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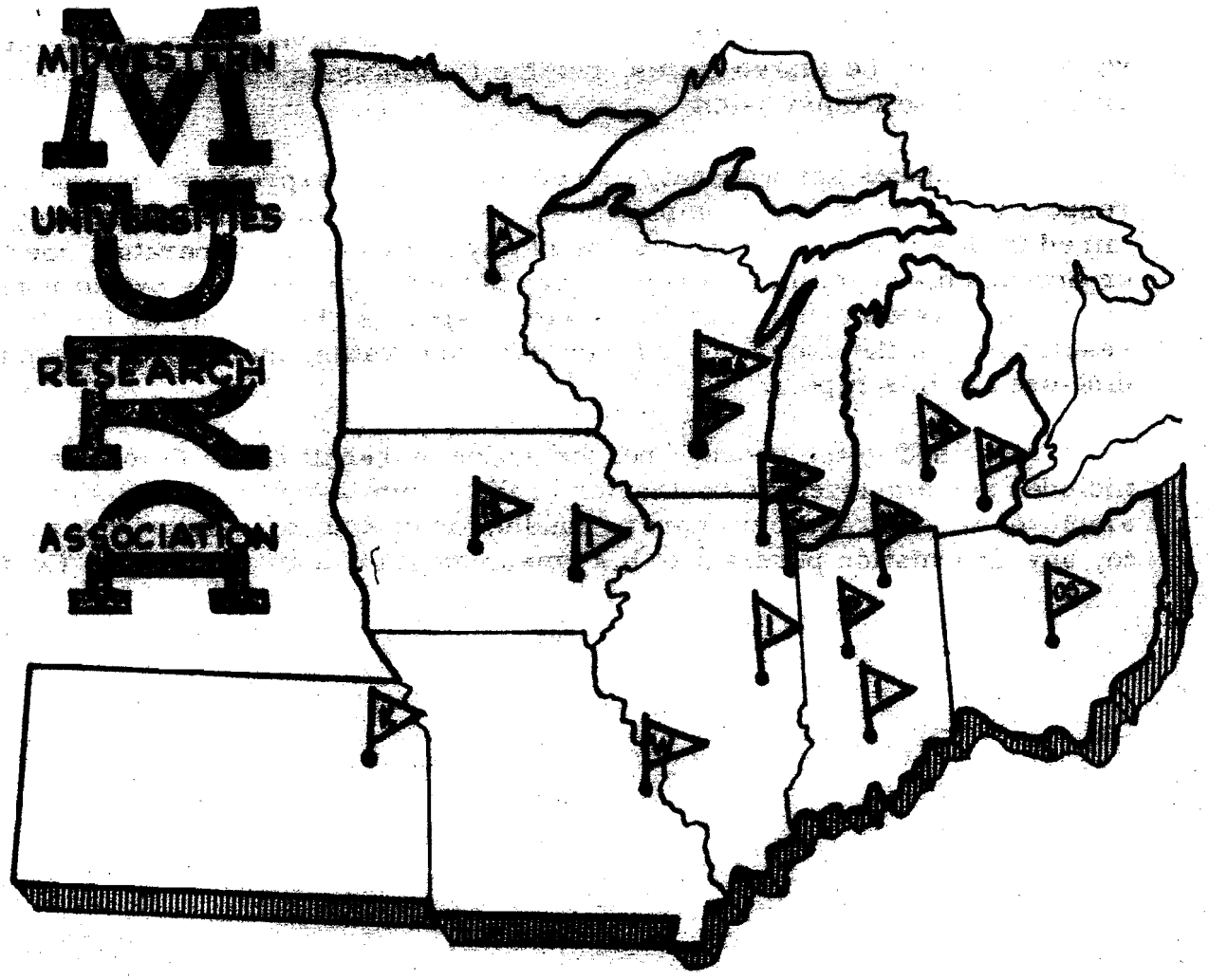

CURRENT SOURCES FOR SCALING MAGNETIC FIELDS (II)

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ABSTRACT

In spiral sector FFAG guide fields it has been customary to locate the potential grading pole face windings in azimuthal slots. It is shown that, for the limit in which the potential grading slots are considered to form a continuum, a scaling magnetic field is obtained if these slots are located along any smooth family of curves crossing between the feeder and return spiral slots. Specifically, the curves are considered to be members of the logarithmic spiral family. For generality, the results are extended to the case in which the pole face windings only partially fill the associated slots.

