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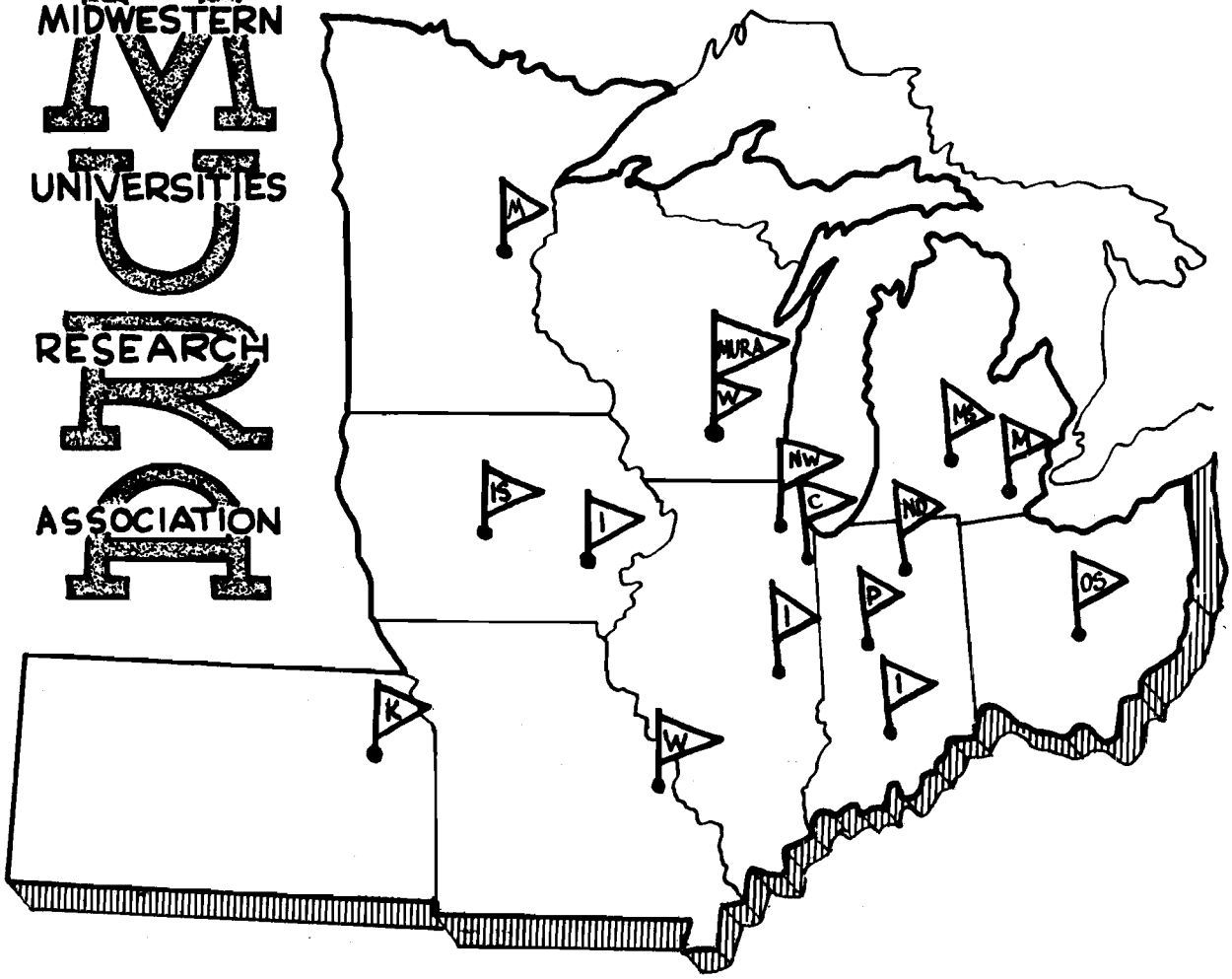

BEAT FACTORS FOR AN ACCELERATOR WITH A GENERAL MAGNETIC FIELD

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ABSTRACT

Expressions are obtained for the beat factors of an accelerator with a general magnetic field. Results are also given for the particular case of the scaling FFAG accelerator.

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In this report, results are obtained for the beat factors of an accelerator having a general magnetic field. In an accelerator, where the median plane field has the period $2\pi/N$ in θ, the betatron oscillation has the form

$$u = A(\theta) \cos[\nu\theta - \varphi(\theta)] . \tag{1.1}$$

$A(\theta)$ and $\varphi(\theta)$ have the period $2\pi/N$. For some values of θ, $A(\theta)$ will have its maximum and minimum values, A_{max} and A_{min}. The beat factor, B, is defined by

$$B = \frac{A_{max}}{A_{min}} . \tag{1.2}$$

We will assume that the magnetic field in the median plane is given by

$$H_z = H_0(r) + 2 H_1(r) \cos[N\theta - \beta_1(r)] . \tag{1.3}$$

We will neglect the effect of the higher harmonics, and we will make the same assumptions about the magnetic field as was assumed in MURA-397, 451.

