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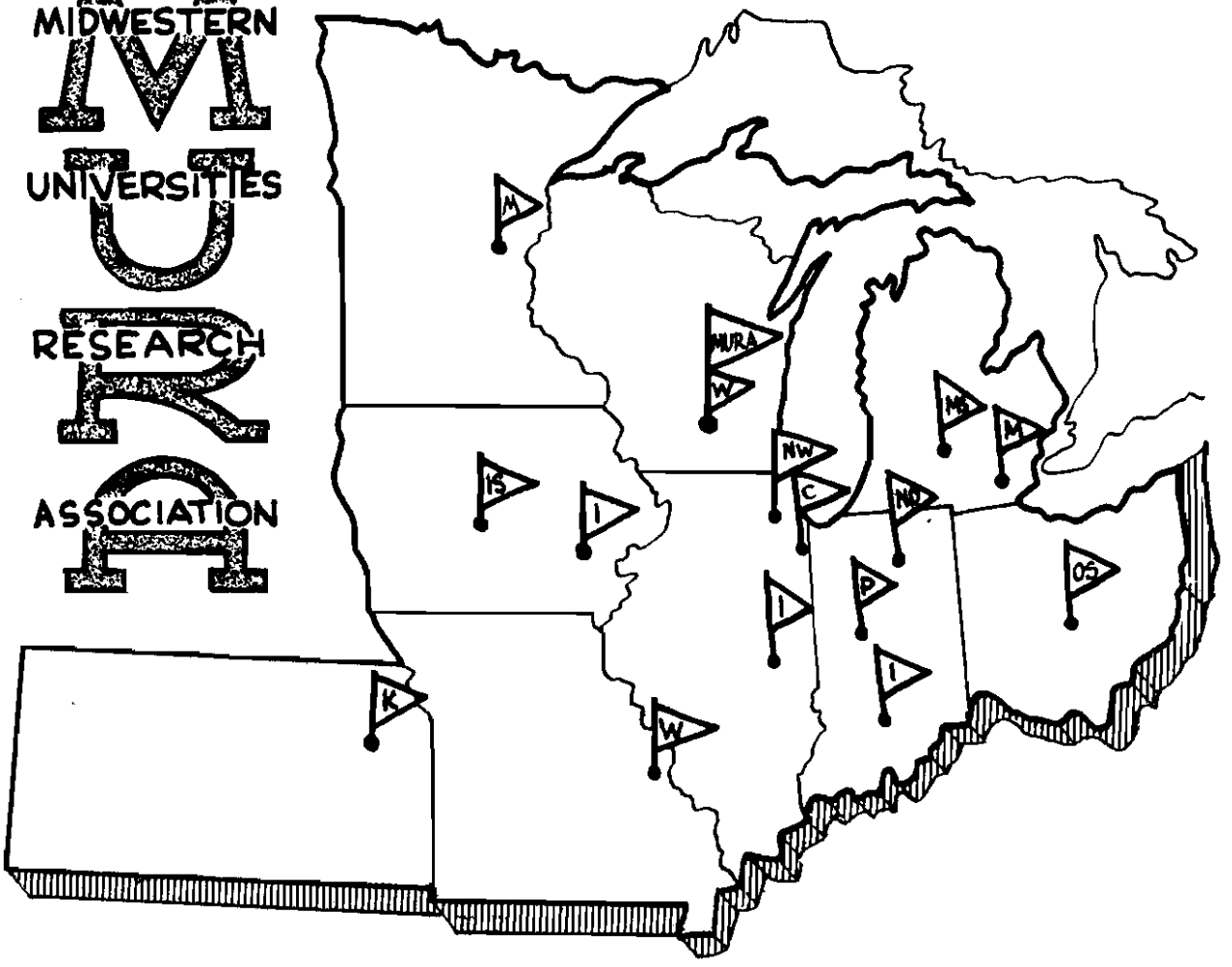

ON STURROCK'S PERTURBATION THEORY

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REPORT **NUMBER** 556

MURA - 556
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ON STURROCK'S PERTURBATION THEORY

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January 22, 1960

ABSTRACT

It has been found here that Sturrock's perturbation invariants are approximate quantities obtained by truncating the perturbation series expansion of the familiar Poincaré's integral invariants. Application of the second order invariant so constructed has been examined in a simple example.

