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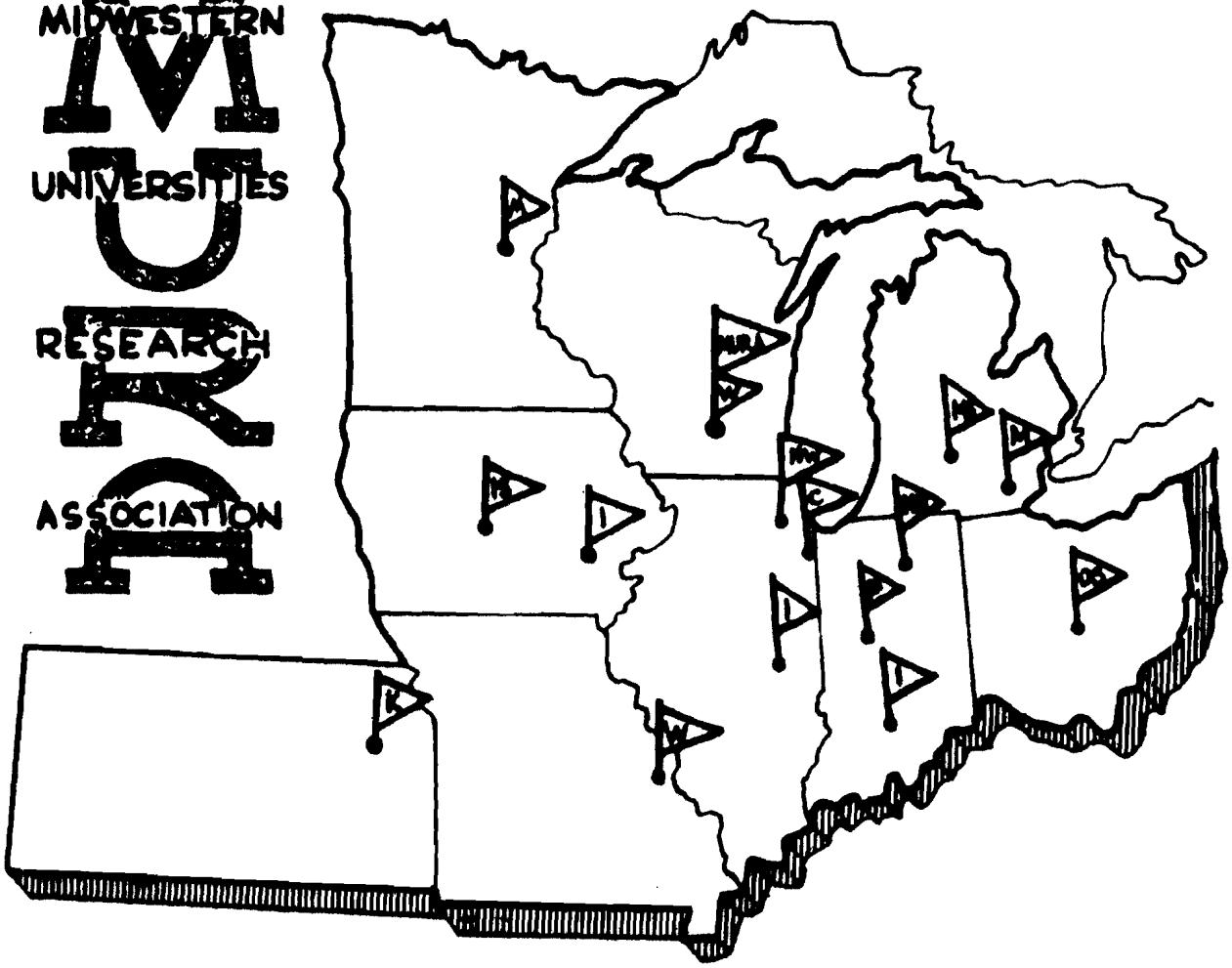

CURRENT SOURCES FOR SCALING MAGNETIC FIELDS

S. C. Snowdon

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2203 University Avenue, Madison, Wisconsin

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S. C. Snowdon**

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ABSTRACT

It is shown that the solution of magnetostatic problems in the presence of distributed current sources may be expressed in terms of a scalar-type potential and scalar-type quantities representing the current density. Application is made to the solution of magnetostatic problems connected with scaling-type magnetic fields employed in FFAG guide fields.

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I. INTRODUCTION

In the design of FFAG magnetic guide fields, it has been customary to employ fields which scale, i.e., fields in which the equilibrium orbits for various average radii appear as magnified and rotated images of one another. A method is introduced which allows the determination of magnetic fields entirely in terms of scalar-type quantities. An idealized configuration of current distributions and infinite permeability iron is treated which gives rise to a scaling magnetic field, following which application is made to a spiral sector FFAG magnet. The general treatment follows from a discussion given for two-dimensional magnetic fields by Laslett.[1] The notion of an operator representing the coordinate scaling transformation is used following which scaling magnetic fields are considered to be eigenvectors of the scaling operator.

