0 \leqslant \vartheta \leqslant \vartheta_{max}) \tag{4.11}$$

<u>$E'_{max} > m\gamma_c$:</u> the range of E' must be split in two,

a) $m\gamma_c \geqslant E' \geqslant E'_o$ $\quad (0 \leqslant \vartheta \leqslant \pi/2)$

b) $E'_{max} \geqslant E' \geqslant m\gamma_c$ $\quad (0 \leqslant \vartheta \leqslant \pi)$.

The contribution from range (a) is given by (4.11) with $E'_{max} = m\gamma_c$. In range (b), ϑ'_+ contributes to the forward cone ($\vartheta \leqslant \pi/2$) and ϑ'_- to the backward cone ($\vartheta \geqslant \pi/2$). Therefore

$$A(\vartheta) = \int_{E'_o}^{E'_{max}} \sigma'[\vartheta'_+, E'] \left| \frac{d(\cos\vartheta'_+)}{d(\cos\vartheta)} \right| dE' + \int_{E'_o}^{m\gamma_c} \sigma'[\vartheta'_-, E'] \left| \frac{d(\cos\vartheta'_-)}{d(\cos\vartheta)} \right| dE' \tag{4.12}$$

for $0 \leqslant \vartheta \leqslant \pi/2$

and

$$= \int_{m\gamma_c}^{E'_{max}} \sigma'[\vartheta'_-, E'] \left| \frac{d(\cos\vartheta'_-)}{d(\cos\vartheta)} \right| dE'$$

for $\pi/2 \leqslant \vartheta \leqslant \pi$

5. APPLICATIONS

The production of pions and anti-nucleons are two examples of the nucleon-nucleon collisions discussed above:

$$N + N \rightarrow N + N + \pi \tag{5.1}$$

$$N + N \rightarrow 3N + \bar{N} \tag{5.2}$$

From the kinematic point of view, the chief difference between these processes is that the maximum CMS energy of the pion is greater that $m_\pi \gamma_c$ whereas that of the anti-nucleon is less than $M_{\bar{N}} \gamma_c$ (m_π, $M_{\bar{N}}$ are the pion, anti-nucleon masses respectively). Consequently, while the pion may be produced at all angles in the LS, the anti-nucleon is restricted to a forward cone.

Consider a bombarding nucleon of 10 BEV kinetic energy (the target nucleon being at rest): then equations (1.3), (1.5) become

$$T \approx 11\,M \qquad \beta_c \approx 0.92$$
$$E'_{cms} \approx 5.1\,M \qquad \gamma_c \approx 2.55 \tag{5.3}$$

The pion has a maximum CMS energy (see 2.7),

$$E'_{\pi\,max} = 2.2\,M \tag{5.4}$$

which is greater than $m_\pi \gamma_c$ since $m \approx \frac{1}{7} M$; its LS energy lies between (see (2.14), (2.15))

$$10.5\,M \geqslant E_\pi \geqslant m_\pi \tag{5.5}$$

Since the anti-nucleon's maximum CMS energy

$$E'_{\bar{N}\,max} = 1.8\,M \tag{5.6}$$

is less than $M_{\bar{N}} \gamma_c$ ($M_{\bar{N}} = M$), it will appear in an LS cone of semi-angle

$$\theta_{max}(E'_{max}) = \tan^{-1}\sqrt{\frac{5}{8}} \approx 38^\circ \tag{5.7}$$

(see 3.2) and the limits on its LS energy are

$$7.9\,M \geqslant E_{\bar{N}} \geqslant 1.1\,M \tag{5.8}$$

From (3.13), (5.3), (5.4), (5.6), the largest LS angles at which pions and anti-nucleons of given LS energy may appear are

$$\Theta_{\pi} = \cos^{-1}\left(\frac{E_{\pi} - 0.863}{\sqrt{E_{\pi}^2 - m_{\pi}^2}}\right) \quad ; \quad \Theta_{\bar{N}} = \cos^{-1}\left(\frac{E_{\bar{N}} - 0.706}{\sqrt{E_{\bar{N}}^2 - M^2}}\right) \tag{5.9}$$

The values of Θ_{π}, $\Theta_{\bar{N}}$ for various energies, together with the ratios of the centre-of-mass velocity to the pion and anti-nucleon LS velocities (see 4.2), are given in Table I.

