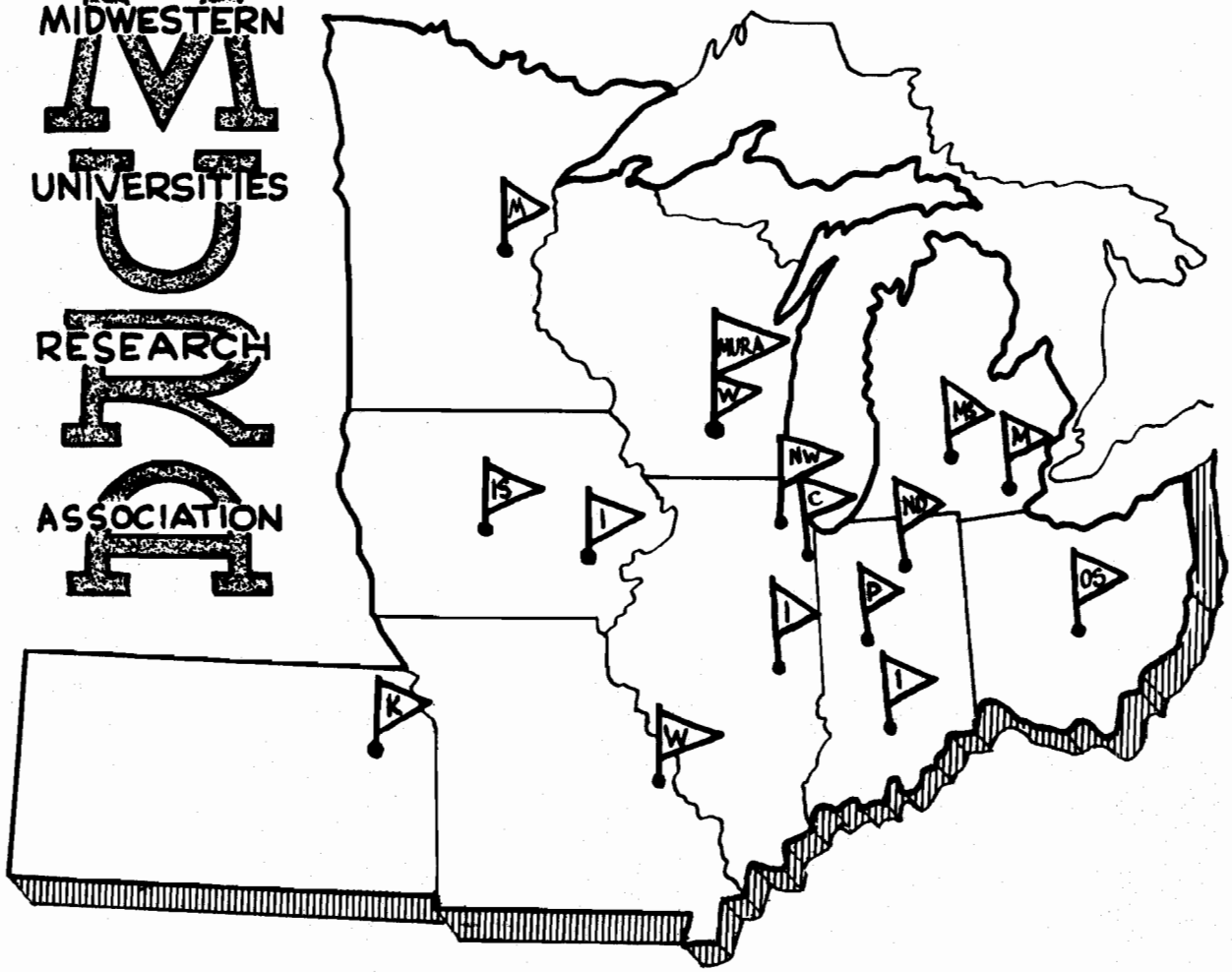
560

Fermilab Library
0 1160 0038846 6

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



ADIABATIC BEHAVIOR NEAR TRANSITION ENERGY

G. Bronca

REPORT **NUMBER** 560

MURA-560
UC-28, Particle Accelerators and
High Voltage Machines
TID-4500 (14th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

ADIABATIC BEHAVIOR NEAR TRANSITION ENERGY

G. Bronca**

March 1, 1960

ABSTRACT

Computer results for the behavior of particles near transition energy are compared with analytical studies of the limit of adiabatic motion.

*AEC Research and Development Report. Research supported by the U.S. Atomic Energy Commission, Contract No. AT(11-1)-384.

**On leave from Centre d'Etudes Nucléaires, Saclay, France.

I. CONDITIONS FOR ADIABATIC MOTION

In FFAG accelerators the behavior of particles is described[1] by the Hamiltonian

$$H = \Gamma\varphi + V\cos\varphi - \pi h f_s' W^{*2} \tag{1}$$

where φ is the phase of the rf oscillator,

$$W = \int_{E_0}^{E} \frac{dE}{f(E)} \tag{2}$$

is the conjugate momentum and

$$\Gamma = \sin\varphi_s, \tag{3}$$

$$W^* = W - W_s(t), \tag{4}$$

$$f_s' = \left(\frac{df}{dW}\right)_{W=W_s} \tag{5}$$

The subscript s refers to the synchronous particle.