II. SCALAR FUNCTIONS IN MAGNETOSTATICS

In order to produce a scaling type of magnetic field with distributed currents in the presence of iron, it is necessary to consider current flowing in essentially two different types of regions. The feeder current lines are considered to be located in slots that are bounded by scaling surfaces. In this region the permeability may be taken as unity. The feeder lines deliver current to and return current from the potential grading slots. Although, in practice, these slots are non-scaling, a limiting scaling situation is envisaged in which the slots are increased in number and reduced in width until a continuous current carrying medium is produced. The perme-

ability of this medium may be represented by a tensor which has the property of allowing a finite magnetic field transverse to the fine-grained slots, reducing to zero all other components.

The problem first considered is the determination of a magnetic field in the scaling slots where $\mu = 1$. This problem is specified by

$$\nabla \times \vec{H} = 4\pi \vec{J} \quad , \quad \nabla \cdot \vec{B} = 0 \quad , \quad \vec{B} = \vec{H} . \tag{1}$$

These equations may be treated without the use of a vector potential if one considers an operator $\vec{\mathcal{L}}$ defined as

$$\vec{\mathcal{L}} = \vec{u} \times \nabla , \tag{2}$$

where $\vec{u}$ in general is taken to be the gradient of some scalar function. For example, for problems in spherical coordinates $\vec{u}$ is taken to be the radius vector $\vec{r}$.[2] For the problems to be considered, it is first convenient to consider $\vec{u} = \vec{k}$, the unit vector along the z-axis of a cylindrical coordinate system. In a later section it proves convenient to let $\vec{u} = \vec{\imath}_r$, the radial unit vector in cylindrical coordinates. In particular the properties of $\vec{\mathcal{L}}$ allow solenoidal vectors to be written as, for example,

$$\vec{J} = \vec{\mathcal{L}} \sigma + \nabla \times \vec{\mathcal{L}} \tau . \tag{3}$$

A similar decomposition of the vector potential which would normally be introduced to account for the solenoidal nature of $\vec{B}$ shows after using Eqs. (1) and (3) that

$$\vec{H} = 4\pi \vec{\mathcal{L}} \tau + \nabla \times \vec{\mathcal{L}} U , \tag{4}$$

where

$$\nabla^2 U = -4\pi\sigma . \tag{5}$$

The operator $\nabla\times\vec{\mathcal{L}}$ may be expanded to give for Eq. (4)

$$\vec{H} = 4\pi\vec{\mathcal{L}}\tau - 4\pi\vec{k}\sigma - \nabla\frac{\partial U}{\partial z} , \tag{6}$$

if, for the vector $\vec{u}$, one chooses $\vec{k}$. Outside of the sources σ and τ the magnetic field is seen to be the negative gradient of a scalar function, V, where

$$V = \frac{\partial U}{\partial z} . \tag{7}$$

III. SCALING TRANSFORMATION

A consideration of the dynamics of a charged particle moving in a magnetic field shows that much simplification of the orbit properties is obtained if the magnetic field is an eigenvector of the scaling operator. This operator may be considered loosely as a combined magnification and rotation operator. Specifically, for cylindrical coordinates, the scaling operation replaces the coordinates (r, θ, z) by $(\alpha r, \theta + \frac{1}{\omega N}\ln\alpha, \alpha z)$. Hence, although the magnification is arbitrary, the rotation is related to the magnification in a definite manner. Powell[3] and Laslett[4] have shown that the eignefunctions of the scaling operator may be put in the form

$$\rho^p F(\xi, \eta) , \tag{8}$$

where

$$\rho = \frac{r}{r_0} \ , \ \xi = \frac{1}{w} \ln \frac{r}{r_0} - N\theta \ , \ \eta = \frac{z}{r} \ , \tag{9}$$

and p is any real constant.

An eigenvector may be defined by a suitable extension of the notion of an eigenfunction of the scaling operator. For example, in the case of the magnetic field, the eigenvector form would be

$$\vec{H} = \rho^k \vec{h}(\xi, \eta) \ , \tag{10}$$

where $\vec{h}(\xi, \eta)$ may be considered as the scaled magnetic field. In this paper the connection between $\vec{H}$ and $\vec{J}$ is considered such that H is an eigenvector of the scaling operator. To this end, a mixed coordinate system is employed. The unit vectors are those of a cylindrical coordinate system, whereas the components, which would normally be expressed in terms of r, θ, z are considered as functions of the scaling coordinates ρ, ξ, η. Thus:

$$r = r_0 \rho \ , \ \theta = \frac{1}{wN} \ln \rho - \frac{1}{N} \xi \ , \ z = r_0 \rho \eta \ , \tag{11}$$

$$\frac{\partial}{\partial r} = \frac{1}{r_0} \left[\frac{\partial}{\partial \rho} + \frac{1}{w\rho} \frac{\partial}{\partial \xi} - \frac{\eta}{\rho} \frac{\partial}{\partial \eta} \right] , \tag{12}$$

$$\frac{\partial}{\partial \theta} = -N \frac{\partial}{\partial \xi} \tag{13}$$

$$\frac{\partial}{\partial z} = \frac{1}{r_0 \rho} \frac{\partial}{\partial \eta} \ . \tag{14}$$

$$\vec{\mathcal{L}} = \vec{k} \times \nabla = \vec{i}_\theta \frac{\partial}{\partial r} - \vec{i}_r \frac{1}{r} \frac{\partial}{\partial \theta} \ ,$$