Table I: Maximum LS angle as a function of LS energy

E_{π}/M	1	4	7	10
δ_{π}	~0.92	0.92	0.92	0.92
Θ_{π}	81°	31°	18°	7°
$E_{\bar{N}}/M$	2	4	6	
$\delta_{\bar{N}}$	1.06	0.95	0.933	
$\Theta_{\bar{N}}$	36°	22.5°	13°	

The angular distribution of pions and anti-nucleons with given LS energies are of particular inportance for the problems of internal targets in an accelerator, and can be obtained from the CMS differential cross-sections for (5.1), (5.2). Although not much is known about these cross-sections at present, one or more of the following possibilities may not be too inconsistent with existing experimental data:

$$\sigma'(\theta', E') \sim \begin{cases} \text{Constant} & \text{(a)} \\ 1/E'^2 & \text{(b)} \\ \cos^2\theta' & \text{(c)} \\ \cos^2\theta'/E'^2 & \text{(d)} \end{cases} \tag{5.10}$$

From (4.3), (4.4), the corresponding LS cross-sections are

$$\sigma(\theta, E) \sim \begin{cases} J(\theta, E) & \text{(a)} \\ \dfrac{J(\theta, E)}{(E - \beta_c p \cos\theta)^2} & \text{(b)} \\ \dfrac{J(\theta, E)\,(\cos\theta - \delta)^2}{\gamma_c^{-2}(\cos\theta - \delta)^2 + \sin^2\theta} & \text{(c)} \\ \dfrac{J(\theta, E)\,(\cos\theta - \delta)^2}{(E - \beta_c p \cos\theta)^2\left(\gamma_c^2(\cos\theta - \delta)^2 + \sin^2\theta\right)} & \text{(d)} \end{cases} \tag{5.11}$$

and the angle θ never exceeds Θ (see 5.9)

All four possibilities in (5.11) have their largest values at

$\vartheta = 0$ and tend to decrease with increasing ϑ. Therefore a rough measure of their behaviour is given by the ratio

$$S_t(E) = \sigma_t(\Theta, E) \Big/ \sigma_t(0, E) \qquad (t \equiv a, b, c, d) \qquad (5.12)$$

The values of $S_t(E)$ for both pion and anti-nucleon production are given in Table II. At very high energies, $S_t(E)$ does not change very much for different choices of t, mainly because Θ is small; at lower energies, it varies considerably

Table II.

LS Energy	$S_t(E)$ a	b	c	d	
1	0.09	0.0008	0.07	0.0006	π
4	0.4	0.05	0.04	0.005	π
7	0.7	0.03	0.05	0.02	π
10	0.9	0.08	0.7	0.6	π
2	0.2	0.06	0.1	0.03	$\bar{N}$
4	0.33	0.13	0.01	0.004	$\bar{N}$
6	0.7	0.4	0.1	0.07	$\bar{N}$

APPENDIX

PROOF OF MAXIMUM ENERGY IN CMS.

From (2.5), (2.6) the maximum CMS energy of the jth secondary occurs when the right-hand side of

$$E'_{CMS} - (m_j^2 + \ell_j'^2 + t_j'^2)^{1/2} = \sum_{i \neq j} (m_i^2 + \ell_i'^2 + t_i'^2)^{1/2} \tag{A.1}$$

is a minimum, subject to the conditions:

$$\sum_{i \neq j} \ell_i' = -\ell_j' \quad ; \quad \sum_{i \neq j} t_i' = -t_j' \tag{A.2}$$

Let H denote the right-hand side of (A.1): any extremum of H must be a minimum because H tends to infinity when any t_i', ℓ_i' tend to infinity. By the Lagrange method of undertermined multipliers, this minimum occurs when

$$\left.\begin{aligned} \ell_i' + \lambda (m_i^2 + \ell_i'^2 + t_i'^2)^{1/2} &= 0 \\ t_i' + \mu (m_i^2 + \ell_i'^2 + t_i'^2)^{1/2} &= 0 \end{aligned}\right\} \quad (i \neq j) \tag{A.3}$$

The solution of (A.3) is

$$\frac{\ell_i'}{t_i'} = \frac{\lambda}{\mu} \quad ; \quad \frac{\ell_i'}{m_i} = \frac{\ell_k'}{m_k} \quad (i, k \neq j) \tag{A.4}$$

and from (A.2)

$$\ell_i' = \frac{-m_i}{(\sum_{i \neq j} m_i)} \ell_j' \quad ; \quad t_i' = \frac{-m_i}{(\sum_{i \neq j} m_i)} t_j' \tag{A.5}$$

Substituting (A.5) in (A.1) and squaring both sides, we find that

$$E'_{j\,max} = \frac{E'^2_{CMS} + m_j^2 - (\sum_{i \neq j} m_i)^2}{2 E'_{CMS}} \tag{A.6}$$

REFERENCES

(1) K. Dedrick, Internal Memorandum, M.L. Report 574, Standford University (February 1959).
See also "The Theory of Relativity", C. Møller, Oxford University Press, 1952, p. 72.