1. Summary of Results

The radial beat factor B_r is given by

$$B_r = 1 + 4 \frac{|g_1|}{N^2 - 4\nu_r^2} , \tag{1.4}$$

$$|g_1| = \frac{eR}{pc} \left\{ \left(R H_1' + \frac{3}{2} H_1\right)^2 + R^2 \beta_1'^2 H_1^2 \right\}^{\frac{1}{2}} \tag{1.5}$$

where H_1 and β_1 are evaluated at $r = R$, the average radius of equilibrium orbit and p the momentum is related to R by

$$p = \frac{eR}{2c} \left\{ H_0 + \left\{ H_0^2 + \frac{8}{N^2} \left(R H_1' H_1 + \frac{3}{2} H_1^2\right) \right\}^{1/2} \right\} \tag{1.6}$$

The vertical beat factor B_z is given by

$$B_z = 1 + 4 \frac{|f_1|}{N^2 - 4\nu_z^2}, \tag{1.7}$$

$$|f_1| = \frac{eR}{pc} \left\{ \left(R H_1' + \frac{1}{2} H_1\right)^2 + R^2 \beta_1'^2 H_1^2 \right\}^{1/2} \tag{1.8}$$

For a scaling FFAG accelerator, whose medium plane field is given by

$$H_z = -B_0 \left(\frac{r}{r_1}\right)^K \left\{ h_0 + 2h_1 \cos\left[N\theta - \frac{1}{w} \ln \frac{r}{r_1} - \alpha_1\right] + \ldots\ldots \right\}, \tag{1.9}$$

the beat factors are given by

$$B_r = 1 + \frac{8h_1}{h_0 + \left[h_0^2 + 8h_1^2 (K + \frac{3}{2})/N^2\right]^{1/2}} \frac{\left[(K + 3/2)^2 + (1/w)^2\right]^{1/2}}{N^2 - 4\nu_r^2}, \tag{1.10}$$

$$B_z = 1 + \frac{8h_1}{h_0 + \left[h_0^2 + 8h_1^2 (K + \frac{3}{2})/N^2\right]^{1/2}} \frac{\left[(K + \frac{1}{2})^2 + 1/w^2\right]^{1/2}}{N^2 - 4\nu_z^2} \tag{1.11}$$

The two-way machine is included in the last results by simply putting $h_0 = 0$.

2. Derivation of Beat Factors

The derivation will use the results found in MURA-397, 451 and which are summarized in MURA-454.

The Floquet solution of the linear radial equation is given by (see MURA-454)

$$u = e^{i\nu_r \theta} h(\theta), \tag{2.1a}$$

$$h(\theta) = 1 - \frac{g_1}{(N + \nu_r)^2 - E_0} e^{iN\theta} - \frac{g_{-1}}{(N - \nu_r)^2 - E_0} e^{-iN\theta} \tag{2.1b}$$

where g_1 and E_0 are given in MURA-454. We have dropped the second harmonic term $\bar{g}_2$ given there although this may be appreciable for a small two-way FFAG accelerator.

The linear radial motion is thus given by

$$u = a e^{i \nu_r \theta} h(\theta) + c.c. \tag{2.2}$$

which can be written as

$$u = A(\theta) \cos(\nu_r \theta - \varphi(\theta)), \tag{2.3}$$

where

$$A(\theta) = 2|a|\,|h(\theta)| \tag{2.4}$$

The beat factor is then given by

$$B_r = \frac{|h_{max}|}{|h_{min}|} \tag{2.5}$$

The maximum values of $h(\theta)$ will occur when all the terms are positive and the minimum value when they are all negative. Thus

$$B_r = \frac{1 + \dfrac{|g_1|}{(N+\nu_r)^2 - E_0} + \dfrac{|g_{-1}|}{(N-\nu_r)^2 - E_0}}{1 - \dfrac{|g_1|}{(N+\nu_r)^2 - E_0} - \dfrac{|g_{-1}|}{(N-\nu_r)^2 - E_0}} \tag{2.6}$$

This result for B_r can be simplified in some cases. In particular if we assume that ν_r is not near $N/2$, then we can put $(N \pm \nu_r)^2 - E_0 = N(N \pm \nu_r)$. If we further assume that

$$\frac{|g_1|}{N(N-2\nu_r} << 1 \tag{2.7}$$

then we can write

$$B_r = 1 + \frac{4\,|g_1|}{N^2 - 4\nu_r^2}\,, \tag{2.8a}$$

$$|g_1| = \frac{eR}{pc}\left\{\left(RH_1' + \frac{3}{2}H_1\right)^2 + R^2\beta_1'^2 H_1^2\right\}^{1/2}. \tag{2.8b}$$

Eq. (2.8) is perhaps the most useful result, being simple.

The beat factor for the vertical motion, B_z, is also given by Eq. (2.6) or Eq. (2.8) if one replaces g_1, E_0, ν_r by f_1, E_0', ν_z respectively. The result corresponding to Eq. (2.8) becomes

$$B_z = 1 + 4\,\frac{|f_1|}{N^2 - 4\nu_z^2}\,, \tag{2.9a}$$

$$|f_1| = \frac{eR}{pc}\left\{\left(RH_1' + \frac{1}{2}H_1\right)^2 + R^2 H_1^2\beta_1'^2\right\}^{1/2}. \tag{2.9b}$$