or

$$\vec{\mathcal{L}} = \frac{1}{r_0}\left[\vec{i}_\theta\left(\frac{\partial}{\partial\rho} + \frac{1}{w\rho}\frac{\partial}{\partial\xi} - \frac{\eta}{\rho}\frac{\partial}{\partial\eta}\right) + \vec{i}_r\frac{N}{\rho}\frac{\partial}{\partial\xi}\right] . \tag{15}$$

$$\mathcal{L}^2 = \nabla^2 - \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

or

$$\mathcal{L}^2 = \frac{1}{r_0^2\rho^2}\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{2\rho}{w}\frac{\partial^2}{\partial\rho\partial\xi} - 2\eta\rho\frac{\partial^2}{\partial\rho\partial\eta}\right.$$

$$\left. + \left(\frac{1}{w^2} + N^2\right)\frac{\partial^2}{\partial\xi^2} - 2\frac{\eta}{w}\frac{\partial^2}{\partial\xi\partial\eta} + \eta\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)\right] . \tag{16}$$

Finally, it is useful to define a unit vector along the spiral curves defined by the intersection of the spiral cylinder, ξ = constant, and the cone, η = constant.[1] This is accomplished by first considering the differential displacement vector $\vec{ds}$.

$$\vec{ds} = \vec{i}_r\, dr + \vec{i}_\theta\, r\, d\theta + \vec{k}\, dz . \tag{17}$$

Equation (11) may be used to express Eq. (17) as

$$\vec{ds} = r_0\left[\left(\vec{i}_r + \frac{1}{wN}\vec{i}_\theta + \eta\vec{k}\right)d\rho - \frac{\rho}{N}\vec{i}_\theta\, d\xi + \rho\vec{k}\, d\eta\right] . \tag{18}$$

For $d\xi = d\eta = 0$, the unit vector under consideration is seen to be

$$\vec{n} = \frac{\vec{i}_r + \frac{1}{wN}\vec{i}_\theta + \eta\vec{k}}{\sqrt{1 + \left(\frac{1}{wN}\right)^2 + \eta^2}} . \tag{19}$$

IV. SCALING MAGNETOSTATIC PROBLEM

As was stated previously, for the problem under consideration, it is necessary to represent all quantities in cylindrical coordinates using the variables ρ, ξ, η. The current density as given in Eq. (3) requires the use of $\vec{\mathcal{L}}$ as given in Eq. (15). The operator $\nabla \times \vec{\mathcal{L}}$ may be obtained from

$$\nabla \times \vec{\mathcal{L}} = \vec{k}\nabla^2 - \nabla\frac{\partial}{\partial z} = \vec{k}\left(\mathcal{L}^2 + \frac{\partial^2}{\partial z^2}\right) - \nabla\frac{\partial}{\partial z}$$

or

$$\nabla \times \vec{\mathcal{L}} = -\vec{i}_r \frac{\partial^2}{\partial r \partial z} - \vec{i}_\theta \frac{1}{r}\frac{\partial^2}{\partial \theta \partial z} + \vec{k}\,\mathcal{L}^2 . \tag{20}$$

Thus:

$$\begin{aligned}\vec{J} = {} & \vec{i}_r\left\{\frac{N}{r_0\rho}\frac{\partial\sigma}{\partial\xi} - \frac{1}{r_0^2\rho^2}\left[\rho\frac{\partial^2\tau}{\partial\rho\partial\eta} + \frac{1}{w}\frac{\partial^2\tau}{\partial\xi\partial\eta} - \frac{\partial}{\partial\eta}\left(\eta\frac{\partial\tau}{\partial\eta}\right)\right]\right\} \\ & + \vec{i}_\theta\left\{\frac{1}{r_0\rho}\left[\rho\frac{\partial\sigma}{\partial\rho} + \frac{1}{w}\frac{\partial\sigma}{\partial\xi} - \eta\frac{\partial\sigma}{\partial\eta}\right] + \frac{N}{r_0^2\rho^2}\frac{\partial^2\tau}{\partial\xi\partial\eta}\right\} \\ & + \vec{k}\left\{\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tau}{\partial\rho}\right) + \frac{2\rho}{w}\frac{\partial^2\tau}{\partial\rho\partial\xi} - 2\rho\eta\frac{\partial^2\tau}{\partial\rho\partial\eta} + \left(\frac{1}{w^2} + N^2\right)\frac{\partial^2\tau}{\partial\xi^2}\right. \\ & \left. - \frac{2\eta}{w}\frac{\partial^2\tau}{\partial\xi\partial\eta} + \eta\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\tau}{\partial\eta}\right)\right\} .\end{aligned} \tag{21}$$

In order to express the current density in a scaling manner, it is sufficient to put σ and τ in the form of Eq. (8). Thus, choose

$$\sigma = r_0\,\rho^k f(\xi,\eta), \tag{22}$$

and

$$\tau = r_0^2\,\rho^{k+1} g(\xi,\eta). \tag{23}$$

The magnetic field described in Eq. (6) may be expressed in a manner similar to that of the current density:

$$\vec{H} = \vec{\imath}_r \left\{ \frac{4\pi N}{r_0 \rho} \frac{\partial \tau}{\partial \xi} - \frac{1}{r_0 \rho} \left[\rho \frac{\partial V}{\partial \rho} + \frac{1}{w} \frac{\partial V}{\partial \xi} - \eta \frac{\partial V}{\partial \eta} \right] \right\}$$

$$+ \vec{\imath}_\theta \left\{ \frac{4\pi}{r_0 \rho} \left[\rho \frac{\partial \tau}{\partial \rho} + \frac{1}{w} \frac{\partial \tau}{\partial \xi} - \eta \frac{\partial \tau}{\partial \eta} \right] + \frac{N}{r_0 \rho} \frac{\partial V}{\partial \xi} \right\}$$

$$- \vec{k} \left\{ 4\pi\sigma + \frac{1}{r_0 \rho} \frac{\partial V}{\partial \eta} \right\} . \tag{24}$$