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I. INTRODUCTION

In a previous paper[1] the problem of finding a magnetic field from prescribed but restricted scaling current distributions has been solved. This problem permits of certain useful generalizations with relatively little additional theoretical consideration. For practical reasons, the potential grading slots in an FFAG guide field are distributed in a discrete manner. This results in a magnetic field that deviates from the scaling condition. In order that this deviation shall be as small as possible, it is expedient to choose a direction for the potential grading slots such that the particle orbits traverse the slots at the largest possible aspect, other factors being equal. To test the effect of the inclination of these slots on the potential distribution in the scaling limit, a continuous medium of fine-grained slots is introduced in which the direction of the current flow is along a family of spirals crossing the main feeder and return current spirals.

In order to have more flexibility in the design of an FFAG magnet, the previous results are also extended to the case in which the distributed currents partially fill the pole face slots.

II. POTENTIAL GRADING SPIRALS

The fine-grained potential grading slots are treated by introducing a permeable anisotropic medium in which the relation between the magnetic field $\vec{H}$ and the magnetic induction $\vec{B}$ is[1]

$$\vec{H} = \vec{\vec{\gamma}} \cdot \vec{B}, \tag{1}$$

where the dyadic

$$\vec{\mathcal{T}} = \frac{1}{\mu} \nabla q \nabla q \tag{2}$$

has the effect of reducing to zero all components of $\vec{H}$ except the one in the direction of ∇q. If one puts

$$q = \frac{1}{w} \ln \frac{r}{r_0} + N\theta, \tag{3}$$

then, q = constant, is the equation of a spiral cylinder and ∇q is a vector normal to this cylinder. Since $\vec{B}$ cannot be found in this medium, the magnetostatic problem reduces to solving

$$\nabla \times \vec{H} = 4\pi \vec{J}, \tag{4}$$

where, if Eqs. (1) and (2) are used

$$\nabla \times \left[H_q \frac{\nabla q}{|\nabla q|} \right] = 4\pi \vec{J}. \tag{5}$$

The magnetic field H_q is the field in the direction of ∇q. Using for the operator $\vec{\mathcal{L}}$ [1]

$$\vec{\mathcal{L}} = \nabla q \times \nabla \tag{6}$$

and decomposing $\vec{J}$ into

$$\vec{J} = \vec{\mathcal{L}} \sigma + \nabla \times \vec{\mathcal{L}} \tau, \tag{7}$$

one finds

$$\mathcal{L}^2 \left[\frac{H_q}{|\nabla q|} + 4\pi\sigma \right] = 0. \tag{8}$$

The solution of Eq. (8) is

$$H_q = \left[-4\pi\sigma + f(q) \right] \cdot |\nabla q|. \tag{9}$$

At the depth $\eta = \eta_1$ of the region traversed by the potential grading currents,

the magnetic field is to vanish. Hence, if f(q) is set equal to zero, and a scaling form for H_q is required

$$\sigma = -\frac{r_0^2}{\frac{1}{w}+\frac{1}{w'}}\,\rho^{K+1}(\eta-\eta_1)\,h(\xi), \tag{10}$$

where several constants have been introduced for convenience in later formulas. This gives for H_q

$$H_q = 4\pi r_0 \frac{\sqrt{\frac{1}{w^2}+N^2}}{\frac{1}{w}+\frac{1}{w'}}\,\rho^{K}(\eta-\eta_1)\,h(\xi). \tag{11}$$