(2) M. Gell-Mann and A. H. Rosenfeld, Ann. Rev. Nuc. Sci 7, 407 (1957).

(3) H. C. Meyer III and W. G. Holladay, Sup. Nuovo Cim. (In Press).

(4) M. F. Kaplon and T. Yamanouchi, University of Rochester, N.Y.O. - 2622 (1959).

FIGURE CAPTIONS

Fig. ii Mapping of ϑ' onto ϑ for various δ'. Corresponding angles are shown with the same number of arrowheads.

$$\cos \vartheta'_m = -1/\delta' \quad , \quad \cos \vartheta'_\perp = -\delta'$$

Fig. iii Tan ϑ as a function of ϑ'

Fig. iv. LS energy as a function of LS angle for given CMS energies

$$\delta' > 1 \text{ implies } \quad E' < m\gamma_c$$

$$\delta' < 1 \text{ implies } \quad E' > m\gamma_c$$

Fig. v. LS angles and energies allowed when transverse momentum is bounded. Points marked $\pm$, correspond to the solutions $(E_\pm, \vartheta_\pm)$ of (3.17).

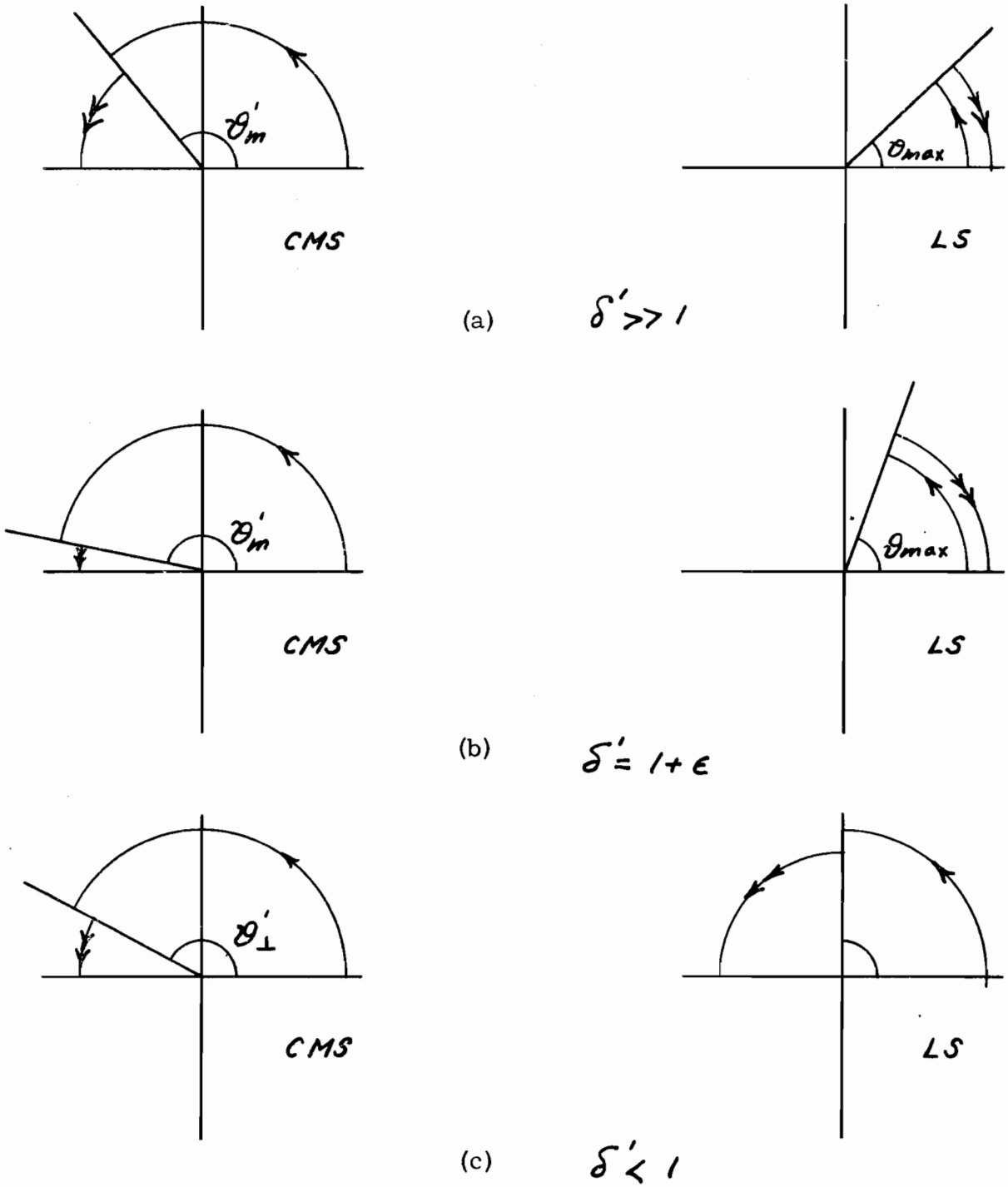
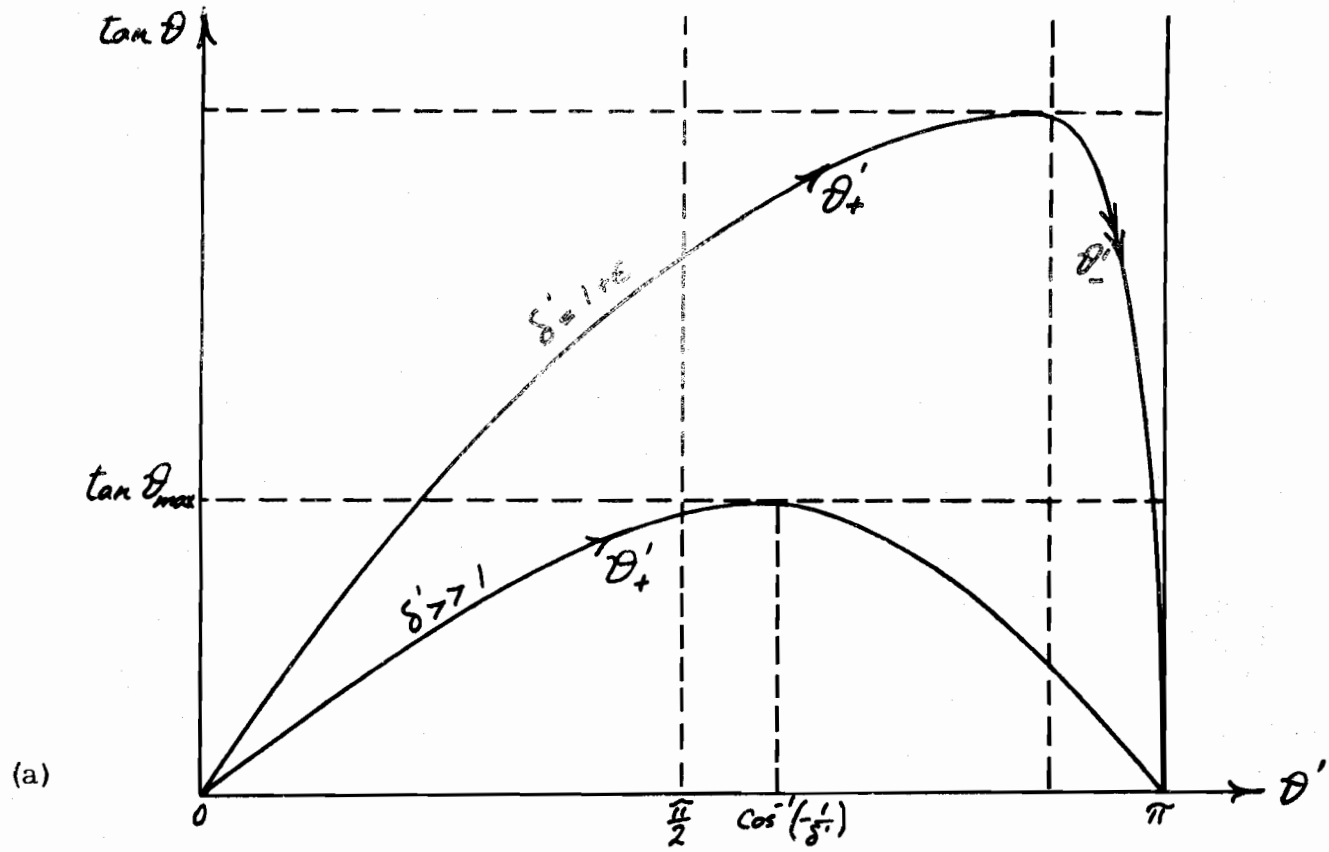
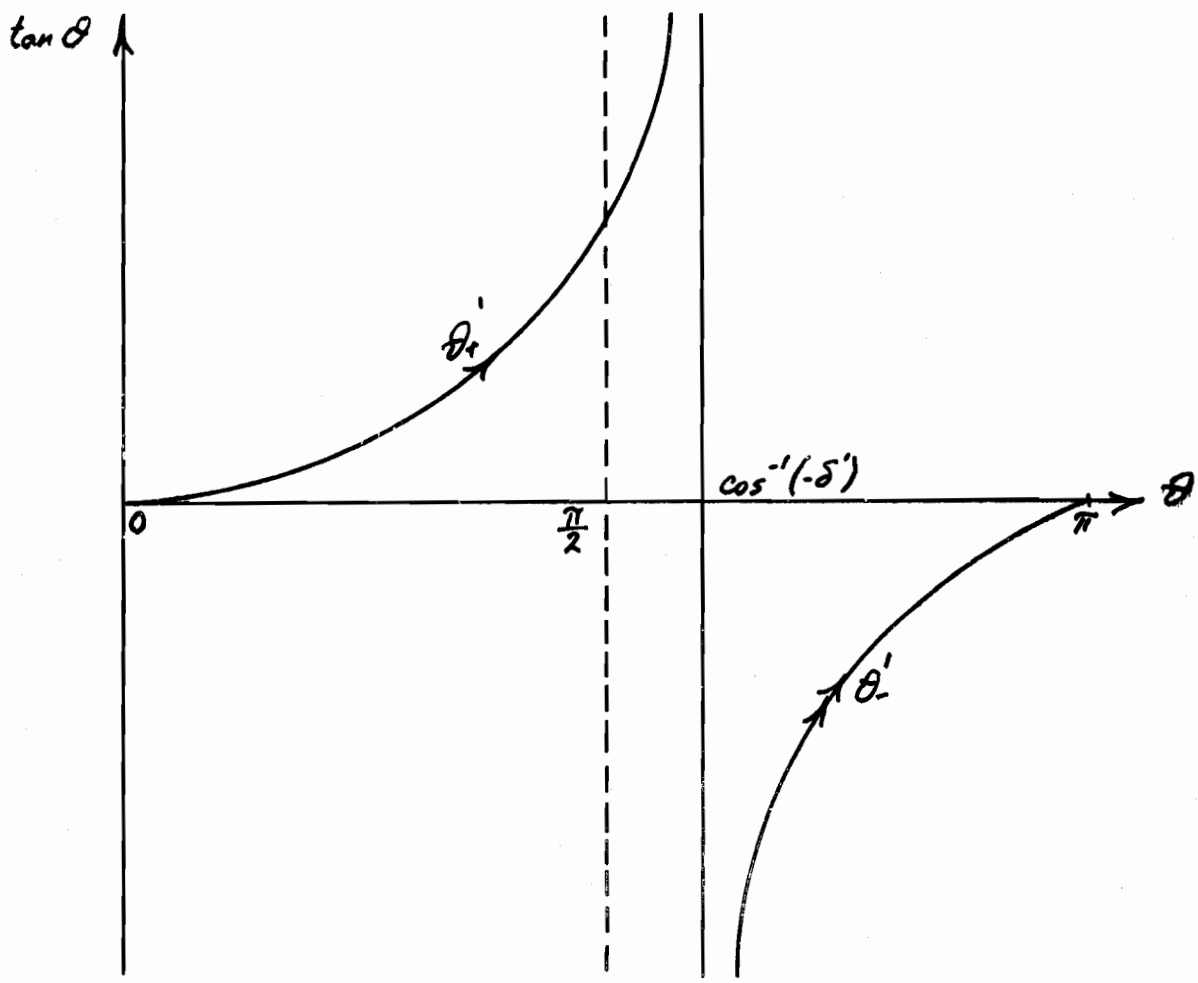