When V and φ_s are constant, the motion will be adiabatic if f_s' varies slowly with time. At the transition point $f_s' = 0$ and near transition energy, this condition is not fulfilled; the motion is no longer adiabatic. A condition of adiabaticity can be found by writing that the fractional variation during one phase oscillation of the coefficient f_s' is small compared to 1. If ν_p is the frequency of the phase oscillation, this condition is

$$\frac{1}{f_s'}\frac{1}{\nu_p}\frac{df_s'}{dt} \ll 1 \tag{6}$$

Near transition energy we can approximate f(E) by a parabola and we write[1]

$$f \simeq f_t \left[1 - \frac{(E-E_t)^2}{k E_t^2} \right] \simeq f_t \left[1 - \frac{f_t^2 (W-W_t)^2}{k E_t^2} \right] \tag{7}$$

and

$$f_s' \simeq -\frac{2 f_t^2}{k E_t^2} (E - E_t) . \tag{8}$$

The frequency of the phase oscillation is

$$\nu_p = \left[\frac{h V f_s' \cos \varphi_s}{2\pi} \right]^{1/2} , \tag{9}$$

so if we use the relation

$$\frac{dE}{dt} = f_t V \sin \varphi_s \tag{10}$$

the adiabatic condition (6) becomes

$$\left(\frac{|E - E_t|}{E_t} \right)^{3/2} \gg \left(\frac{\pi k V \sin^2 \varphi_s}{h E_t |\cos \varphi_s|} \right)^{1/2} . \tag{11}$$

Note that we must keep the power 3/2 in (11) to find the right order of magnitude. Equation (11) is the condition given in reference 2 for the limit of the adiabatic motion.

Kolomenski and Sabsovich[3] treated the problem in another way. From the Hamiltonian (1), they got the differential equation in φ and they wrote the solution by means of Bessel functions of order 2/3:

$$\varphi - \varphi_s = A_1 u^{2/3} J_{2/3}(u) + A_2 u^{2/3} J_{-2/3}(u), \tag{12}$$

where

$$u = \frac{2}{3} g |t|^{3/2}, \tag{13}$$

and

$$g^2 = 2\pi h b V |\cos\varphi_s| . \tag{14}$$

It is assumed that near transition energy f_s' varies with time according to the relation

$$f_s' = -bt \tag{15}$$

where $t = 0$ at the transition point and b is positive, and is given by intergrating (8) and (10):

$$b = 2\frac{f_t^3}{h}\frac{V}{E_t^2}\sin\varphi_s . \tag{16}$$

For large values of u (and t), we can use the asymptotic expression for the Bessel function. Then the solution (12) becomes

$$\varphi - \varphi_s = A_1' \sqrt{\frac{2}{\pi}} \left(\frac{2}{3} g\right)^{1/6} |t|^{1/4} \cos\left(\frac{2}{3} g |t|^{3/2} + \theta_1'\right). \tag{17}$$

A_1' and θ_1' represent the initial conditions. Equation (17) is also the expression for the solution we get directly from the Hamiltonian when the motion is adiabatic. From this result we can say that the adiabaticity condition corresponds to the case where in the asymptotic expression for Bessel function only the first term is important.

For order ν, this condition is

$$\frac{4\nu^2 - 1}{8u} \ll 1, \tag{18}$$

and for $\nu = 2/3$

$$u \gg \frac{7}{72} \tag{19}$$

and

$$|t|^{3/2} \gg \frac{7}{48g} \quad . \tag{20}$$

Equation (20) is the relation given by Kolomenski and Sabsovich. Note that we can limit the asymptotic expression for Bessel function to the first term only when u > 1. $|t|^{3/2}$ must be at least one order of magnitude greater than $7/48g$.

Using (14), (16), and (10), we write the adiabaticity condition

$$\left(\frac{|E - E_t|}{E_t}\right)^{3/2} \gg \frac{7}{2\pi 48}\left(\frac{\pi k V \sin^2 \varphi_s}{h E_t |\cos \varphi_s|}\right)^{1/2} . \tag{21}$$

Equations (11) and (21) are different by the coefficient $\frac{7}{2\pi \cdot 48}$.

II. COMPUTER RESULTS

Calculations were made by means of the computer using the TTT program. Phases and energies were plotted for many synchronous particle energies near the transition points. Comparisons were made with the curves we should get if we assume the motion was adiabatic. These curves are obtained in the following way.

For small phase oscillations, a particle describes, in the phase plane, an ellipse given by writing H = a constant and ($\varphi - \varphi_s$) small.

Then the axes of the ellipse are given by the relations

$$(\varphi - \varphi_s)_{max} = a_1 = \sqrt{-\frac{2H}{V\cos\varphi_s}} \tag{22}$$

$$W^*_{max} = a_2 = \sqrt{-\frac{H}{\pi h f_s'}} \tag{23}$$

and the ratio

$$\frac{a_1}{a_2} = \frac{2\pi h f_s'}{V} . \tag{24}$$

By Liouville's theorem, the area of the ellipse remains constant during the motion so

$$\pi a_1 a_2 = A \tag{25}$$

then

$$a_1 = \left(\frac{A^2\, 2h\, f_s'}{\pi V \cos\varphi_s}\right)^{1/4} . \tag{26}$$

Starting with points on the ellipse with φ axis a_{10}, the ellipse for adiabatic motion has a φ axis given by

$$a_1 = a_{10}\left(\frac{f_s'}{f_{s0}'}\right)^{1/4} . \tag{27}$$

Figures 1,2,3,4,5,6,7, and 8 illustrate the distribution of the points compared to the adiabatic ellipse. Let us define Δ as the parameter

$$\Delta = \left(\frac{|E - E_t|}{E_t}\right)^{3/2} . \tag{28}$$

We shall use the subscript D and K when in equations (11) and (21) we replace the inequality sign by equality.