*AEC Research and Development Report. Research supported by the U.S. Atomic Energy Commission, Contract No. AT(11-1) - 384.

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INTRODUCTION

This report summarizes a study, at the suggestion of Dr. Cole, on Sturrock's Perturbation Theory,[1] in anticipation of its possible application to future work.

It is proved here that Sturrock's perturbation invariants, which he constructed by solving partial differential equations[2], can also be obtained from the conceptually more familiar integral invariants of Poincaré.[3]

Some practical aspects regarding the application of the theory are discussed and the sample equation [4]

$$\frac{d^2q}{dt^2} + \omega^2 q + \frac{1}{2}\sin 2t\, q^2 = 0,$$

studied by Dr. Laslett in great detail in the neighborhood of the third subharmonic resonance, is used as a test. Actual calculation up to the second order in perturbation and first order in the displacement of the working point from the third subharmonic resonance (i.e., to first order in ϵ, defined by $\omega = \frac{2}{3} + \epsilon$) indicates that the invariant obtained is in acceptable agreement with Laslett's result by Moser's method.

II THEORETICAL CONSIDERATIONS

We consider a one dimensional problem. The unperturbed problem, supposed to have been rigorously solved, is described by a Lagrangian $L(q\, q'\, t)$ while the perturbed problem is represented by the

Lagrangian

$$L^*(q^* q^{*\prime} t) = L(q^* q^{*\prime} t) + \Delta L(q^* q^{*\prime} t) \tag{1}$$

wherein ΔL is regarded as small in the sense that if the solutions of the two problems are related as

$$q^* = q + \Delta q \, , \tag{2}$$

then Δq is so small that its higher powers are negligible for all time within our interest. Both of these Lagrangians are periodic with the same period.

We fix the initial time at t=0 and calculate the difference of actions expended along actually possible trajectories of the two problems after a time τ. This is given by

$$\begin{aligned} \Delta V \equiv V^* - V &= \int_0^\tau dt \left\{ L^*(q^* q^{*\prime} t) - L(q q' t) \right\} \\ &= \int_0^\tau dt \left\{ \Delta L(q q' t) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\Delta q \frac{\partial}{\partial q} + \Delta q' \frac{\partial}{\partial q'} \right)^n L^*(q q' t) \right\}. \end{aligned} \tag{3}$$

The symbol $\delta_\alpha \equiv \delta\alpha \frac{\partial}{\partial \alpha}$ is used to designate the variational operation with respect to variational parameters enumerated by α and implicit in q & q' (e.g. the initial parameters in the solved problem). The recognition that this variational operation and the perturbation are independent of each other implies the equality

$$\delta_\alpha \Delta V = \Delta(\delta_\alpha V). \tag{4}$$

The right hand side can be evaluated in the usual manner taking into consideration that the Euler-Lagrangian equations pertaining to the two problems must be satisfied. With the introduction of the following definitions,

$$p \equiv \frac{\partial L(q q' t)}{\partial q'} \, , \quad p^* \equiv \frac{\partial L^*(q^* q^{*\prime} t)}{\partial q^{*\prime}} \quad \& \quad p^* \equiv p + \Delta p \tag{5}$$

we arrive at

$$\delta_\alpha \Delta V(\alpha\tau) = (\Delta p\,\delta_\alpha q + p\,\delta_\alpha \Delta q + \Delta p\,\delta_\alpha \Delta q)\Big|_0^\tau \tag{6}$$

We may conceive a closed curve C in the q-p phase space as generated by varying α (5) and note that the operation δ_α is contemporaneous. Then by integrating (6) along this C which at time t=0 corresponds to C_0, say, we have

$$\oint_C \delta_\alpha \Delta V = \oint_C (\Delta p\,\delta_\alpha q + p\,\delta_\alpha \Delta q + \Delta p\,\delta_\alpha \Delta q) - \oint_{C_0} (\Delta p\,\delta_\alpha q + p\,\delta_\alpha \Delta q + \Delta p\,\delta_\alpha \Delta q). \tag{7}$$