To put this in a scaling form it is only necessary to use

$$V = \rho^{k+1} \Psi(\xi, \eta) . \tag{25}$$

For reasons of mathematical simplicity, and in order to obtain practical methods of achieving the current distribution, the problem is specialized through the use of three conditions. The first restriction insures that the magnitude of the z-component of current density is just sufficient to allow the current to flow locally in planes defined by the unit vectors $\vec{n}$ and $\vec{\imath}_\theta$. This condition is specified by requiring that

$$(\vec{\imath}_\theta \times \vec{n}) \cdot \vec{J} = 0 . \tag{26}$$

If Eq. (19) is used for $\vec{n}$, Eq. (26) becomes

$$J_z = \eta J_r . \tag{27}$$

The second condition actually is not restrictive and is introduced to permit a unique determination of the magnetostatic potential in view of the fact that

only one scalar function V is needed whereas two source densities, σ and τ, are to be specified. From Eq. (6) it is to be noted that the condition, that $\nabla \cdot \vec{B}$ or $\nabla \cdot \vec{H}$ shall equal zero, can be insured by choosing[5]

$$\nabla^2 V = 0 \ , \tag{28}$$

if the source density σ does not contain z. In terms of the variables (ρ, ξ, η), this condition may be stated by requiring that Eq. (22) be restricted to

$$\sigma = r_0 \, \rho^k f(\xi) \, . \tag{29}$$

If Eq. (29) for σ and Eq. (23) for τ are substituted into Eq. (27) using Eq. (21) to obtain the components of the current density, a relation for $g(\xi, \eta)$ results which is homogeneous in η to the first degree. For simplicity then, a third condition is imposed which is consistent with practical wiring schemes. Let

$$\tau = r_0^2 \, \rho^{k+1} \eta \, g(\xi) \, . \tag{30}$$

Equation (27) then connects the allowed forms for $f(\xi)$ and $g(\xi)$ according to

$$N f'(\xi) = k(k+1) g(\xi) + \frac{2k+1}{w} g'(\xi) + \left(\frac{1}{w^2} + N^2\right) g''(\xi) \, . \tag{31}$$

Under the restrictions imposed, the current density in Eq. (21) becomes

$$\vec{J} = \rho^{K-1} \left\{ \left(\vec{i}_r + \frac{1}{wN}\vec{i}_\theta + \eta\vec{k}\right)\left[N f'(\xi) - k g(\xi) - \frac{1}{w} g'(\xi)\right] \right.$$

$$\left. + \vec{i}_\theta \left[k f(\xi) + \frac{1}{N}\left(\frac{1}{w^2} + N^2\right) g'(\xi) + \frac{K}{wN} g(\xi) \right] \right\} . \tag{32}$$

In a similar manner Eq. (24), for the magnetic field, becomes

$$\vec{H} = \rho^{K} \left\{ \vec{i}_r \left[4\pi r_0 N \eta g'(\xi) - \frac{1}{r_0}\left((K+1)\Psi + \frac{1}{w}\frac{\partial \Psi}{\partial \xi} - \eta \frac{\partial \Psi}{\partial \eta} \right) \right] \right.$$

$$+ \vec{i}_\theta \left[4\pi r_0 \eta \left(k g(\xi) + \frac{1}{w} g'(\xi) \right) + \frac{N}{r_0}\frac{\partial \Psi}{\partial \xi} \right]$$

$$\left. - \vec{k} \left[4\pi r_0 f(\xi) + \frac{1}{r_0}\frac{\partial \Psi}{\partial \eta} \right] \right\} . \tag{33}$$

The total current flowing in the feeder slot is

$$I = \iint \vec{i}_r \cdot \vec{J}\, r\, d\theta\, dz = r_0 \rho \iint \vec{i}_r \cdot \vec{J} \frac{\partial(\theta, z)}{\partial(\xi, \eta)}\, d\xi\, d\eta , \tag{34}$$

or

$$I = \frac{r_0^2}{N} \rho^{K+1} \iint \left[N f'(\xi) - k g(\xi) - \frac{1}{w} g'(\xi) \right] d\xi\, d\eta . \tag{35}$$

If Eq. (31) is used to eliminate $f'(\xi)$, Eq. (35) becomes

$$I = \frac{r_0^2}{N} \rho^{K+1} \iint \left[k^2 g(\xi) + \frac{2k}{w} g'(\xi) + \left(\frac{1}{w^2} + N^2\right) g''(\xi) \right] d\xi\, d\eta , \tag{36}$$

a relation which shows that a degree of arbitrariness still remains in the distribution of current in the feeder slots.