$$\Delta_D = \left(\frac{\pi k V \sin^2\varphi_s}{h E_t |\cos\varphi_s|}\right)^{1/2} \tag{29}$$

$$\Delta_K = \frac{7}{2\pi 48}\left(\frac{\pi k V \sin^2\varphi_s}{h E_t |\cos\varphi_s|}\right)^{1/2} \tag{30}$$

In Figs. 1,2,3,4,5,6,7,and 8 we see that for $\Delta \simeq \Delta_D$ the motion is still adiabatic. From (30) this corresponds to $\Delta \simeq 43\Delta_K$. This suggests that Kolomenski's condition with two orders of magnitude is the best expression for the limit of adiabatic motion.

$$\Delta = 100\,\Delta_K \tag{31}$$

As $\frac{E-E_t}{E_t}$ depends on the parameter Δ by the power 2/3, equation (31) becomes

$$\frac{E-E_t}{E_t} \simeq 20\left(\frac{E-E_t}{E_t}\right)_K . \tag{32}$$

We can also use equation (11) without requiring Δ much greater than Δ_D but only $\Delta \geqslant \Delta_D$. A similar result for the adiabaticity condition has been found in the study of adiabatic capture in rf buckets,[4] for the voltage turn-on time.

REFERENCES

1. K. R. Symon and A. M. Sessler, "Methods for Radio Frequency Acceleration in Fixed Field Accelerators," Proceedings of the CERN Symposium on High Energy Accelerators and Pion Physics, p. 44 (1956).
2. Margaret Foster and Gareth Guest, "A Study of the RF Phase Plane Near Transition with Frequency Modulation," MURA-423.
3. A. A. Kolomenski and L. L. Sabsovich, "On the Decay of the Passage Through the Transition Energy in Accelerators," Proceedings of the CERN Symposium on High Energy Accelerators and Pion Physics, p. 112 (1956).
4. C. G. Lilliequist and K. R. Symon, "Deviations from Adiabatic Behavior during Capture of Particles into an RF Bucket," MURA-491.

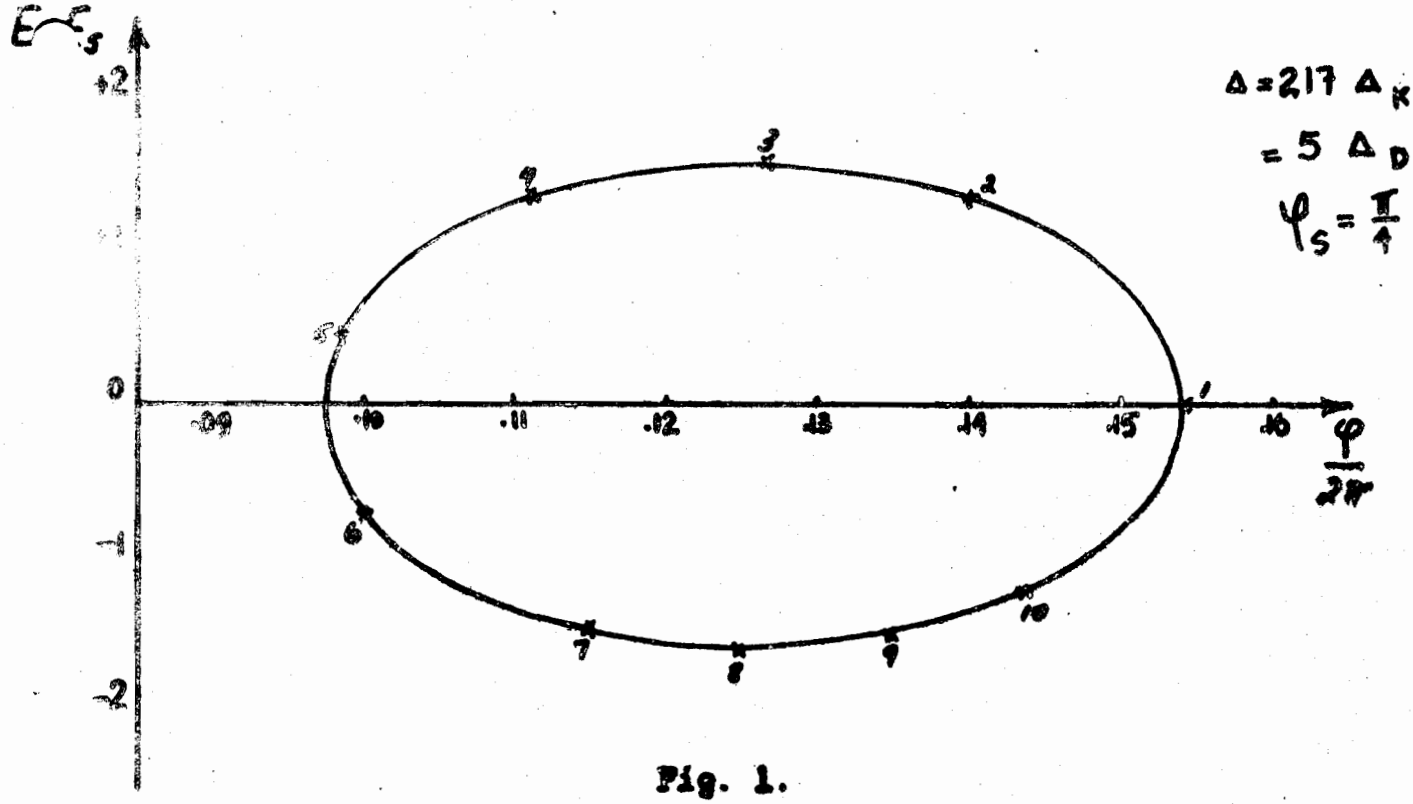

$E - E_s$
+2
0
-1
-2
.09
.10
.11
.12
.13
.14
.15
.16
$\frac{\varphi}{2\pi}$
$\Delta = 2.17\ \Delta_K$
$= 5\ \Delta_D$
$\varphi_s = \frac{\pi}{4}$

Fig. 1.