The question whether this integral exists for at least some choice of C will be discussed later. Let $F(q(\alpha,t), q'(\alpha,t), t)$ be the abbreviation of the whole integrand in (3) then

$$\oint_C \delta_\alpha \Delta V = \oint_C \delta_\alpha \int_0^\tau dt\; F(q(\alpha,t), q'(\alpha,t), t)$$

$$= \int_0^\tau dt \oint_{C_t} \delta_\alpha F(q(\alpha,t), q'(\alpha,t), t) = \int_0^\tau dt \left[F(q(\alpha,t), q'(\alpha,t), t) \right]_{\alpha_0}^{\alpha_1}$$

where the limits α_1 and α_0, by definition of C, satisfy

$$q(\alpha_1) = q(\alpha_0), \quad q'(\alpha_1) = q'(\alpha_0) \qquad \text{for fixed } t,$$

so the integrand of the last result is zero; consequently

$$\oint_C \delta_\alpha \Delta V = K(c)$$

where K(c) is a constant depending on our choice of C. Hence (7) tells us that

$$\Delta\Phi(c) \equiv \oint_C (\Delta p\,\delta_\alpha q + p\,\delta_\alpha \Delta q + \Delta p\,\delta_\alpha \Delta q + \delta_\alpha G) \tag{8}$$

is an integral invariant for any arbitarily chosen closed C in p-q space, G being any arbitrary gauge function of p, q, τ. Although we are

supposed to have full knowledge of the functions $q(\alpha,t)$ and $p(\alpha,t)$ we are unable to evaluate (8) due to the presence of perturbation increments Δq and Δp whose functional dependence on q and p has yet to be found in some approximate manner. The approximation is done by subdividing the increments as

$$\Delta q = \Delta^1 q + \Delta^2 q + \Delta^3 q + \cdots$$
$$\Delta p = \Delta^1 p + \Delta^2 p + \Delta^3 p + \cdots \tag{9}$$

and $$\Delta V = \Delta^1 V + \Delta^2 V + \Delta^3 V + \cdots .$$

The notion of order of a quantity is defined as the total incremental power appearing in it. In the following, we shall not subdivide ΔL but consider it as of increment power 1.

For example

$$\Delta p = \frac{\partial L^*(q^* q^{*\prime} t)}{\partial q^{*\prime}} - \frac{\partial L(q q' t)}{\partial q'}$$
$$= \left[\frac{\partial \Delta L}{\partial q'} + \left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right)\frac{\partial L}{\partial q'} \right] + \left[\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right)\frac{\partial \Delta L}{\partial q'} \right.$$
$$\left. + \left(\Delta^2 q \frac{\partial}{\partial q} + \Delta^2 q' \frac{\partial}{\partial q'}\right)\frac{\partial L}{\partial q'} + \frac{1}{2}\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q}\right)^2 \frac{\partial L}{\partial q'} \right] + \cdots \tag{10}$$

and the first two square brackets define $\Delta^1 p$ and $\Delta^2 p$ respectively. A similar expansion of ΔV in (3) is:

$$\Delta V = \int_0^\tau dt \left[\Delta L + \left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right) L \right] + \int_0^\tau dt \left[\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right) \Delta L \right.$$
$$\left. + \frac{1}{2}\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right)^2 L + \left(\Delta^2 q \frac{\partial}{\partial q} + \Delta^2 q' \frac{\partial}{\partial q'}\right) L \right] + \cdots \tag{11}$$

and the first two integrals are respectively $\Delta' V$ and $\Delta^2 V$. By expanding formula (8) likewise, we find that the nth order invariant is given by by

$$\Delta^n \Phi(c) = \oint_C \sum_{\mu+\nu=n} \Delta^\mu p \, \delta_\alpha \Delta^\nu q \qquad \text{where} \quad \begin{matrix} \Delta^0 p \equiv p \\ \Delta^0 q \equiv q \end{matrix} \tag{12}$$

With these prescriptions and the specification, without loss of generality, that at t=0, $\Delta q = 0$ & $\Delta p = 0$, we may write down the first two orders of equation (6):

$$\delta_\alpha \int_0^\tau dt \left[\Delta L + \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right) L = p \, \delta_\alpha \Delta' q + \Delta' p \, \delta_\alpha q \right. \tag{13a}$$

$$\begin{aligned} \delta_\alpha \int_0^\tau dt \Big[& \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right) \Delta L + \frac{1}{2} \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right)^2 L \\ & + \left(\Delta^2 q \frac{\partial}{\partial q} + \Delta^2 q' \frac{\partial}{\partial q'}\right) L = p \, \delta_\alpha \Delta^2 q + \Delta' p \, \delta_\alpha \Delta' q + \Delta^2 p \, \delta_\alpha q . \end{aligned} \tag{13b}$$