θ'_m
CMS
θ_{max}
LS
(a)
$\delta' >> 1$
θ'_m
CMS
θ_{max}
LS
(b)
$\delta' = 1 + \epsilon$
$\theta'_\perp$
CMS
LS
(c)
$\delta' < 1$

Fig. 11



(a)



(b)

Fig. 11

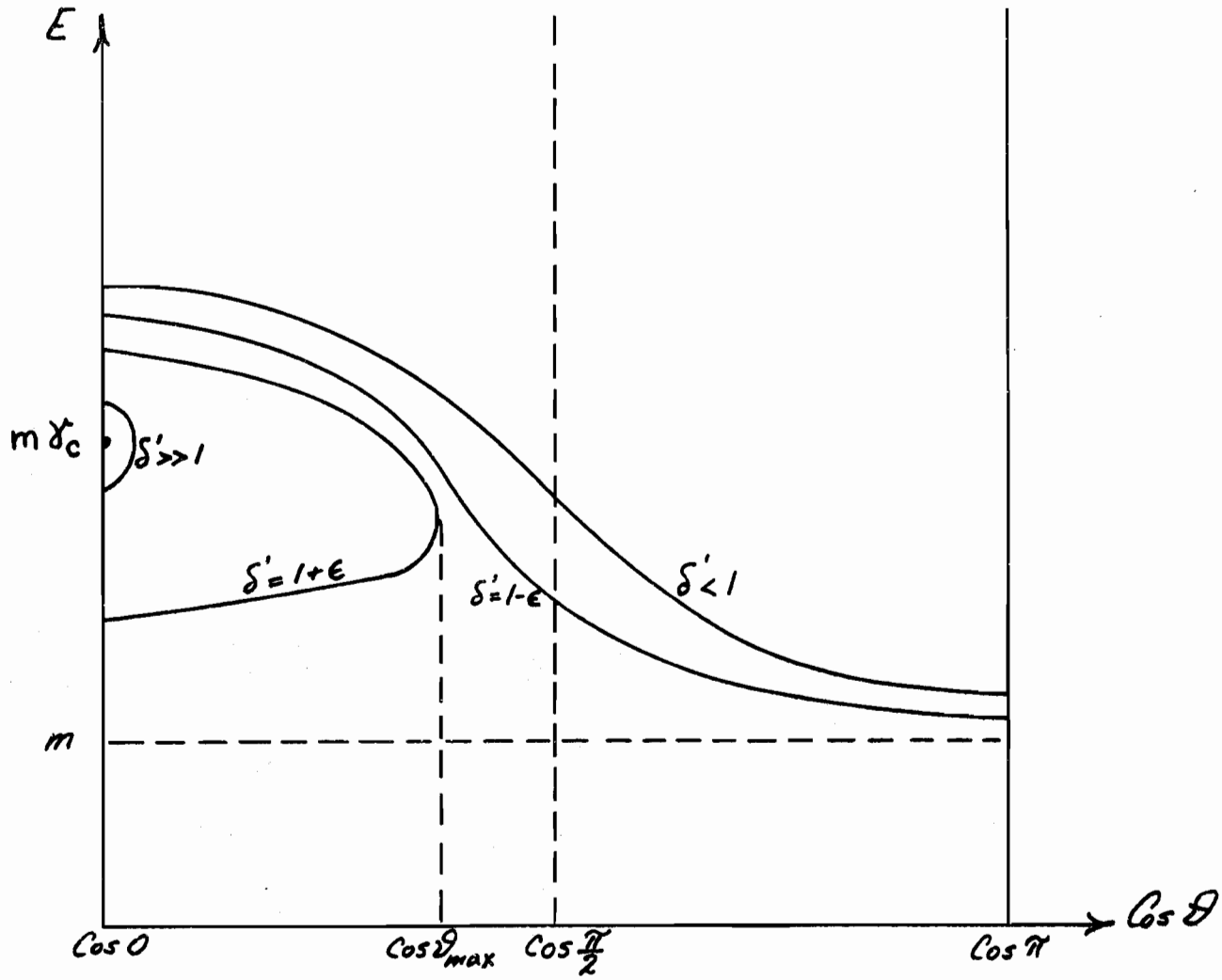

E
$m\gamma_c$
$\delta' \gg 1$
$\delta' = 1+\epsilon$
$\delta' = 1-\epsilon$
$\delta' < 1$
m
Cos 0
$\cos\vartheta_{max}$
$\cos\frac{\pi}{2}$
$\cos\pi$
$\cos\vartheta$