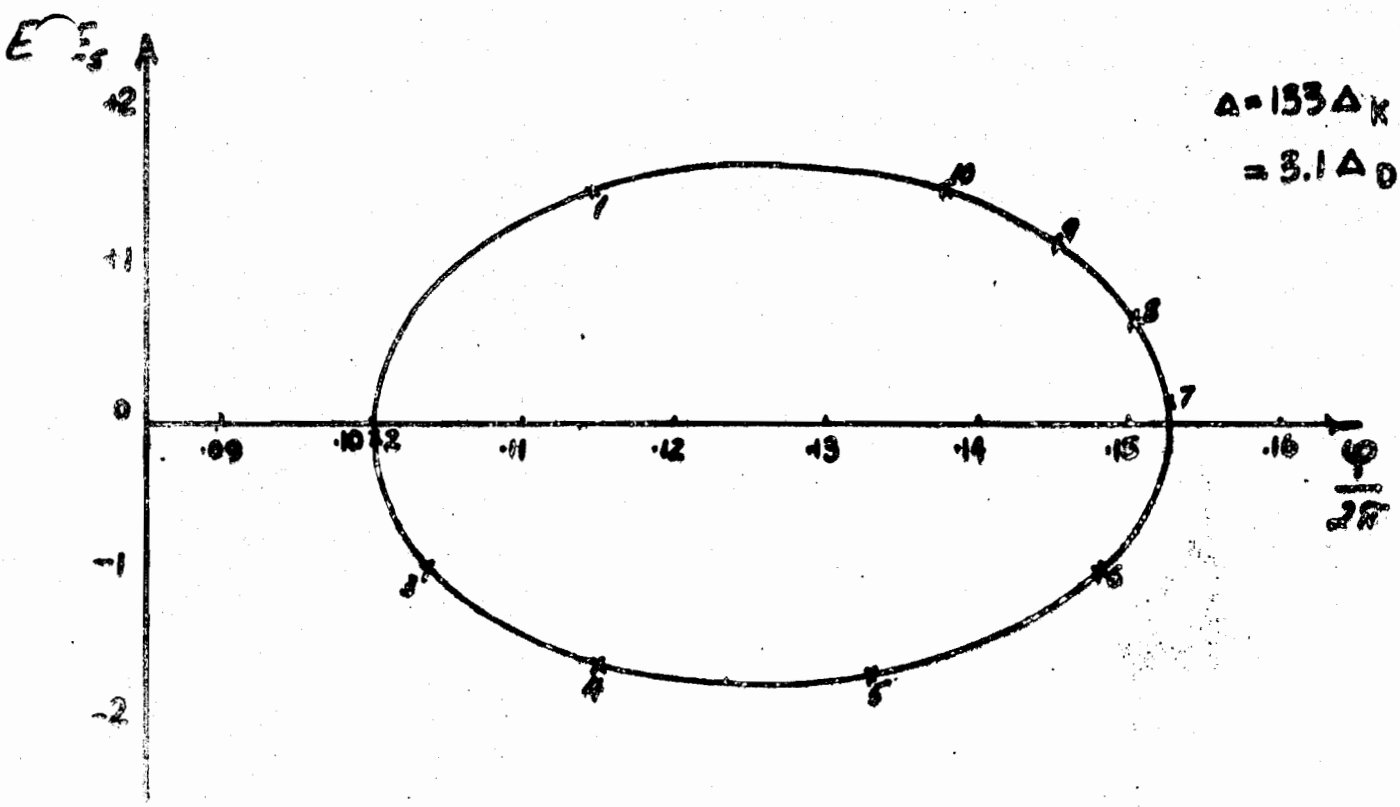

$E - E_s$
+2
+1
0
-1
-2
.09
.10
.11
.12
.13
.14
.15
.16
$\frac{\varphi}{2\pi}$
$\Delta = 1.33\ \Delta_K$
$= 3.1\ \Delta_D$

Fig. 2.