The two invariants associated with (13) are just integrals of the left-hand sides along a closed curve C. However to save labor in actual evaluation, we should first eliminate any total differentials existing under the integral signs since they eventually will contribute nothing to the integral. For example, in (13a), by definition

$$\left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right) L = \Delta' q \frac{dp}{dt} + \frac{d\Delta' q}{dt} p = \frac{d}{dt}(\Delta' q \, p)$$

and in (13b) by virtue of (10) and the Euler-Lagrangian equations

$$\begin{aligned} \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right)^2 L &= \Delta' q \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right) \frac{\partial L}{\partial q} + \Delta' q' \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right) \frac{\partial L}{\partial q'} \\ &= \Delta' q \left(\frac{d \Delta' p}{dt} - \frac{\partial \Delta L}{\partial q} \right) + \Delta' q' \left(\Delta' p - \frac{\partial \Delta L}{\partial q'} \right) \\ &= \frac{d}{dt} (\Delta' q \, \Delta' p) - \left(\Delta' q \frac{\partial}{\partial q} + \Delta' q' \frac{\partial}{\partial q'}\right) \Delta L \end{aligned}$$

so that the first and second order invariants are

$$\Delta^1\Phi = \oint_C \delta_\alpha \int_0^\tau dt\, \Delta L \equiv \oint_C dt\, \Delta L$$
$$\Delta^2\Phi = \oint_C \delta_\alpha \int_0^\tau dt\, \frac{1}{2}\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right)\Delta L \equiv \oint dt\, \frac{1}{2}\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right)\Delta L \tag{14}$$

They are in complete accord with Sturrock's results obtained by solving partial differential equations.

With the separation of the total differentials, we note that (13) can now be more symmetrically put as:

$$\delta_\alpha \int_0^t dt\, \Delta L = \Delta^1 p\, \delta_\alpha q - \Delta^1 q\, \delta_\alpha p$$
$$\delta_\alpha \int_0^t dt \left[\frac{1}{2}\left(\Delta^1 q \frac{\partial}{\partial q} + \Delta^1 q' \frac{\partial}{\partial q'}\right)\Delta L\right] + \frac{1}{2}\left(\Delta^1 q\, \delta_\alpha \Delta^1 p - \Delta^1 p\, \delta_\alpha \Delta^1 q\right) = \Delta^2 p\, \delta_\alpha q - \Delta^2 q\, \delta_\alpha p \tag{15}$$

If we stop the approximation at 2nd order, then equations (15) are the equations of motion for the perturbation increments Δp & Δq and formulas (14) supply a recipe for the construction of perturbation invariants to study the stability of the perturbed dynamical system in the usual way.

For actual application, let us consider the solution

$$q = q(\gamma, J, t) \quad , \quad p = p(\gamma, J, t) \tag{16}$$

where γ, J are initial parameters which for periodic problems may be chosen to be the familiar angle and action variables. We shall regard (16) as a transformation from p-q space to γ- J space and shall choose the Jacobian of the transformation to have its determinant equal to one:

$$\mathcal{J} \equiv \begin{pmatrix} \frac{\partial q}{\partial \gamma} & \frac{\partial p}{\partial \gamma} \\ \frac{\partial q}{\partial J} & \frac{\partial p}{\partial J} \end{pmatrix} \qquad \text{and det. } \mathcal{J} = 1 \tag{17}$$

so that we have for this transformation:

$$\Delta p\,\delta q - \Delta q\,\delta p = \Delta J\,\delta\gamma - \Delta\gamma\,\delta J \quad , \quad \Delta p\frac{\partial}{\partial p} + \Delta q\frac{\partial}{\partial q} = \Delta J\frac{\partial}{\partial\gamma} + \Delta\gamma\frac{\partial}{\partial J} \quad . \tag{18}$$