Fig. 1

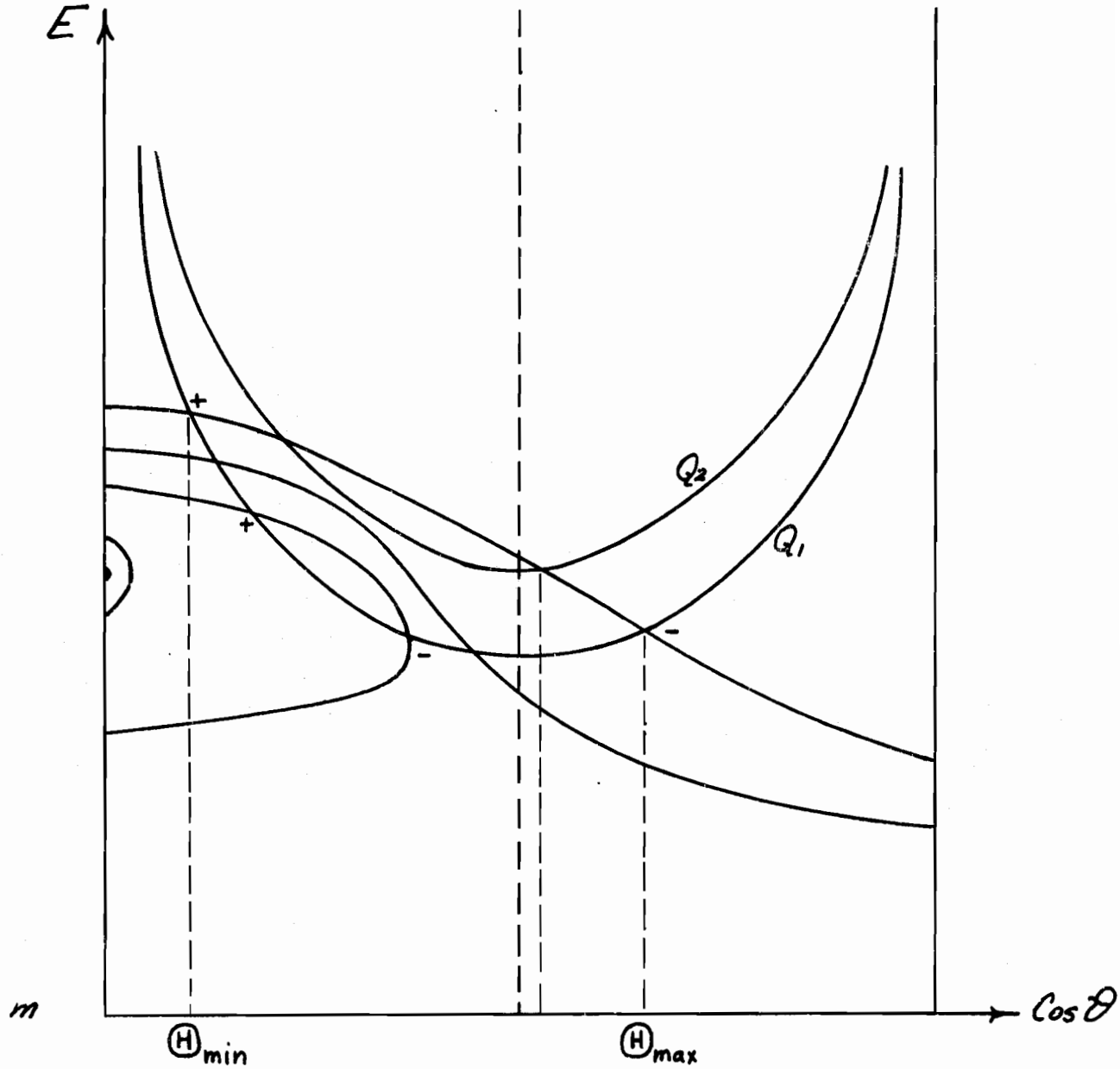


Fig. v