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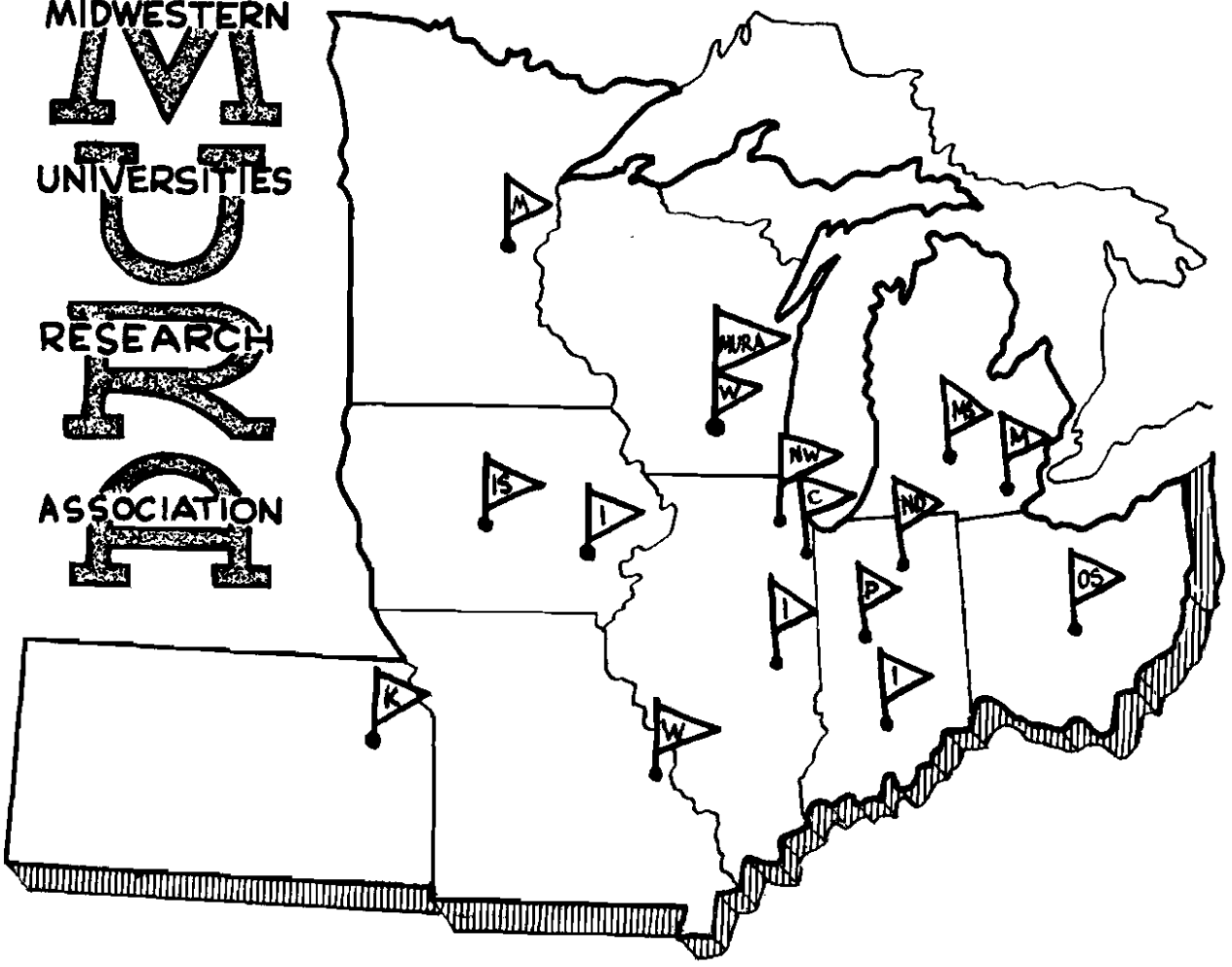

MAGNETIC FIELD CALCULATIONS USING DISTRIBUTED CURRENTS

S. C. Snowdon

REPORT **NUMBER** 553

MURA - 553
Internal

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2203 University Avenue, Madison, Wisconsin

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S. C. Snowdon**

January 8, 1960

ABSTRACT

A modification of the standard algorism for obtaining solutions of the scaled Laplace equation is introduced in order to permit the use of this equation in the solution of certain magnetostatic problems involving distributed currents. Application is made to the case of an FFAG spiral sector magnet.

*AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission, Contract No. AT(11-1)-384.