If we agree to define an incremental Jacobian matrix of the νth order

$$\Delta^{\nu} g \equiv \begin{pmatrix} \frac{\partial\Delta^{\nu} q}{\partial\gamma} & \frac{\partial\Delta^{\nu} p}{\partial\gamma} \\ \frac{\partial\Delta^{\nu} q}{\partial J} & \frac{\partial\Delta^{\nu} p}{\partial J} \end{pmatrix}$$

then the differential form $\Delta p\,\delta\Delta q - \Delta q\,\delta\Delta p$ can be symbolically expressed as

$$(\Delta J \quad -\Delta\gamma)\ \widetilde{g^{-1}}\ (\widetilde{\Delta g})\ \widetilde{(\delta\gamma \quad \delta J)}$$

with (ΔJ $-\Delta\gamma$) representing a row matrix and the tilde, the transpose operation on the matrix under it. We now delete the label α and write equations (15) as

$$\delta\int_0^t dt\,\Delta L = \Delta' J\,\delta\gamma - \Delta'\gamma\,\delta J \tag{19a}$$

$$\delta\int_0^t dt\,\tfrac{1}{2}\left(\Delta' q\frac{\partial}{\partial q} + \Delta' q'\frac{\partial}{\partial q'}\right)\Delta L - \tfrac{1}{2}(\Delta J \quad -\Delta\gamma)\ \widetilde{g^{-1}}\ (\widetilde{\Delta' g})\ \widetilde{(\delta\gamma \quad \delta J)}$$
$$= \Delta^2 J\,\delta\gamma - \Delta^2\gamma\,\delta J \quad . \tag{19b}$$

In (19a), if we identify γ with the variational parameter α , we have

$$\frac{\partial\Delta L}{\partial\gamma} = \frac{d\Delta' J}{dt} \tag{20a}$$

and if J is identified with α , we have

$$\frac{\partial\Delta L}{\partial J} = -\frac{d\Delta'\gamma}{dt} \quad . \tag{20b}$$

These are the equations of motion of the first order perturbation increments. These $\Delta' J$ and $\Delta'\gamma$ are used in (19b) for the second order calculations or in (14) to evaluate the second order invariant. In the following, several aspects in connection with the operation:

$$\oint_C dt\, F(qq't) \tag{21}$$

are discussed.

It may be pointed out that the variational parameter α employed previously can be identified with the angle variable γ while the closed curve C here is most conveniently taken to be J = const. in γ-J space. To investigate whether this choice of C will guarantee the existence of the integral (21), let us assume the Lagrangian to have a period T_L and the period associated with the angle variable be T_ω. Then $Q \equiv \frac{T_L}{T_\omega}$ is the number of oscillations of q per Lagrangian period due to the variation in γ and for each additional Lagrangian period, the phase of q undergoes a shift equal to $2\pi Q$. Let $Q = N + \delta$ where N is an integer or zero and δ the remainder less than 1. If δ is zero, then (21) certainly exists since we just have to integrate it from 0 to T_L and at these limits q & p each assumes equal values. Then for all subsequent Lagrangian period, the state of affairs is just a repetition of the first Lagrangian period. If $\delta \neq 0$, let us say that there exist two integers $r < n$ such that

$$\delta = \frac{r}{n}.$$

Then the operation in (22) is equivalent to

$$\sum_{\nu=0}^{n} \int_0^{T_L} dt\, F(J, \gamma_\nu, t)$$

where $\gamma_\nu \equiv \gamma + 2\pi\nu Q$ so that $\gamma_n = \gamma + 2\pi r$ and since r is an integer, the phases of q & p at the integration limits differ only by 0 (mod. 2π).

In the case that δ is not expressible as a rational fraction, integral (22) does not exist. Or, of more practical interest to us,

when we want δ to be expressed as $\delta = \frac{n}{n} + \epsilon$, $\epsilon \ll \frac{n}{n}$ with a preassigned n, in order to study the problem in the neighborhood of the nth subharmonic resonance, we must reformulate the unperturbed problem and treat the effect of ϵ as another source of perturbation.

III PRACTICAL CONSIDERATIONS

To apply the results just developed, we shall formulate the problem in such a way that the starting point of the problem is made as simple as possible. The unperturbed problem is supposed to be a linear problem.