The arbitrariness in the choice of σ and τ is best resolved by setting τ equal to zero. In this case the current density becomes

$$\vec{J} = \vec{\nabla}\sigma = \left(\frac{1}{w r}\vec{i}_r + \frac{N}{r}\vec{i}_\theta\right)\times\left(\vec{i}_r\frac{\partial\sigma}{\partial r} + \vec{i}_\theta\frac{1}{r}\frac{\partial\sigma}{\partial\theta} + \vec{k}\frac{\partial\sigma}{\partial z}\right)$$

or

$$\vec{J} = \frac{N}{r}\left[\vec{i}_r\frac{\partial\sigma}{\partial z} - \frac{1}{wN}\vec{i}_\theta\frac{\partial\sigma}{\partial z} + \vec{k}\left(\frac{1}{w'Nr}\frac{\partial\sigma}{\partial\theta} - \frac{\partial\sigma}{\partial r}\right)\right],$$

which, if the variables ρ, ξ, η [1] are used, becomes

$$\vec{J} = \frac{N}{r_0^2\rho^2}\left\{\vec{i}_r\frac{\partial\sigma}{\partial\eta} - \frac{1}{wN}\vec{i}_\theta\frac{\partial\sigma}{\partial\eta} - \vec{k}\left[\rho\frac{\partial\sigma}{\partial\rho} + \left(\frac{1}{w}+\frac{1}{w'}\right)\frac{\partial\sigma}{\partial\xi} - \eta\frac{\partial\sigma}{\partial\eta}\right]\right\}. \tag{12}$$

In order that the current density $\vec{J}$ be directed along the spiral, q = constant, η = const, it is sufficient to let

$$\rho\frac{\partial\sigma}{\partial\rho} + \left(\frac{1}{w}+\frac{1}{w'}\right)\frac{\partial\sigma}{\partial\xi} - \eta\frac{\partial\sigma}{\partial\eta} = -\eta\frac{\partial\sigma}{\partial\eta}$$

or

$$\rho\frac{\partial\sigma}{\partial\rho} + \left(\frac{1}{w}+\frac{1}{w'}\right)\frac{\partial\sigma}{\partial\xi} = 0. \tag{13}$$

Equation (13) substituted into Eq. (12) gives

$$\vec{J} = \frac{N}{r_0^2 \rho^2}\left(\vec{i}_r - \frac{1}{w'N}\vec{i}_\theta + \eta\vec{k}\right)\frac{\partial\sigma}{\partial\eta}\,, \tag{14}$$

which demonstrates that $\vec{J}$ is directed along

$$\vec{m} = \frac{\vec{i}_r - \frac{1}{w'N}\vec{i}_\theta + \eta\vec{k}}{\sqrt{1+\left(\frac{1}{w'N}\right)^2+\eta^2}}\,, \tag{15}$$

a unit vector along the curve, q = constant, η = constant. Equation (13) together with Eq. (10) determines the function $h(\xi)$:

$$\left(\frac{1}{w}+\frac{1}{w'}\right)h'(\xi) + (k+1)h(\xi) = 0\,. \tag{16}$$

Equation (14), for the current density, becomes

$$\vec{J} = -\left(\vec{i}_r - \frac{1}{w'N}\vec{i}_\theta + \eta\vec{k}\right)\cdot\frac{N}{\frac{1}{w}+\frac{1}{w'}}\cdot\rho^{k-1}h(\xi)\,. \tag{17}$$