**On leave from the Bartol Research Foundation of the Franklin Institute, Swarthmore, Pennsylvania.

I. INTRODUCTION

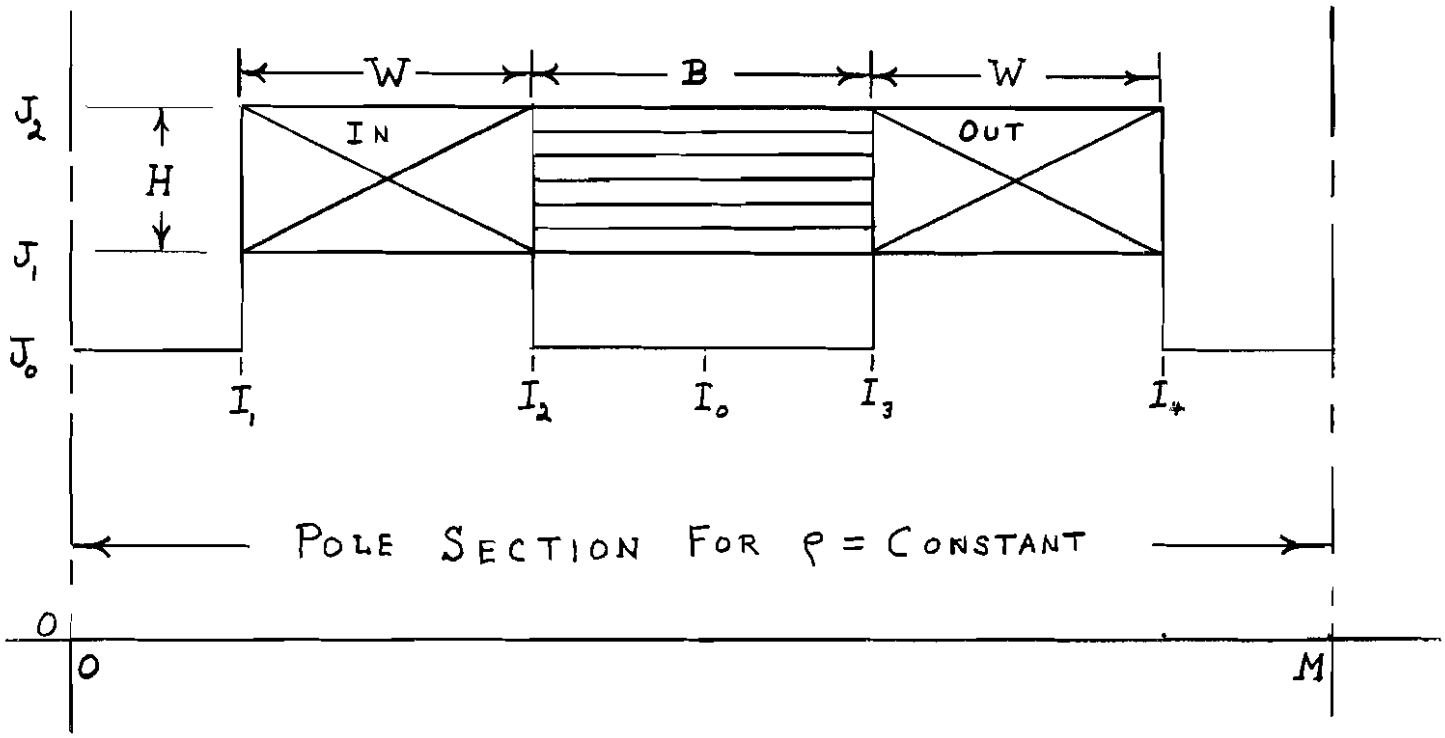
Laslett[1] has developed algorisms for the determination of the magnetostatic scalar potential characterizing a spiral sector FFAG magnetic guide field (MARK V). His solutions are obtained in terms of pre-assigned boundary potentials. In addition, Laslett[2] has considered a modification of the previous algorism in order to incorporate the effect of distributed currents on the potentials. The example, however, pertains only to two dimensional fields. His method has been generalized to the case of three dimensional scaling potentials.[3,4] In the following, this generalization is adapted to the relaxation computing technique used in the FOROCYL-GOLLYCONDER program.[5]

II. BOUNDARY POTENTIALS

Previous calculations[3,4] have demonstrated that magnetic fields generated by distributed currents may be calculated in terms of a scalar potential function. The specifications of boundary values common to most potential problems, in this case must be supplemented by a specification of the discontinuity in the potential in passing from the free space region to the current carrying region. In this section only the boundary potentials are discussed.