V. POTENTIAL GRADING SLOTS

As mentioned in Section II, the windings, which are used to grade the potential, are to be considered as if they were continuously distributed through a fine-grained slot region in order to use a scaling treatment throughout. In the limit these fine-grained potential grading slots are represented by a permeable anisotropic medium where the relation between $\vec{H}$ and $\vec{B}$ is given by

$$\vec{H} = \vec{\vec{\gamma}} \cdot \vec{B} . \tag{37}$$

For simplicity, the potential grading slots are considered to be in the azimuthal direction. In this case

$$\vec{\vec{\gamma}} = \frac{1}{\mu} \vec{\imath}_r \vec{\imath}_r ,$$

a dyadic which has the effect of reducing to zero all components of $\vec{H}$ except H_r. It is to be noted that, for a given $\vec{B}$, $\vec{H}$ may be determined but that, for a given $\vec{H}$, the magnetic induction is not unique.

The magnetic field, $\vec{H}$, may be found by considering the single relation

$$\nabla \times \vec{H} = 4\pi \vec{J} , \tag{38}$$

where Eq. (37) shows that only H_r is needed. Thus

$$\nabla \times \vec{\imath}_r H_r = 4\pi \vec{J} . \tag{39}$$

In general, as in Eq. (3), $\vec{J} = \vec{\mathcal{L}}\sigma + \nabla \times \vec{\mathcal{L}}\tau$. However, since the construction of the operator $\vec{\mathcal{L}}$ is somewhat flexible, it proves convenient in this instance to choose

$$\vec{\mathcal{L}} = \vec{i}_r \times \nabla . \tag{40}$$

Since $\nabla \times \vec{i}_r = 0$, Eq. (39) may be written as

$$\vec{\mathcal{L}} H_r = -4\pi \vec{J} . \tag{41}$$

Taking the dot product of Eq. (41) with $\vec{\mathcal{L}}$, it is seen that

$$\mathcal{L}^2 \left(H_r + 4\pi\sigma \right) = 0 , \tag{42}$$

and that the arbitrariness in the choice of σ and τ is best resolved by setting $\tau = 0$. Since the operator $\mathcal{L}^2$ operates only on the transverse variables, the solution of Eq. (42) is simply expressed by

$$H_r = -4\pi\sigma + f(r) . \tag{43}$$

The current density $\vec{J}$ is given by

$$\vec{J} = \vec{\mathcal{L}}\sigma = \vec{i}_r \times \nabla\sigma = \vec{k}\,\frac{1}{r}\frac{\partial\sigma}{\partial\theta} - \vec{i}_\theta \frac{\partial\sigma}{\partial z} . \tag{44}$$

For the azimuthal slots contemplated, it is desired to have the current flowing azimuthally; hence, the form for σ must be independent of θ and linear in z to match onto the form in the feeder slot region. If $\eta = \eta_1$ is the surface bounding the depth of the azimuthal slot region, a choice of $f(r) = -\frac{1}{r_0}(k+1)e^{k}\Phi_0$ and

$$\sigma = -r_0 e^{k}(\eta - \beta) h_0 , \tag{45}$$

gives a scaling current density. If, furthermore, the magnetic field, H_r, is set equal to zero for $\eta = \eta_1$, the depth of the azimuthal slots,

$$\vec{J} = \vec{\tau}_\theta \rho^{k-1} h_0 \, , \tag{46}$$

and

$$H_r = 4\pi r_0 h_0 \rho^k (\eta - \eta_1) \, . \tag{47}$$

Equations (46) and (47) are both independent of Φ_0 . If $\vec{H}$ is expressed in the general form of Eq. (6), it is seen that the magnetostatic potential in this region may be taken as zero, the field being given entirely by the supplemental terms in σ and τ .

The current flowing per unit increment in ρ through the azimuthal slots is given by considering

$$\Delta I = \iint \vec{\tau}_\theta \cdot \vec{J} \, dr \, dz = r_0^2 \iint \rho \, \vec{\tau}_\theta \cdot \vec{J} \, d\eta \, d\rho \, , \tag{48}$$

or

$$\frac{\partial I}{\partial \rho} = r_0^2 h_0 \rho^k (\eta_1 - \eta_0) \, , \tag{49}$$

where the surface $\eta = \eta_0$ is considered to be the beginning of the azimuthal slot region.

VI. BOUNDARY CONDITIONS

The remaining problem is to connect the magnetic fields determined in the previous two regions with one another and with the field in the free-space region between the pole faces. In the scaling problem the boundaries of interest are: (1) those for which ρ and ξ are constant; and (2) those for which ρ and η are constant. For case (1) the tangential component of

$\vec{H}$ surrounding the feeder slot region is from Eq. (33)

$$\vec{i}_\theta \cdot \vec{H} = 4\pi r_0 \rho^k \eta \left[k g(\xi) + \frac{1}{\nu} g'(\xi) \right] + \frac{N}{r_0} \rho^k \frac{\partial \Psi}{\partial \xi} \quad . \qquad [\eta = \text{const.}] \tag{50}$$

The tangential component of $\vec{H}$ for case (2) is from Eq. (33)

$$\vec{k} \cdot \vec{H} = -4\pi r_0 \rho^k f(\xi) - \frac{1}{r_0} \rho^k \frac{\partial \Psi}{\partial \eta} \quad . \qquad [\xi = \text{const.}] \tag{51}$$