III. BOUNDARY CONDITIONS

In a previous paper,[1] it was shown that the relevant components of $\vec{H}$ for the boundaries of interest in a scaling problem are $\vec{i}_\theta\cdot\vec{H}$, $\vec{k}\cdot\vec{H}$, and $\vec{n}\cdot\vec{H}$. For the region treated in Section II, these components are

$$\vec{i}_\theta\cdot\vec{H} = \vec{i}_\theta\cdot\nabla\varphi\frac{H_\varphi}{|\nabla\varphi|} = 4\pi r_0\frac{N}{\frac{1}{w}+\frac{1}{w'}}\rho^k(\eta-\eta_1)h(\xi)\,, \tag{18}$$

$$\vec{k}\cdot\vec{H} = 0\,, \tag{19}$$

$$\vec{n}\cdot\vec{H} = \frac{1}{\sqrt{1+\left(\frac{1}{w'N}\right)^2+\eta^2}}\cdot 4\pi r_0\rho^k(\eta-\eta_1)h(\xi)\,. \tag{20}$$

An examination of the matching conditions in the previous paper[1] reveals that, after replacing Eqs. (53-55) of that paper by Eqs. (18-20) above, the

present boundary conditions are satisfied if, for the $f(\xi)$ and $g(\xi)$ governing the current distribution in the spiral feeder slots, one takes

$$f(\xi) = \begin{cases} \dfrac{h(\xi_0)}{k(\xi_1-\xi_0)}\left[\xi_1 - \xi + \dfrac{1}{k(k+1)w}\right] & \text{for } \xi_0 < \xi < \xi_1 \\[2ex] \dfrac{h(-\xi_0)}{k(\xi_1-\xi_0)}\left[\xi_1 + \xi - \dfrac{1}{k(k+1)w}\right] & \text{for } -\xi_1 < \xi < -\xi_0 \end{cases} \tag{21}$$

and

$$g(\xi) = \begin{cases} -\dfrac{N h(\xi_0)}{k^2(k+1)(\xi_1-\xi_0)} & \text{for } \xi_0 < \xi < \xi_1 \\[2ex] \dfrac{N h(-\xi_0)}{k^2(k+1)(\xi_1-\xi_0)} & \text{for } -\xi_1 < \xi < -\xi_0 \end{cases} \tag{22}$$

In particular, matching the tangential components of $\vec{H}$ in the region $-\xi < \xi < \xi_0$ for $\eta = \eta_0$ gives from Eqs. (18) and (20)

$$\frac{\partial \Omega}{\partial \xi} = -4\pi r_0^2(\eta_1-\eta_0)\cdot\frac{1}{\frac{1}{w}+\frac{1}{w_1}}\cdot h(\xi) \tag{23}$$

and

$$\Omega = 4\pi r_0^2(\eta_1-\eta_0)\frac{1}{k+1}h(\xi)\,. \tag{24}$$

By differentiating Eq. (24) and making use of Eq. (16), Eqs. (23) and (24) are seen to be compatible. Since the solution of Eq. (16) is

$$h(\xi) = h_0\, e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w_1}}\xi}\,, \tag{25}$$

it is convenient to introduce a reference potential defined as

$$\Omega_0 \equiv 4\pi r_0^2(\eta_1-\eta_0)\frac{h_0}{k+1}\,, \tag{26}$$

the significance of which from Eq. (24) is that of a scaled potential at $\xi = 0$ and $\eta = \eta_0$.

The boundary potentials may be summarized as follows: For $\xi_0 < \xi < \xi_1$

$$\Psi(\xi_0 \eta) = -\frac{\Omega_0 e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left\{\eta\left[\xi_1-\xi_0+\frac{1}{kw}\right]-k(\eta_1-\eta)(\xi_1-\xi_0)\right\}, \tag{27}$$

$$\Psi(\xi_1 \eta) = -\frac{\Omega_0 \eta e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k^2 w(\xi_1-\xi_0)(\eta_1-\eta_0)}, \tag{28}$$

$$\Psi(\xi \eta_1) = -\frac{\Omega_0 \eta_1 e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1-\xi+\frac{1}{kw}\right], \tag{29}$$

and

$$\Omega(\xi \eta_0) - \Psi(\xi \eta_0) = \frac{\Omega_0 \eta_0 e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1-\xi+\frac{1}{kw}\right]. \tag{30}$$