The variables ξ and η used in previous calculations,[3,4] while convenient for formal development, are not precisely the variables required in the FOROCYL agendum. If the ξ and η variables required in the FOROCYL program are here designated by x and y, the relations between the previous ξ and η and the present x and y are:

$$\xi = 2\pi x \quad , \quad \eta = \frac{2\pi}{\sqrt{\frac{1}{w^2} + N^2}}\, y \ . \tag{1}$$



The unit cell in the FOROCYL two dimensional grid has dimensions

$$\Delta x = h \quad , \quad \Delta y = \ell \ . \tag{2}$$

Since the origin of coordinates in the FOROCYL program is at the left of a sector ; whereas the origin of the ξ or x coordinate is in the center of the pole, let.

$$x = (I - I_0)h \quad , \quad y = J\ell \ , \tag{3}$$

where I and J are integers and I_0 is at the center of the pole.

The boundary potentials are:

$$\Psi(I, J_1) = \frac{\Omega_2 J_1}{2\pi h k^2 w W H} \ , \tag{4}$$

$$\Psi(I, J_2) = \frac{\Omega_2 J_2}{2\pi h k^2 w W H} \ , \tag{5}$$

$$\Psi(I_2 J_1) = \Omega_2 - \frac{\Omega_2 J_1}{kWH}\left[W - \frac{1}{2\pi h k w}\right], \tag{6}$$

$$\Psi(I_2 J_2) = -\frac{\Omega_2 J_2}{kWH}\left[W - \frac{1}{2\pi h k w}\right], \tag{7}$$

$$\Omega(0\, J_0) = \Omega(M\, J_0) = 0, \tag{8}$$

$$\Omega(I_1 J_0) = \Omega(I_4 J_0) = 0, \tag{9}$$

$$\Omega(I_1 J_1) = \Omega(I_4 J_1) = 0, \tag{10}$$

$$\Omega(I_2 J_0) = \Omega(I_2 J_1) = \Omega_0 e^{\frac{\pi h (k+1)}{\frac{1}{w}+\frac{1}{w'}} B} \equiv \Omega_2, \tag{11}$$

$$\Omega(I_3 J_0) = \Omega(I_3 J_1) = \Omega_0 e^{-\frac{\pi h (k+1)}{\frac{1}{w}+\frac{1}{w'}} B} \equiv \Omega_3, \tag{12}$$

$$\Omega(I J_0) = \Omega_0 e^{-\frac{2\pi h (k+1)}{\frac{1}{w}+\frac{1}{w'}}(I - I_0)} \quad \text{for } I_0 - \frac{B}{2} < I < I_0 + \frac{B}{2}, \tag{13}$$

$$\Psi(I_3 J_1) = \Omega_3 - \frac{\Omega_3 J_1}{kWH}\left[W + \frac{1}{2\pi h k w}\right], \tag{14}$$

$$\Psi(I_3 J_2) = -\frac{\Omega_3 J_2}{kWH}\left[W + \frac{1}{2\pi h k w}\right], \tag{15}$$

$$\Psi(I_4, J_1) = -\frac{\Omega_3 J_1}{2\pi h k^2 w W H} , \tag{16}$$

$$\Psi(I_4, J_2) = -\frac{\Omega_3 J_2}{2\pi h k^2 w W H} , \tag{17}$$

where W, H, B, etc. are shown in the accompanying figure and Eqs. (11-12) define the potentials Ω_2 and Ω_3. Only the potentials at the corners of the various regions are recorded. A linear interpolation between any two adjacent potentials serves to specify the boundary potential completely. Of course, a linear interpolation of the current-free space interface is not permitted; this face is handled by a discontinuity in potential that will be discussed separately.

III. CURRENT VALUES

The algorism adopted for the solution of the boundary value problem must be modified to account for the discontinuity in potential at the current-free space interface.[3,4] Laslett[2] has termed such a modification a current value since the correction to the standard algorism accounts for the presence of the current. For the standard point, the algorism developed by Laslett[2] is:

$$\mathcal{D}\Psi_{00} = N_{01}\Psi_{01} + N_{-10}\Psi_{-10} + N_{10}\Psi_{10} + N_{0-1}\Psi_{0-1}$$
$$+ N_{-11}(\Psi_{-11} + \Psi_{1-1}) + N_{11}(\Psi_{11} + \Psi_{-1-1}) . \tag{18}$$

Equation (18) must be modified for two rows of mesh points within the current carrying region. The simplest alteration occurs for the row $J = J_1 + 1$.