For the ξ = constant boundary, $\vec{n} \cdot \vec{H}$ is also a tangential component, which, from Eqs. (19) and (33), is

$$\vec{n} \cdot \vec{H} = \frac{\rho^k}{\sqrt{1 + \left(\frac{1}{\nu N}\right)^2 + \eta^2}} \left\{ -\frac{k+1}{r_0} \Psi - 4\pi r_0 \eta f(\xi) + \frac{4\pi r_0 \eta}{N} \left[\frac{k}{\nu} g(\xi) + \left(\frac{1}{\nu^2} + N^2 \right) g'(\xi) \right] \right\} \tag{52}$$

$$[\xi = \text{const.}]$$

In the azimuthal slot region, the relevant components of $\vec{H}$ for matching purposes are

$$\vec{i}_\theta \cdot \vec{H} = 0 \tag{53}$$

$$\vec{k} \cdot \vec{H} = 0 \tag{54}$$

$$\vec{n} \cdot \vec{H} = \frac{H_r}{\sqrt{1 + \left(\frac{1}{\nu N}\right)^2 + \eta^2}} \quad , \tag{55}$$

where H_r is given by Eq. (47).

Finally, in the free-space region, the magnetic field is obtained as the negative gradient of the magnetostatic potential. In this region let the magnetostatic potential V be represented by

$$V = \rho^{k+1} \Omega(\xi, \eta) \, , \tag{56}$$

in terms of which the relevant field components are

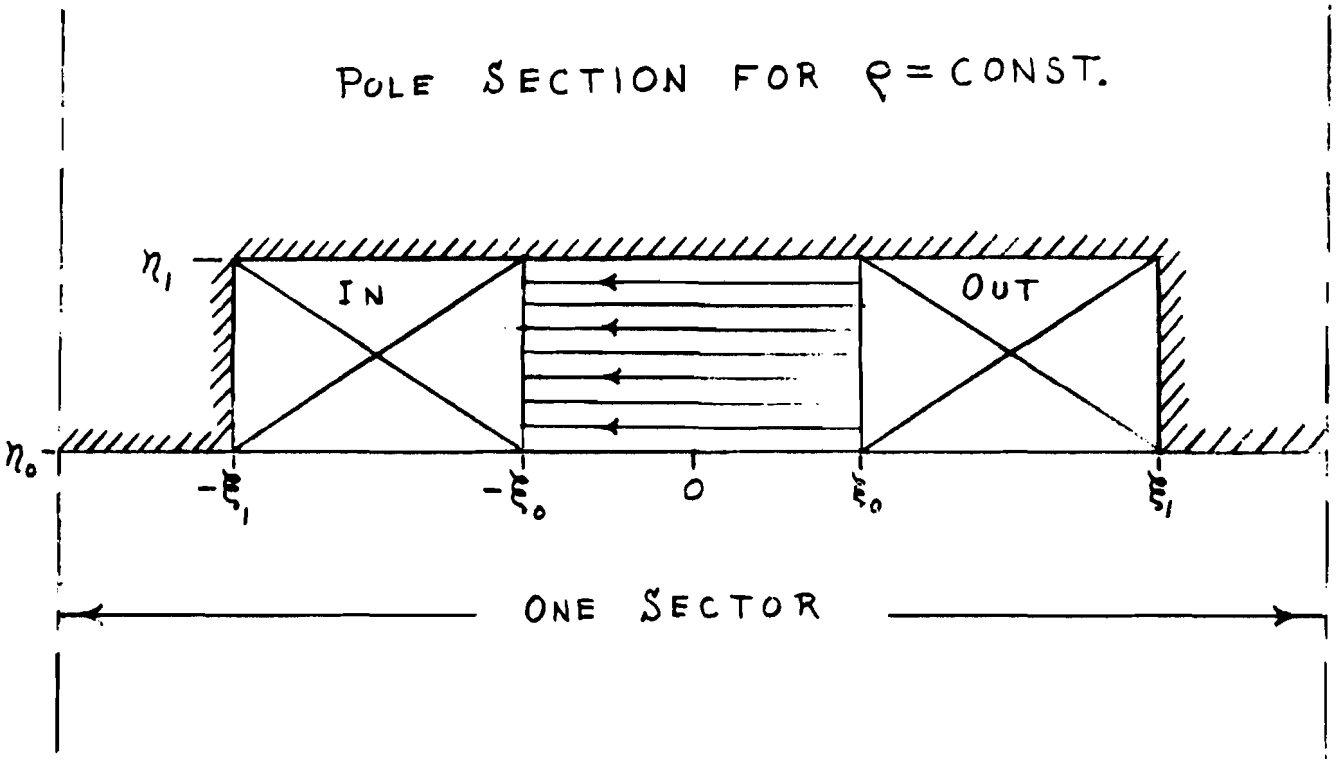
$$\vec{\lambda}_\theta \cdot \vec{H} = \frac{N}{r_0} \rho^\kappa \frac{\partial \Omega}{\partial \xi} , \tag{57}$$

$$\vec{K} \cdot \vec{H} = - \frac{\rho^\kappa}{r_0} \frac{\partial \Omega}{\partial \eta} , \tag{58}$$

and

$$\vec{n} \cdot \vec{H} = - \frac{\rho^\kappa}{\sqrt{1 + \left(\frac{1}{\omega N}\right)^2 + \eta^2}} \cdot \frac{\kappa + 1}{r_0} \cdot \Omega . \tag{59}$$

The condition that the tangential components of the magnetic field be continuous across boundaries gives the following results.