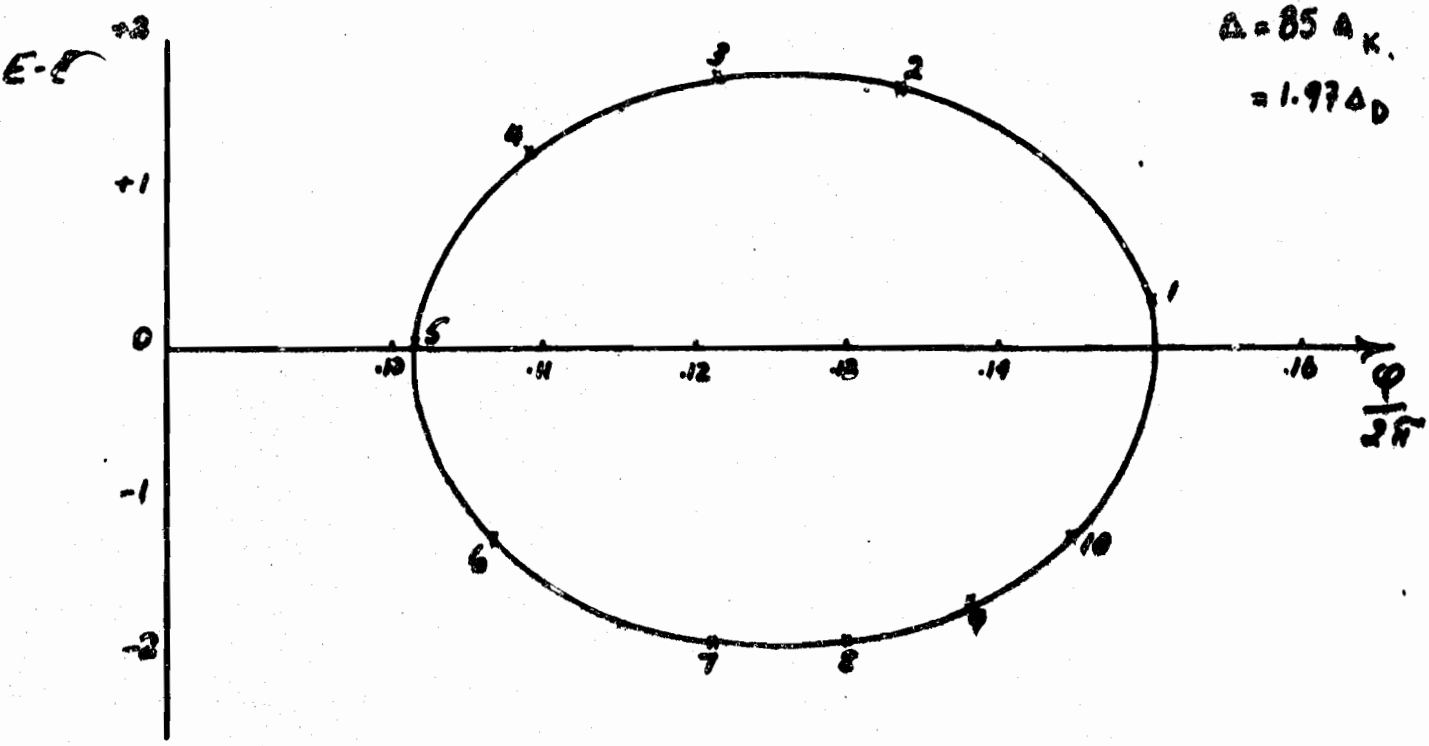

E-E
+2
+1
0
-1
-2
.10
.11
.12
.13
.14
.16
φ/2π
Δ = 85 Δ_K
= 1.97 Δ_D
1
2
3
4
5
6
7
8
9
10

Fig. 3.

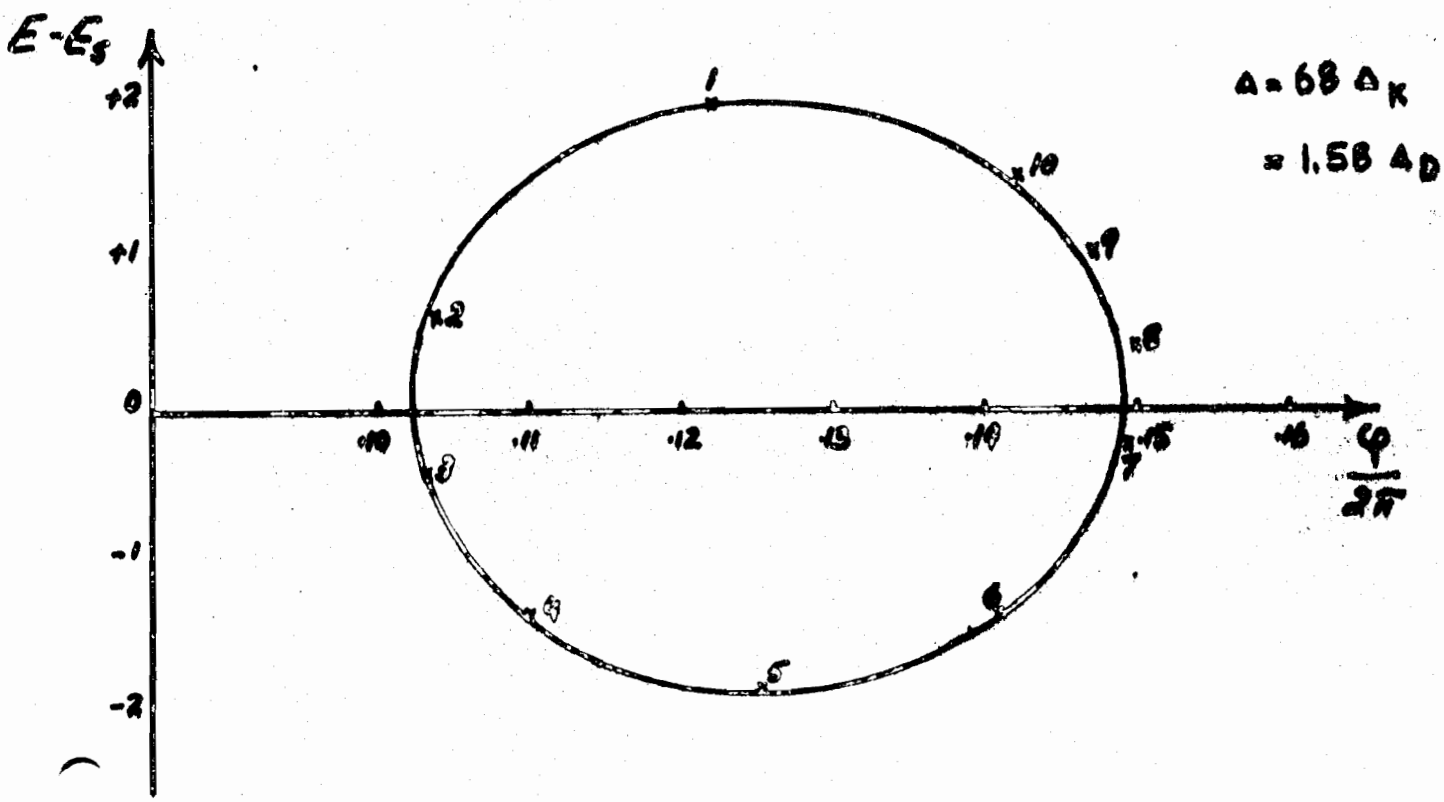

E-E_s
+2
+1
0
-1
-2
.10
.11
.12
.13
.14
.15
.16
φ/2π
Δ = 68 Δ_K
= 1.58 Δ_D
1
2
3
4
5
6
7
8
9
10

Fig. 4.