In this case, Ψ_{-1-1} , Ψ_{0-1} , and Ψ_{1-1} lie on the boundary $J = J_1$. On this boundary two functions are to be specified, Ψ and Ω , the difference being a pre-assigned quantity in the problem. If no current values are to be associated with Ω in the free space region, the computer must store Ω and not Ψ in the row $J = J_1$. Hence, for the row $J = J_1 + 1$, the computer will pick up Ω_{-1-1}, Ω_{01}, and Ω_{1-1} instead of the corresponding Ψ values. If the discontinuity on the boundary is Δ ,

$$\Omega = \Psi + \Delta , \tag{19}$$

and for row $J = J_1 + 1$

$$\Psi_{00} = [\text{Std. Algorism}] + [CV]_{I\,J_1+1} \tag{20}$$

where

$$\mathcal{D}[CV]_{I\,J_1+1} = -N_{11}\,\Delta(I-1) - N_{-11}\,\Delta(I+1) - N_{-10}\,\Delta(I) . \tag{21}$$

From Eqs. (30) and (35) of a previous paper,[4] after using Eqs. (1-3), the current values for $J = J_1 + 1$ become:

$$[CV]_{I_1 J_1+1} = \frac{\Omega_2 J_1}{\mathcal{D} k W H}\left[\left(N_{11} + N_{-11} + N_{0-1}\right)\frac{1}{2\pi h k w} + \left(N_{11} - N_{-11}\right)\right], \tag{22}$$

$$[CV]_{I_2 J_1+1} = -\frac{\Omega_2 J_1}{\mathcal{D} k W H}\left[\left(N_{11} + N_{-11} + N_{0-1}\right)\left(W - \frac{1}{2\pi h k w}\right) - \left(N_{11} - N_{-11}\right)\right], \tag{23}$$

$$[CV]_{I_3 J_1+1} = -\frac{\Omega_3 J_1}{\mathcal{D} k W H}\left[\left(N_{11} + N_{-11} + N_{0-1}\right)\left(W + \frac{1}{2\pi h k w}\right) + \left(N_{11} - N_{-11}\right)\right], \tag{24}$$

$$[CV]_{I_4 J_1+1} = -\frac{\Omega_3 J_1}{\mathcal{D} k W H}\left[\left(N_{11}+N_{-11}+N_{0-1}\right)\frac{1}{2\pi h k w} + \left(N_{11}-N_{-11}\right)\right], \tag{25}$$

where only the end point values have been recorded since a linear interpolation serves to specify these current values. The only values retained, however, are the interpolated points, there being no current values at the actual end points.

The development of a correction to the standard algorism for points along the boundary $J = J_1$ is not as easily determined as for the row $J = J_1 + 1$. It is necessary to predict what Ω would be if it were continued smoothly into the Ψ region. A Taylor series expansion of Ω about each boundary point is required. In addition to the boundary conditions developed in a previous report,[3] it is necessary to match the normal components of $\vec{H}$ in order to obtain $\frac{\partial \Omega}{\partial \eta}$ as a function of ξ. Thus, since

$$\left(\vec{l}_\theta \times \vec{n}\right)\cdot \vec{H} = \frac{1}{\sqrt{1+\left(\frac{1}{wN}\right)^2+\eta^2}}\left(-\vec{k}+\eta\,\vec{l}_r\right)\cdot \vec{H}, \tag{26}$$

using Eqs. (33), (76), and (77) of MURA - 535 gives, after replacing η_0 by η_1, as is required in MURA - 539

$$\frac{\partial \Omega}{\partial \eta} - \eta_1\left[(k+1)\Omega + \frac{1}{w}\frac{\partial \Omega}{\partial \xi} - \eta_1\frac{\partial \Omega}{\partial \eta}\right] = 4\pi r_0^2 F(\xi) - \eta_1\left[(k+1)\Psi + \frac{1}{w}\frac{\partial \Psi}{\partial \xi} - \eta_1\frac{\partial \Psi}{\partial \eta}\right]. \tag{27}$$

Relations derived from Eqs. (68), (76), and (83) of MURA - 535 may be used to simplify Eq. (27). Thus, at $\eta = \eta_1$ for $\xi_0 < \xi < \xi_1$

$$\frac{\partial \Omega}{\partial \eta} = \frac{\partial \Psi}{\partial \eta} + \frac{(k+1)\,\Omega_3}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1 - \xi + \frac{1}{k(k+1)w}\right]. \tag{28}$$