At the boundary, $\xi = \xi_1$, since $\vec{K} \cdot \vec{H} = 0$, Eq (51) gives

$$\frac{\partial \Psi}{\partial \eta} + 4\pi r_0^2 f(\xi_1) = 0 . \tag{60}$$

From Eq. (52), since $\vec{n} \cdot \vec{H} = 0$,

$$\Psi + \frac{4\pi r_0^2 \eta}{k+1} f(\xi_1) - \frac{4\pi r_0^2 \eta}{N(k+1)}\left[\frac{k}{w} g(\xi_1) + \left(\frac{1}{w^2} + N^2\right) g'(\xi_1)\right] = 0 . \tag{61}$$

Differentiate Eq. (61) and compare with Eq (60) to get

$$kN f(\xi_1) + \frac{k}{w} g(\xi_1) + \left(\frac{1}{w^2} + N^2\right) g'(\xi_1) = 0 . \tag{62}$$

Substituting this into Eq. (61) gives for the boundary Ψ

$$\Psi(\xi_1, \eta) + 4\pi r_0^2 \eta f(\xi_1) = 0 . \tag{63}$$

At the boundary, $\xi = \xi_c$, since $\bar{k} \cdot \bar{H} = 0$, Eqs. (51) and (54) give

$$\frac{\partial \Psi}{\partial \eta} + 4\pi r_0^2 f(\xi_0) = 0 . \tag{64}$$

From Eq. (55), (52), and (47)

$$\Psi + \frac{4\pi r_0^2 \eta}{k+1} f(\xi_0) - \frac{4\pi r_0^2 \eta}{N(k+1)}\left[\frac{k}{w} g(\xi_0) + \left(\frac{1}{w^2} + N^2\right) g'(\xi_0)\right] = -\frac{4\pi r_0^2 h_0}{k+1}(\eta - \eta_1) . \tag{65}$$

Differentiation of Eq. (65) and comparison with Eq (64) gives

$$kN f(\xi_0) + \frac{k}{w} g(\xi_0) + \left(\frac{1}{w^2} + N^2\right) g'(\xi_0) = N h_0 \tag{66}$$

Substituting this into Eq. (65) gives for the boundary Ψ

$$\Psi(\xi_0, \eta) + 4\pi r_0^2\left[\eta f(\xi_0) - \frac{\eta_1 h_0}{k+1}\right] = 0 . \tag{67}$$

At the boundary, $\eta = \eta_0$, $\xi_0 < \xi < \xi_1$, the component $\vec{i}_\theta \cdot \vec{H}$ must be continuous. Equations (50) and (57) give

$$\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Psi}{\partial \xi} + \frac{4\pi r_0^2 \eta_0}{N}\left[k g(\xi) + \frac{1}{w} g'(\xi)\right] . \tag{68}$$

At the boundary, $\eta = \eta_1$, $\xi_0 < \xi < \xi_1$, the potential in the iron is constant. Hence, the boundary condition is

$$\frac{\partial \Psi}{\partial \xi} + \frac{4\pi r_0^2 \eta_1}{N}\left[k\, g(\xi) + \frac{1}{w}\, g'(\xi)\right] = 0 \ . \tag{69}$$

At the boundary, $\eta = \eta_0 \quad -\xi_0 < \xi < \xi_0$ the component $\vec{l}_\theta \cdot \vec{H}$ must be continuous. Equations (53) and (57) give

$$\frac{\partial \Omega}{\partial \xi} = 0 \ . \tag{70}$$

In addition, $\bar{n} \cdot \bar{H}$ must be continuous; hence, Eqs. (55) and (59) give

$$4\pi r_0 h_0 (\eta_0 - \eta_1) \frac{1}{\sqrt{1 + \left(\frac{1}{wN}\right)^2 + \eta^2}} = - \frac{1}{\sqrt{1 + \left(\frac{1}{wN}\right)^2 + \eta^2}} \cdot \frac{K+1}{r_0}\, \Omega \ , \tag{71}$$

or

$$\Omega(\xi\, \eta_0) = \frac{4\pi r_0^2}{K+1} (\eta_1 - \eta_0)\, h_0 \equiv \Omega_0 \ . \tag{72}$$

Equations (72) and (70) are seen to be compatible.

The boundary conditions for the remaining feeder current slot for which $\xi < 0$ may be investigated in a manner similar to that for the, $\xi > 0$, slot. It is sufficient, however, to note that the magnetic fields reflect the solenoidal flow of current down one spiral slot, across the azimuthal slots, and up the other spiral feeder slot. Although, within the scaling requirement, it is possible to distribute the return current in a manner different from that of the feed current, the most natural choice for the return current is to reverse the sign of the factor multiplying the $\bar{n}$ vector and keep the same sign for the excess θ component when ξ is replaced by $-\xi$. From Eq. (32) this condition requires that

$$N f'(\xi) - k\, g(\xi) - \frac{1}{w}\, g'(\xi) \tag{73}$$

be an odd function of ξ and that

$$kNf(\xi) + \frac{k}{w} g(\xi) + \left(\frac{1}{w^2} + N^2\right) g'(\xi) \tag{74}$$

be an even function of ξ .