$$L(qq't) = \frac{1}{2}q'^2 - \frac{1}{2}K(t)q^2 \tag{22}$$

where K(t) is a general focussing function. The perturbing Lagrangian may very generally be put as

$$\Delta L(qq't) = qF(t) + q^3C(t) + q^4Q(t) + \cdots \tag{23}$$

where F(t) is a forcing function (e.g. caused by field deviation from ideal case) and the others are nonlinear contributions. Since the linear problem has been solved, the relevant instantaneous wave length $\beta(t)$ connected to the focussing function K(t) through the equation

$$K(t) = \frac{1}{\beta^2} + \frac{\beta'^2}{4\beta^2} - \frac{\beta''}{2\beta} \qquad \text{(where prime denotes } \frac{d}{dt}\text{)}$$

and the characteristic phase shift per Lagrangian period, μ, defined as

$$\mu \equiv \int_0^{T_L} \frac{dt}{\beta} \qquad \text{(where } T_L \text{ is the Lagrangian period)}$$

are rigorously known. Therefore we may carry out a slightly modified Courant-Snyder transformation: (6)

$$\xi = \frac{1}{\sqrt{\beta\mu}}\, q \quad , \quad \tau \equiv \int \frac{dt}{\beta\mu} \tag{24}$$

which reformulates the problem to one described by a new Lagrangian

$$\Lambda(\xi\xi'\tau) = \frac{1}{2}\xi'^2 - \frac{1}{2}\mu^2\xi^2$$

and

$$\Delta\Lambda(\xi\xi'\tau) = (\mu\beta)^{3/2}\xi\, F(\tau) + (\nu\beta)^{5/2}\xi^3\, C(\tau).$$

So without loss of generality, we always can take a simple harmonic oscillator problem as the starting point of our problem. In the following, a standard linear problem Lagrangian is considered,

$$L(qq't) = \frac{1}{2}q'^2 - \frac{1}{2}\omega^2 q^2 \tag{25}$$

the solution of which is represented by

$$q = \sqrt{\frac{2J}{\omega}}\cos(\omega t + \gamma) \tag{26}$$

$$p = -\sqrt{2J\omega}\sin(\omega t + \gamma)$$

The Jocobian matrix connecting qp to γJ as defined earlier satisfies det. $\mathcal{J} = 1$.

Following Sturrock, we investigate the situation in the neighborhood of the nth subharmonic resonance by introducing a variable ϵ defined by

$$Q = \frac{r}{n} + \epsilon \qquad \text{where } r \text{ is an integer} < n$$

$$\text{and} \quad \epsilon \ll \frac{1}{n}.$$

All calculations will stop at the first order of ϵ and wherever terms involving ϵ appear with terms without ϵ of the same perturbation order, the former will be dropped.

Now we take the equation which has been studied by L. J. Laslett in some detail using Moser's method, as an example to test the present

method. The equation is

$$\frac{d^2q}{dt^2} + \left(\frac{2\nu}{N}\right)^2 q + \frac{1}{2}\sin 2t\, q^2 = 0 \,, \qquad \frac{2\nu}{N} \sim 0.6 \tag{27}$$

corresponding to a Lagrangian

$$L^*(q\,q'\,t) = \frac{1}{2}q'^2 - \frac{1}{2}\omega^2 q^2 - \frac{1}{6}\sin 2t\, q^3 \,, \qquad \omega \equiv \frac{2\nu}{N} \tag{28}$$

and the 3rd subharmonic resonance is under consideration. We note here that the Lagrangian has a period $T_L = \pi$ and the oscillation has a period $T_\omega = \frac{2\pi}{\omega}$ therefore the number of oscillations per Lagrangian period is $Q = \frac{T_L}{T_\omega} = \frac{\omega}{2}$ and the oscillation phase shift per Lagrangian period is $2\pi Q = \pi\omega = \frac{2}{3}\pi$ if $\omega = \frac{2}{3}$.

Consequently, for the unperturbed problem, if at time t = 0, $q \sim \cos\gamma$, then in the first Lagrangian period $q \sim \cos(\omega t + \gamma)$ and in the (n+1)th Lagrangian period $q \sim \cos(\omega t + \gamma_n)$ where $\gamma_n = \gamma + 2n\pi Q$.

For the present problem, at the end of the 3rd Lagrangian period p & q come back to assume their values as at t = 0, so a closed curve in the pq phase space is associated with a time interval of three Lagrangian periods, viz. 3π.