Equation (28) together with appropriately differentiated boundary potentials serve to connect the differential coefficients in the Taylor expansion of Ω

with the coefficients in a similar expansion of Ψ . Thus, if the notation of Eqs. (4-17) is used, for $J = J_1$ and $I_1 < I < I_2$

$$\Delta_{00} = \frac{\Omega_2 J_1}{kWH}\left[I - I_1 - \frac{1}{2\pi h k w}\right], \tag{29}$$

$$h\,\Delta_x = \frac{\Omega_2 J_1}{kWH}, \tag{30}$$

$$\ell\,\Delta_y = \frac{(k+1)\Omega_2}{kWH}\left[I - I_1 - \frac{1}{2\pi h k (k+1) w}\right], \tag{31}$$

$$h\ell\,\Delta_{xy} = \frac{(k+1)\Omega_2}{kWH}, \tag{32}$$

and

$$\ell^2\Delta_{yy} = \frac{(k+1)\Omega_2 J_1 \ell^2}{WH}\cdot\frac{\left[I - I_1 + \frac{1}{2\pi h (k+1) w}\right]}{\left[\frac{\left(\frac{1}{w^2} + N^2\right)}{4\pi^2} + \ell^2 J_1^2\right]}. \tag{33}$$

For $J = J_1$ and $I_3 < I < I_4$

$$\Delta_{00} = \frac{\Omega_3 J_1}{kWH}\left[I_4 - I + \frac{1}{2\pi h k w}\right], \tag{34}$$

$$h\,\Delta_x = -\frac{\Omega_3 J_1}{kWH}, \tag{35}$$

$$\ell\,\Delta_y = \frac{(k+1)\Omega_3}{kWH}\left[I_4 - I + \frac{1}{2\pi h k (k+1) w}\right], \tag{36}$$

$$h\ell\,\Delta_{xy} = -\frac{(k+1)\Omega_3}{kWH}, \tag{37}$$

and

$$\ell^2\Delta_{yy} = \frac{(k+1)\Omega_3 J_1 \ell^2}{WH}\cdot\frac{\left[I_4 - I - \frac{1}{2\pi h (k+1) w}\right]}{\left[\frac{\left(\frac{1}{w^2} + N^2\right)}{4\pi^2} + \ell^2 J_1^2\right]}. \tag{38}$$

Equations (33) and (38) are found by noting that both Ω and Ψ and hence Δ must satisfy the scaled Laplace equation on the boundary $\eta = \eta_1$:

$$(k+1)^2\Omega + \frac{2(k+1)}{w}\frac{\partial\Omega}{\partial\xi} - (2k+1)\eta\frac{\partial\Omega}{\partial\eta} + \left(\frac{1}{w^2}+N^2\right)\frac{\partial^2\Omega}{\partial\xi^2} - \frac{2\eta}{w}\frac{\partial^2\Omega}{\partial\xi\partial\eta} + (1+\eta^2)\frac{\partial^2\Omega}{\partial\eta^2} = 0. \tag{39}$$

Although higher order derivatives may be evaluated by differentiation of Eq. (39), they are not recorded since the standard algorism may only be obtained from a single Taylor expansion if terms higher than the second power are emitted. The algorism of Eq. (18) applied to the Ω function on the boundary is seen to require a correction for the functions Ω_{-11}, Ω_{01}, and Ω_{11} since the computer stores the corresponding Ψ functions at these points. In the notation of Eq. (19), it is necessary to compute Δ_{-11}, Δ_{01}, and Δ_{11}. A Taylor series expansion of Δ then gives:

$$\Delta_{01} = \Delta_{00} + \ell\,\Delta_y + \tfrac{1}{2}\ell^2\Delta_{yy}\,, \tag{40}$$

$$\tfrac{1}{2}(\Delta_{11}+\Delta_{-11}) = \Delta_{00} + \ell\,\Delta_y + \tfrac{1}{2}\ell^2\Delta_{yy} = \Delta_{01}\,, \tag{41}$$

and

$$\tfrac{1}{2}(\Delta_{11}-\Delta_{-11}) = h\,\Delta_x + h\ell\,\Delta_{xy}\,. \tag{42}$$

Equations (18), using the stored functions in the standard algorism becomes:

$$\mathcal{D}\Omega_{00} = \mathcal{D}[\text{Std. Algorism}] + N_{01}\Delta_{01} + (N_{11}+N_{-11})\tfrac{1}{2}(\Delta_{11}+\Delta_{-11}) + (N_{11}-N_{-11})\tfrac{1}{2}(\Delta_{11}-\Delta_{-11})\,, \tag{43}$$

from which the current value is seen to be for $J = J_1$:

$$\mathcal{D}[CV]_{IJ_1} = (N_{01}+N_{11}+N_{-11})\Delta_{01} + (N_{11}-N_{-11})\tfrac{1}{2}(\Delta_{11}-\Delta_{-11})\,. \tag{44}$$

As was the case for the row $J = J_1 + 1$, a linear interpolation of the current values between the end values is the best means of evaluating Eq. (44). It must be remembered, however, that the end values are omitted finally.

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