In order to remove the remaining arbitrariness in the specification of the source functions $f(\xi)$ and $g(\xi)$, note that, for the $\xi > c$ feeder slot, the boundary condition in Eq. (62) demands that the excess θ component of the current density vanish at $\xi = \xi_1$. In a similar manner Eq. (66) requires that the excess θ component match onto the current flowing in the azimuthal slots. Apart from these two requirements, this excess current density may vary in an arbitrary manner with ξ . The simplest variation is linear; hence, from Eqs. (32) and (46) for $\xi > 0$, if this condition is chosen,

$$kNf(\xi) + \frac{k}{w} g(\xi) + \left(\frac{1}{w^2} + N^2\right) g'(\xi) = N h_0 \frac{\xi_1 - \xi}{\xi_1 - \xi_0} . \tag{75}$$

The simultaneous solution of Eqs. (31) and (75) subject to the symmetry conditions of Eqs. (73) and (74) serves to give unique source functions $f(\xi)$ and $g(\xi)$

$$f(\xi) = \frac{h_0}{k(\xi_1 - \xi_0)} \begin{cases} \xi_1 - \xi + \dfrac{1}{k(k+1)w} & \text{for } \xi_0 < \xi < \xi_1 \\ \xi_1 + \xi - \dfrac{1}{k(k+1)w} & \text{for } -\xi_1 < \xi < -\xi_0 \end{cases} \tag{76}$$

and

$$g(\xi) = \frac{N h_0}{k^2(k+1)(\xi_1 - \xi_0)} \begin{cases} -1 & \text{for } \xi_0 < \xi < \xi_1 \\ 1 & \text{for } -\xi_1 < \xi < -\xi_0 \end{cases} . \tag{77}$$

Equation (69) may be integrated to give $\Psi(\xi\,\eta_1)$. It is to be noted that the adjustment of a single integration constant is sufficient to permit the resultant function to reduce to Eqs (67) and (63) at the respective limits $\xi = \xi_0$ and $\xi = \xi_1$. Thus for $\xi_0 < \xi < \xi_1$

$$\Psi(\xi\,\eta_1) = -\frac{4\pi r_0^2 h_0 \eta_1}{\kappa(\kappa+1)(\xi_1-\xi_0)}\left[\xi_1 - \xi + \frac{1}{\kappa w}\right]. \tag{78}$$

Equation (68) may be integrated in a similar manner. Again the adjustment of the single integration constant allows the boundary conditions expressed in Eqs. (63), (67), and (72) at $\xi = \xi_0$ and $\xi = \xi_1$ to be fulfilled. Thus for $\xi_0 < \xi < \xi_1$

$$\Omega(\xi\,\eta_0) = \Psi(\xi\,\eta_0) + \frac{4\pi r_0^2 h_0 \eta_0}{\kappa(\kappa+1)(\xi_1-\xi_0)}\left[\xi_1 - \xi + \frac{1}{\kappa w}\right]. \tag{79}$$

In summary the boundary potentials for $\xi_0 < \xi < \xi_1$ after using Eq. (72) to eliminate the constant h_0 are

$$\Psi(\xi_0\,\eta) = -\frac{\Omega_0}{\kappa(\xi_1-\xi_0)(\eta_1-\eta_0)}\left\{\eta\left[\xi_1-\xi_0+\frac{1}{\kappa w}\right] - \kappa(\eta_1-\eta)(\xi_1-\xi_0)\right\}, \tag{80}$$

$$\Psi(\xi_1\,\eta) = -\frac{\Omega_0\,\eta}{\kappa^2 w(\xi_1-\xi_0)(\eta_1-\eta_0)}, \tag{81}$$

$$\Psi(\xi\,\eta_1) = -\frac{\Omega_0\,\eta_1}{\kappa(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1 - \xi + \frac{1}{\kappa w}\right], \tag{82}$$

and

$$\Omega(\xi\,\eta_0) - \Psi(\xi\,\eta_0) = \frac{\Omega_0\,\eta_0}{\kappa(\xi_1-\xi_0)(\eta_1-\eta_0)}\left[\xi_1 - \xi + \frac{1}{\kappa w}\right]. \tag{83}$$

For $-\xi_1 < \xi < -\xi_0$ the boundary potentials are

$$\Psi(-\xi_0\,\eta) = -\frac{\Omega_0}{\kappa(\xi_1-\xi_0)(\eta_1-\eta_0)}\left\{\eta\left[\xi_1-\xi_0-\frac{1}{\kappa w}\right] - \kappa(\eta_1-\eta)(\xi_1-\xi_0)\right\}, \tag{84}$$

$$\Psi(-\xi_1, \eta) = \frac{\Omega_0 \eta}{\kappa^2 w(\xi_1 - \xi_0)(\eta_1 - \eta_0)} , \tag{85}$$

$$\Psi(\xi \eta_1) = -\frac{\Omega_0 \eta_1}{\kappa(\xi_1 - \xi_0)(\eta_1 - \eta_0)} \left[\xi_1 + \xi - \frac{1}{\kappa w} \right] , \tag{86}$$

and

$$\Omega(\xi \eta_0) - \Psi(\xi \eta_0) = \frac{\Omega_0 \eta_0}{\kappa(\xi_1 - \xi_0)(\eta_1 - \eta_0)} \left[\xi_1 + \xi - \frac{1}{\kappa w} \right] . \tag{87}$$

This completes the specification of the potential problem except for observing that the potentials of interest in FFAG guide fields are odd functions of η and periodic functions of ξ with period 2π.

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