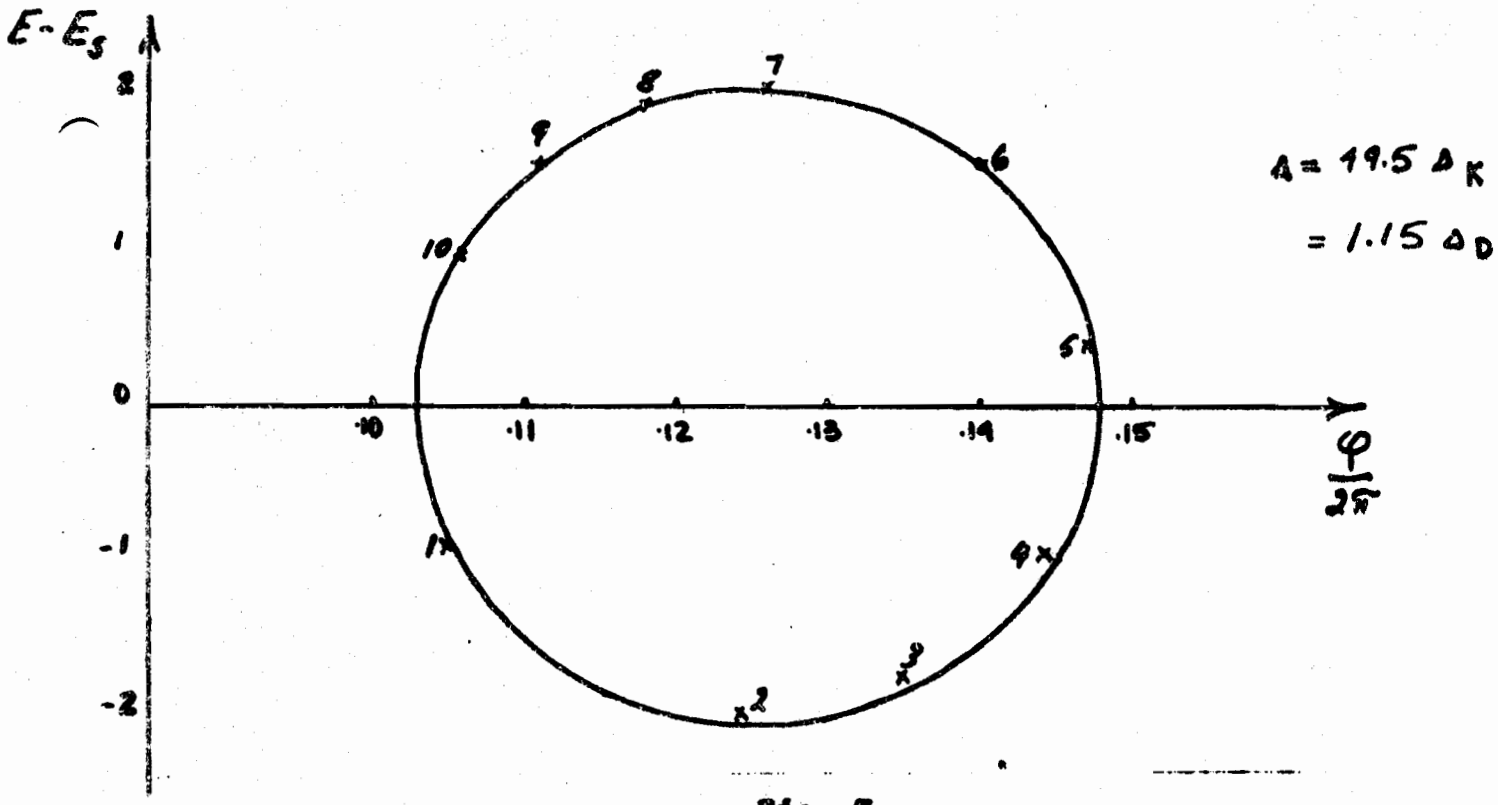

E - E_S
2
1
0
-1
-2
·10
·11
·12
·13
·14
·15
φ/2π
Δ = 19.5 Δ_K
= 1.15 Δ_D

Fig. 5.