For $-\xi_0 < \xi < \xi_0$ the boundary potential is

$$\Omega(\xi \eta_0) = \Omega_0 e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi}. \tag{31}$$

For $-\xi_1 < \xi < -\xi_0$ the boundary potentials are

$$\Psi(-\xi_0 \eta) = -\frac{\Omega_0 e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left\{\eta\left[\xi_1-\xi_0-\frac{1}{kw}\right]-k(\eta_1-\eta)(\xi_1-\xi_0)\right\}, \tag{32}$$

$$\Psi(-\xi_1 \eta) = \frac{\Omega_0 \eta e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k^2 w(\xi_1-\xi_0)(\eta_1-\eta_0)}, \tag{33}$$

$$\Psi(\xi\eta_1) = -\frac{\Omega_0 \eta_1 e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1+\xi-\frac{1}{kw}\right], \tag{34}$$

and

$$\Omega(\xi\eta_0) - \Psi(\xi\eta_0) = \frac{\Omega_0 \eta_0 e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}}{k(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1+\xi-\frac{1}{kw}\right]. \tag{35}$$

In order to determine the distribution of current in the potential grading slots, it is necessary to know the amount of current flowing between two spirals differing by Δq. Thus

$$\Delta I = -\iint \vec{i}_r \cdot \vec{J}\, r\, d\theta\, dz = -\iint \vec{i}_r \cdot \vec{J}\, r\, \frac{\partial(\theta, z)}{\partial(q,\eta)}\, dq\, d\eta\,. \tag{36}$$

Using Eqs. (3) and (14), this becomes

$$\Delta I = -\iint \frac{\partial \sigma}{\partial \eta}\, dq\, d\eta\,, \tag{37}$$

which, from Eqs. (10) and (25), becomes

$$\Delta I = \frac{r_0^2 h_0}{\frac{1}{w}+\frac{1}{w'}} \iint \xi^{k+1} e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi}\, dq\, d\eta$$

or

$$\Delta I = \frac{r_0^2 h_0}{\frac{1}{w}+\frac{1}{w'}} \iint e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}q}\, dq\, d\eta\,. \tag{38}$$

Thus

$$\frac{\partial I}{\partial q} = \frac{\Omega_0}{4\pi}\frac{d}{dq}\left[e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}q}\right], \tag{39}$$

and using for the pole potential

$$V = \rho^{K+1}\Omega_0 e^{-\frac{K+1}{\frac{1}{\nu}+\frac{1}{\nu_1}}\xi} = \Omega_0 e^{\frac{K+1}{\frac{1}{\nu}+\frac{1}{\nu_1}}\varphi}, \tag{40}$$