To start the calculation, we transform pq into γJ. This transformation is canonical, so we have

$$\Delta p\,\delta q - \Delta q\,\delta p = \Delta J\,\delta\gamma - \Delta\gamma\,\delta J. \tag{29}$$

The perturbing Lagrangian is taken as

$$\Delta L(q\,q'\,t) = -\frac{1}{2}\left(\epsilon^2 + \frac{4}{3}\epsilon\right)q^2 - \frac{1}{6}\sin 2t\, q^3 \tag{30}$$

where the ϵ^2-term is immediately dropped, so that the first two terms in (9), with the aid of (8), are given by

$$\oint(\Delta J\,\delta\gamma-\Delta\gamma\,\delta J)=\sum_{n=1}^{3}\int_0^{\pi}dt\;\Delta L(J,\gamma_n,t)$$

$$=\sum_{n=1}^{3}\int_0^{\pi}dt\left\{\frac{-2}{3}\epsilon\left(\frac{2J}{\omega}\right)\cos^2(\omega t+\gamma_n)-\frac{1}{6}\sin 2t\left(\frac{2J}{\omega}\right)^{3/2}\cos^3(\omega t+\gamma_n)\right\}$$

wherein the argument ω in the cosine function is taken equal to $\frac{2}{3}$. The result of the evaluation is

$$-\frac{2\pi\epsilon}{\omega}J+\left(\frac{2}{\omega}\right)^{3/2}J^{3/2}\frac{\pi}{16}\sin 3\gamma. \tag{31}$$

Sturrock's first order perturbation theory stops right here. To calculate the second order contribution, we note in this particular problem the operation $\left(\Delta'\frac{\partial}{\partial q}+\Delta' q'\frac{\partial}{\partial q'}\right)$ equals $\Delta' J\frac{\partial}{\partial J}+\Delta'\gamma\frac{\partial}{\partial\gamma}$.

According to (20)

$$\frac{d\Delta' J}{dt}=\frac{\partial\Delta L}{\partial\gamma}=\frac{1}{4}J^{3/2}\left(\frac{2}{\omega}\right)^{3/2}\sin 2t\,\cos(\omega t+\gamma)\sin(2\omega t+2\gamma)$$

$$\frac{d\Delta'\gamma}{dt}=-\frac{\partial\Delta L}{\partial J}=\frac{1}{4}J^{1/2}\left(\frac{2}{\omega}\right)^{3/2}\sin 2t\,\cos^3(\omega t+\gamma).$$

We have neglected the effect of ϵ in these calculations of high order. Therefore

$$\Delta^2\Phi=\frac{1}{2}\oint_C dt\left(\Delta'\gamma\frac{d\Delta' J}{dt}-\Delta' J\frac{d\Delta'\gamma}{dt}\right)$$

$$\approx\frac{-J^2\left(\frac{2}{\omega}\right)^3 3\pi}{2048}\alpha^* \qquad \text{where} \qquad \alpha^*\approx 2.86.$$

The detailed calculation is shown in the Appendix.

Hence, to second order by this method

$$\Delta\Phi\simeq-3\pi\epsilon J+\left(\frac{2}{\omega}\right)^{3/2}J^{3/2}\frac{3\pi}{48}\sin 3\gamma-\frac{3\pi}{2048}J^2\left(\frac{2}{\omega}\right)^3\alpha^*$$

which on dividing by -3π becomes

$$\frac{\Delta\Phi}{-3\pi}\equiv H_2^*=\epsilon J-\left(\frac{2}{\omega}\right)^{3/2}J^{3/2}\frac{\sin 3\gamma}{48}+\left(\frac{2}{\omega}\right)^3 J^2\frac{\alpha^*}{2048}.$$

This H_2^* compared with Laslett's calculation (H_2 in equation (35), MURA - 452) differs in the coefficient of J^2 - term by nearly a factor $\frac{\alpha^*}{\alpha} = 1.6$ (Laslett's α is approximately 1.75). This discrepancy is imputed to the fact that the 2nd order calculation here has neglected the ϵ contribution in ΔL. Furthermore, the notions of order in the two methods may not be the same.

APPENDIX

We first resolve the trigonometric functions as in the following:

$$\sin 2t \cos^3(\omega t+\gamma) = \frac{1}{8}\Big[-\sin 3\gamma + \sin(3\omega t+2t+3\gamma) - 3\sin(\omega t-2t+\gamma) + 3\sin(\omega t+2t+\gamma)\Big]$$

$$\sin 2t \cos(\omega t+\gamma)\sin(2\omega t+2\gamma) = \frac{1}{4}\Big[\cos 3\gamma - \cos(3\omega t+2t+3\gamma) + \cos(\omega t-2t+\gamma) - \cos(\omega t+2t+\gamma)$$