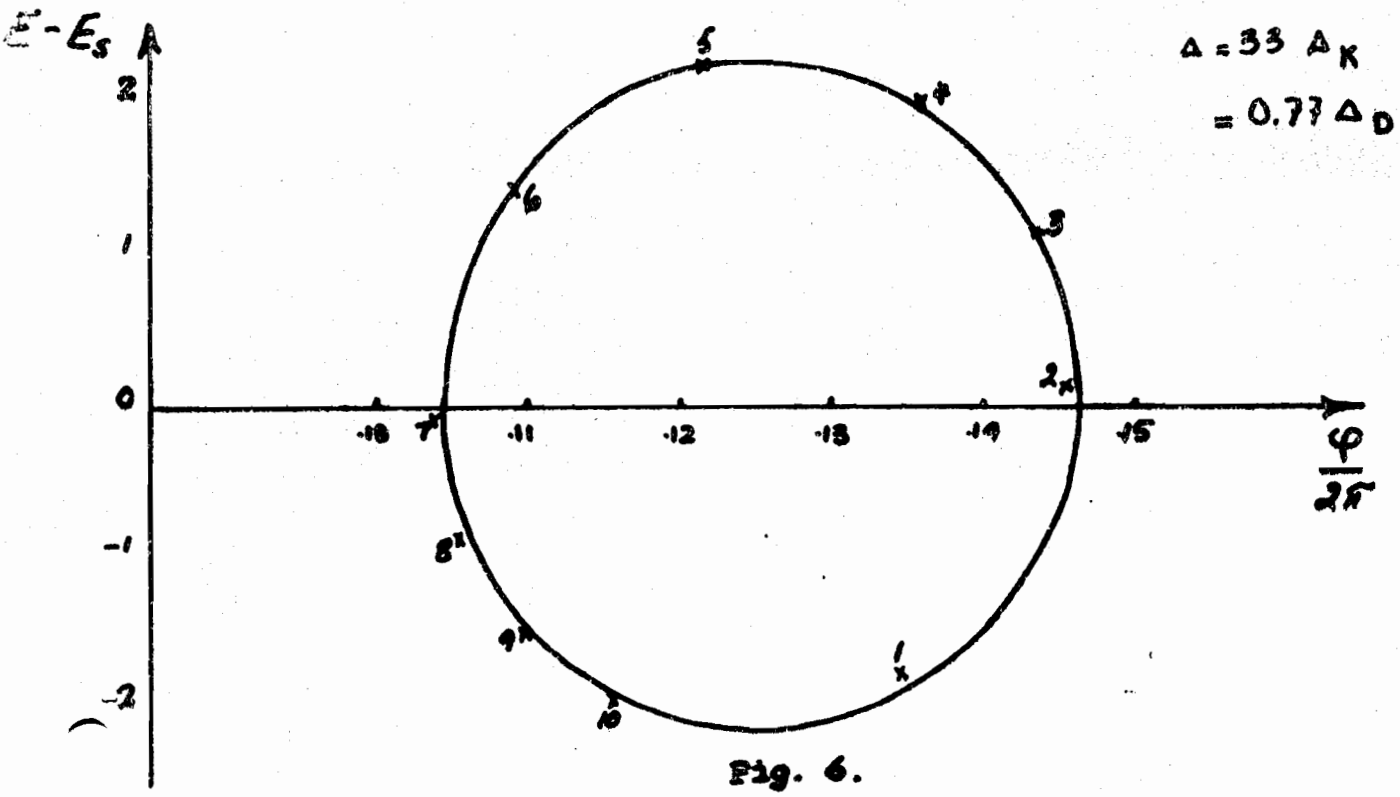

E - E_S
2
1
0
-1
-2
·10
·11
·12
·13
·14
·15
φ/2π
Δ = 33 Δ_K
= 0.77 Δ_D

Fig. 6.