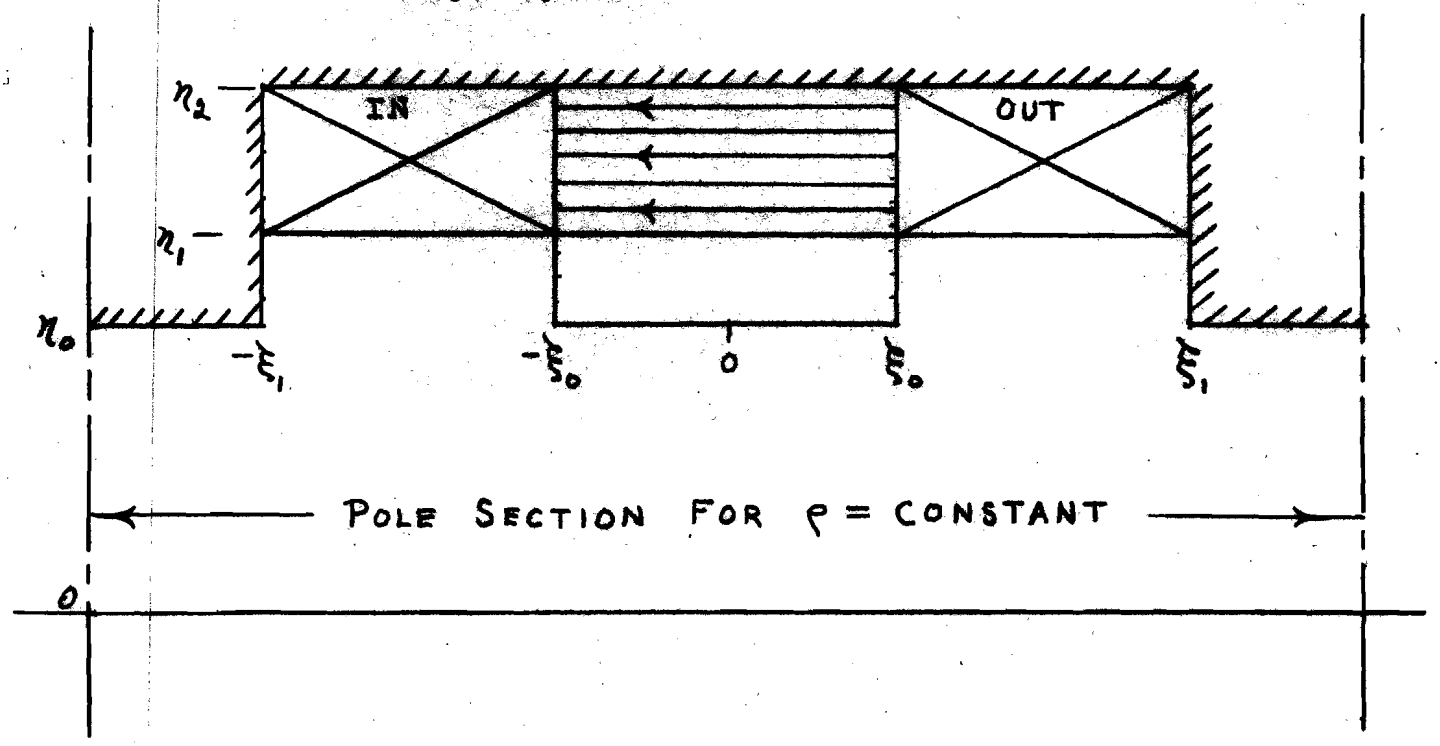
Eq. (39) becomes

$$\frac{\partial I}{\partial \varphi} = \frac{1}{4\pi}\frac{\partial V}{\partial \varphi}, \tag{41}$$

as expected.

IV. PARTIALLY FILLED CURRENT SLOTS

Since it is less likely that imperfections in the pole face windings will produce observable effects if these windings are located deep in the slots, it is desirable to be able to solve the magnetostatic problem in which the currents partially fill the slots. The anisotropic medium that carries the potential grading current will be the same as that discussed in Section II except that it is not completely filled.



The only changes necessary to handle this problem are to replace η_0 and η_1 in Eqs. (27-35) by η_1 and η_2 of the above figure; to extend the zero potential poles down to η_0, and place $\Omega(\pm\xi_0\eta_0)$ of Eq. (31) along the pole extensions at $\xi = \pm\xi_0$. These conditions follow from a consideration of the potential in the unfilled portion of the potential grading slot region. From Eq. (8)

$$\mathcal{L}^2\left[\frac{H_\gamma}{|\nabla\gamma|}\right] = 0 , \tag{42}$$

the solution of which is

$$H_\gamma = f(\gamma)\,|\nabla\gamma| . \tag{43}$$

Equation (43) together with the scaling requirement allows the magnetic field to be derived from the potential

$$V = \Omega_0\, e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\gamma} \tag{44}$$

where the choice of the constant Ω_0 results from matching the tangential components of $\vec{H}$ at $\eta = \eta_0$.

REFERENCES

1. Current Sources for Scaling Magnetic Fields, S. C. Snowdon, MURA-535 (unpublished).