$$\int_0^t dt \sin 2t \cos^3(\omega t+\gamma) = \frac{1}{8}\Big[-t\sin 3\gamma - \frac{\cos(3\omega t+2t+3\gamma)-\cos 3\gamma}{3\omega+2} + \frac{3\cos(\omega t-2t+\gamma)-3\cos\gamma}{\omega-2} - \frac{3\cos(\omega t+2t+\gamma)-3\cos\gamma}{\omega+2}\Big]$$

$$\int_0^t dt \sin 2t \cos(\omega t+\gamma)\sin(2\omega t+2\gamma) = \frac{1}{4}\Big[t\cos 3\gamma - \frac{\sin(3\omega t+2t+3\gamma)-\sin 3\gamma}{3\omega+2} + \frac{\sin(\omega t-2t+\gamma)-\sin\gamma}{\omega-2} - \frac{\sin(\omega t-2t+\gamma)-\sin\gamma}{\omega+2}\Big].$$

For abbreviation, let $\phi_0 \equiv 3\omega t+2t+3\gamma$, $\phi_\pm \equiv \omega t \pm 2t+\gamma$

then in the lengthy expression $\Delta'\gamma \frac{d\Delta' J}{dt} - \Delta' J \frac{d\Delta'\gamma}{dt}$ we first observe that

$$\sum_{3T_L} \int_0^\pi \begin{matrix}\sin\\ \cos\end{matrix}\phi \begin{matrix}\sin\\ \cos\end{matrix} 3\gamma \; dt = 0 \qquad \text{for } \phi_0, \phi_+ \text{ and } \phi_-$$

$$\sum_{3T_L} \int_0^\pi \begin{matrix}\sin\\ \cos\end{matrix}\phi_1 \begin{matrix}\sin\\ \cos\end{matrix}\phi_2 \; dt = 0 \qquad \text{if } \phi_1 \neq \phi_2.$$

In all of them $\omega \simeq 2/3$ is substituted in the argument.

Then

$$\sum_{3T_L} \int_0^{\pi} dt \left(\Delta'\gamma \frac{d\Delta' J}{dt} - \Delta' J \frac{d\Delta'\gamma}{dt} \right) = A \sum_{3T_L} \left\{ \frac{-3\pi}{3\omega+2} - \frac{3\pi}{\omega+2} - \frac{3\pi}{\omega-2} \right.$$

$$+ 3 \sin\frac{2\pi}{3} \left(\frac{1}{\omega+2} - \frac{1}{\omega-2} \right)^2 + \frac{\pi \sin 3\gamma \left(\sin\gamma - \sin\left(\frac{2}{3}\pi+\gamma\right) \right)}{\omega-2} - \frac{\pi \sin 3\gamma \left(\sin\gamma - \sin\left(\frac{2}{3}\pi+\gamma\right) \right)}{\omega+2}$$

$$\left. + \frac{3\pi \cos 3\gamma \left(\cos\gamma - \cos\left(\frac{2}{3}\pi+\gamma\right) \right)}{\omega+2} + \frac{3\pi \cos 3\gamma \left(\cos\left(\frac{2}{3}\pi+\gamma\right) - \cos\gamma \right)}{\omega-2} \right\}$$

where $A \equiv \frac{-1}{512} J^2 \left(\frac{2}{\omega}\right)^3$.

On carrying out $\sum_{3T_L}$ the 4 last terms will vanish and on putting in $\omega \simeq \frac{2}{3}$ for the surviving terms, we have

$$\oint_C dt \, \frac{1}{2} \left(\Delta'\gamma \frac{\partial \Delta L}{\partial \gamma} + \Delta' J \frac{\partial \Delta L}{\partial J} \right) = \frac{-1}{1024} J^2 \left(\frac{2}{\omega}\right)^3 3\pi \, (1.43) = \frac{-J^2 \left(\frac{2}{\omega}\right)^3 3\pi}{2048} \times 2.86.$$

REFERENCES

1. P. A. Sturrock, Annals of Physics 3 , p. 113-189 (1958).

2. Appendix of the last reference.

3. For example, in Goldstein's, Classical Mechanics or Whittaker's Analytical Dynamics, Chapter X.

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5. Such as pictured in a proof used by Whittaker in his book, Analytical Dynamics, Chapter X, p. 115.

6. E. D. Courant & H. S. Snyder, Annals of Physics 3, pp. 1-48 (1958).