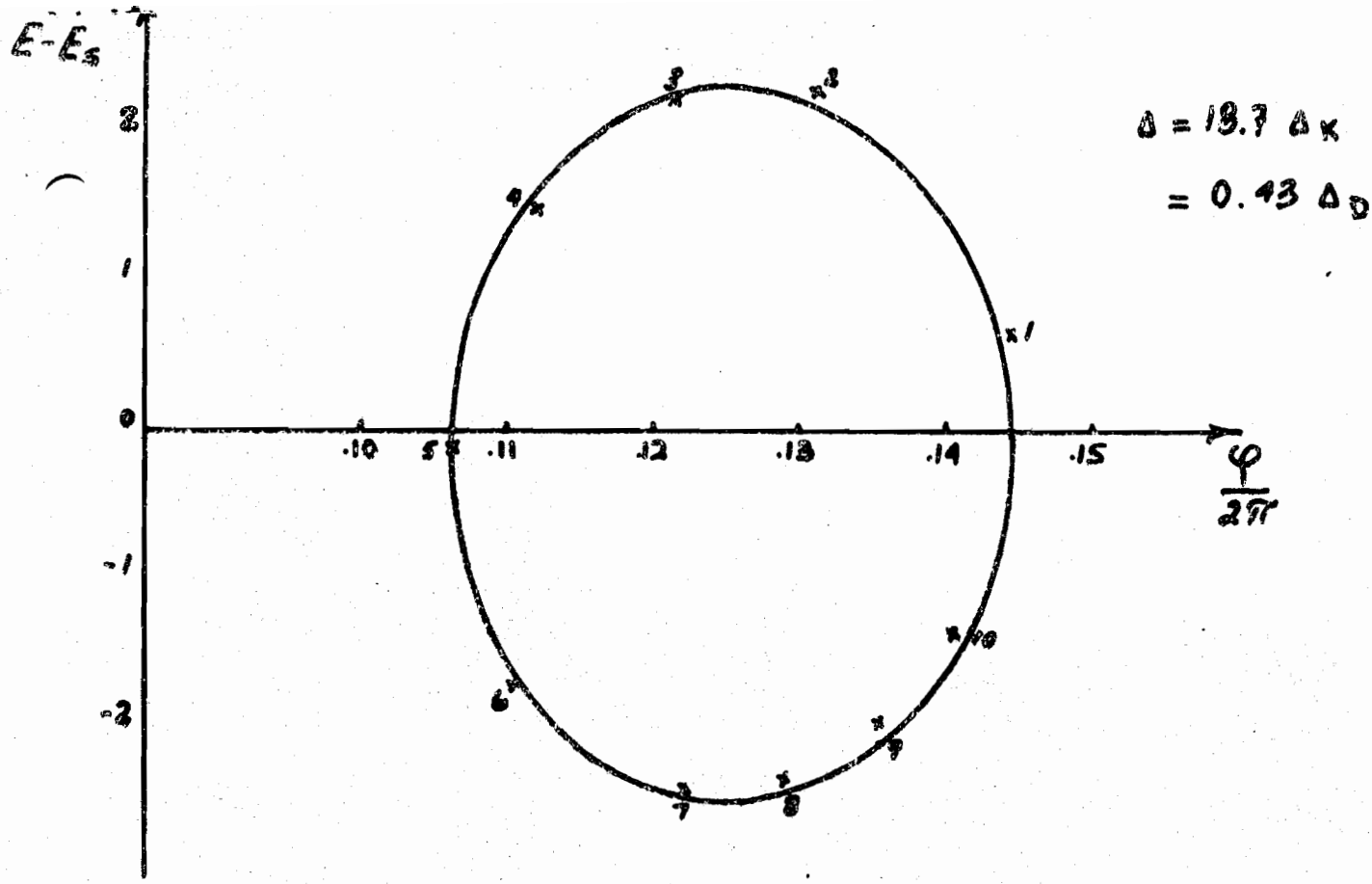

E-E_s
2
1
0
-1
-2
.10
.11
.12
.13
.14
.15
φ/2π
Δ = 18.7 Δ_K
= 0.43 Δ_D

Fig. 7.

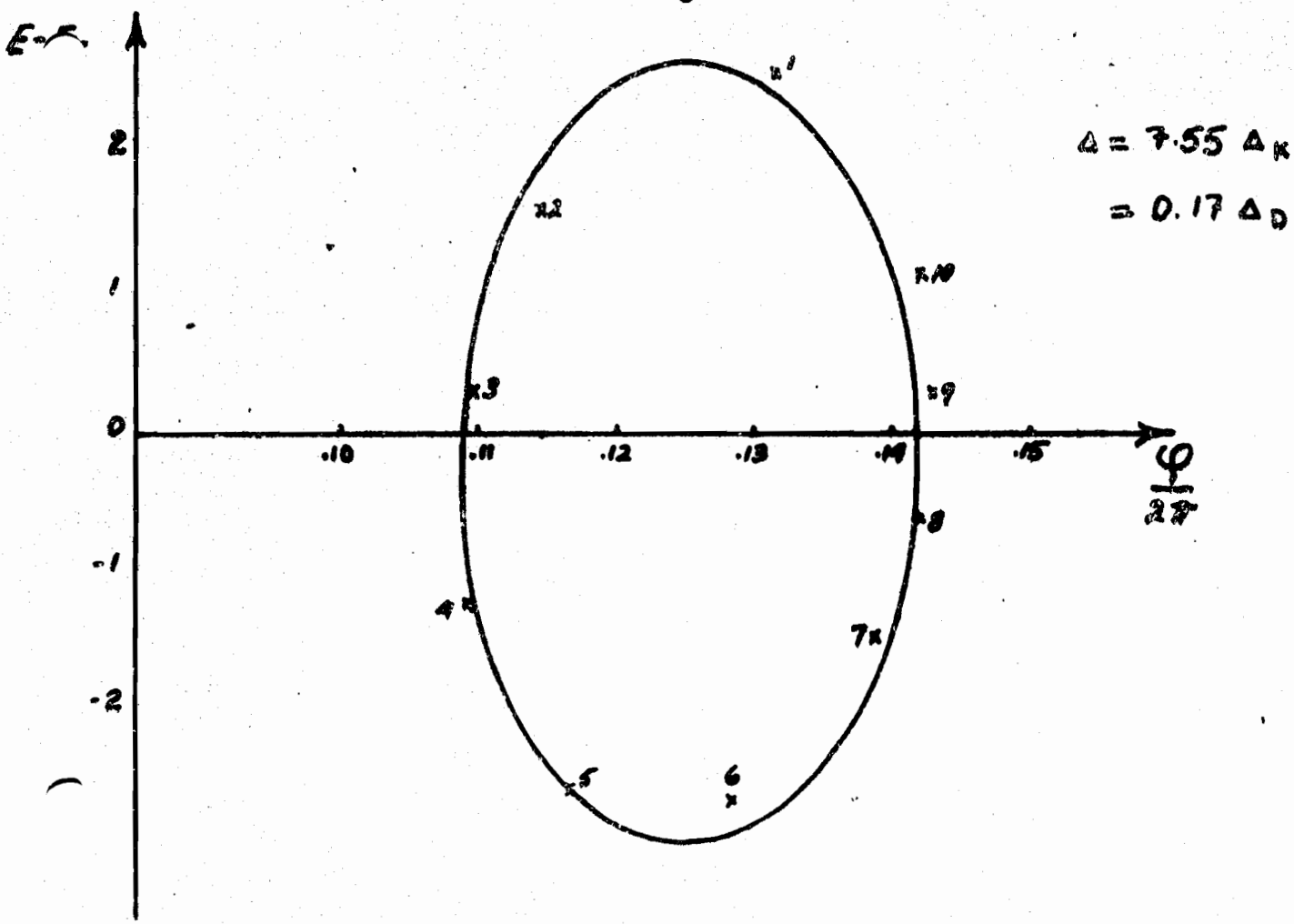

E-E_s
2
1
0
-1
-2
.10
.11
.12
.13
.14
.15
φ/2π
Δ = 7.55 Δ_K
= 0.17 Δ_D

Fig. 8.