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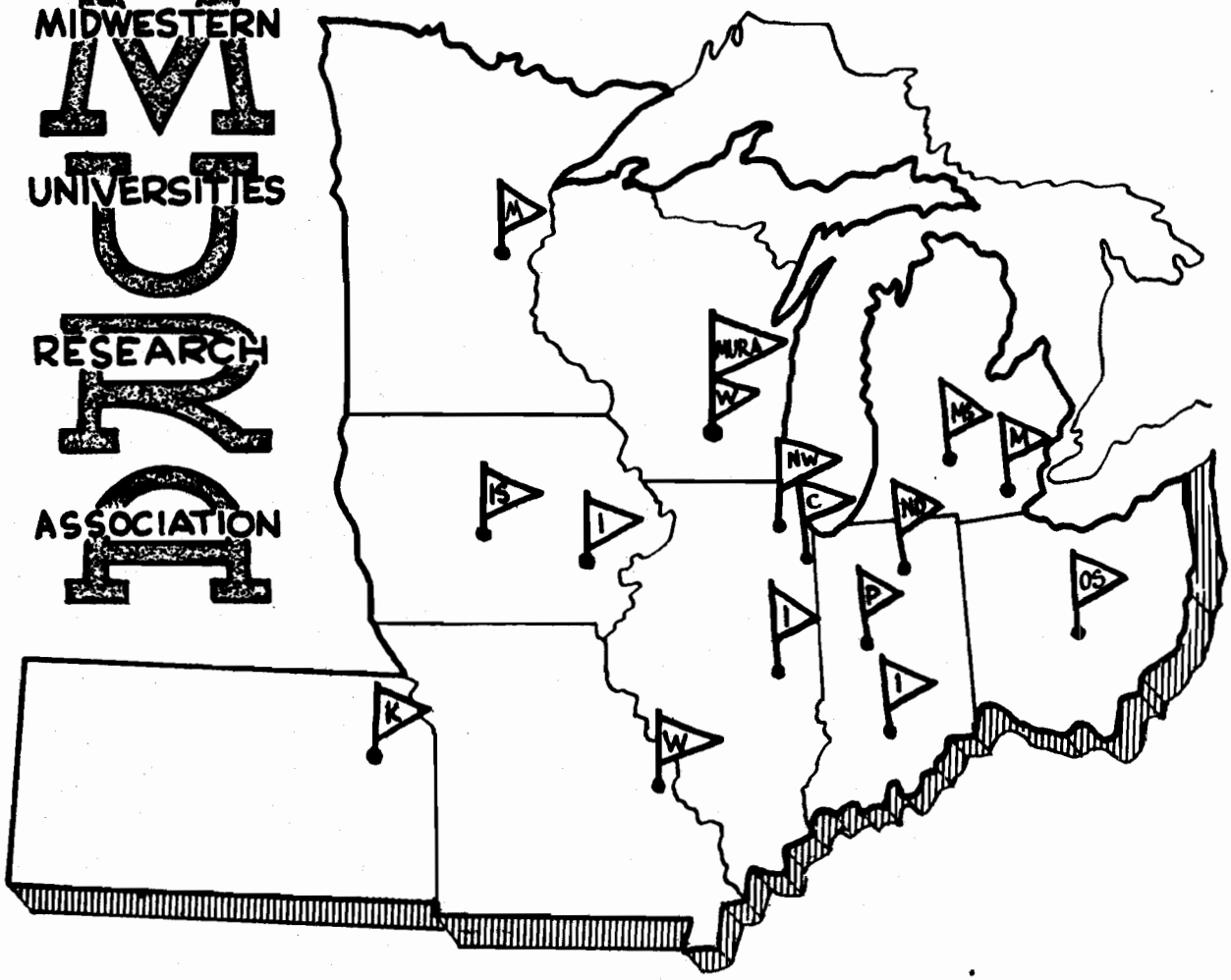

CONCERNING THE $\nu/N \rightarrow 1/3$ RESONANCE, V

ANALYSIS OF THE EQUATION

$$\frac{d^2v}{ds^2} + \left(\frac{2\nu}{N}\right)^2 v - \frac{b}{2}(\cos 2s)\, v^2 - \lambda\left(\cos\frac{2s}{3}\right) = 0$$

L. Jackson Laslett and Seymour J. Wolfson

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August 17, 1959

ABSTRACT

An analytic and computational study has been made of the equation given in the title, specifically for the fixed points in the case $\nu/N = 0.3$, $b = 1.15$, and λ usually equal to 0.006. The equilibrium orbits and the fixed points are found to be obtainable quite accurately by a variational method or by use of harmonic balance if a numerical solution of the simultaneous algebraic equations for the coefficients of the trial function is performed. A straightforward application of the Moser procedure is seen to involve as a first step the elimination of the stable forced equilibrium motion--as is given by the appropriate trial-function solution--and the new differential equation is then found to involve an s-dependent (A-G) coefficient for the linear term. The solution is carried through, by continuation of the Moser method to the same order as in previous reports of this series, aided where appropriate by numerical work for the particular example considered. An alternative, and considerably simpler, analytic method similar to the Moser procedure is also examined and is found to lead to results of reasonable accuracy without requiring extensive numerical work. This last method also permits one to estimate without great effort the critical value of λ at which the stable fixed point and one of the unstable fixed points become coincident.

*Research supported by the Atomic EnergyCommission, Contract No. AT(11-1)-384.
**Department of Physics and Institute for Atomic Research, Iowa State University.
***Summer participant from Wayne State University, Detroit, Michigan.

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A. MOTIVATION

Computer studies, to be reported in detail elsewhere, have been in progress to examine the regions of phase space from which injected particles may be captured into a stable region when a secularly-changing perturbation (decreasing field bump) is applied to an FFAG structure characterized, under certain simplifying assumptions, by a simple non-linear differential equation whose stability limits are determined by the $\nu/N \longrightarrow 1/3$ resonance. In parallel with the computer studies an analytic investigation has been made of unperturbed differential equations, similar to that employed in the computer work, and the results summarized in a series of MURA reports.[1, 2, 3*] It is the purpose of the present report to investigate in a somewhat similar way the character of solutions--particularly of the limiting-amplitude solutions--to an equation of this same form but containing a static perturbation (field bump free of secular change).

B. PROCEDURE

The differential equation which which we shall be concerned in the present report will be taken to be[4]

$$\frac{d^2v}{ds^2} + \left(\frac{2\nu}{N}\right)^2 v - \frac{b}{2}(\cos 2s)\, v^2 - \lambda \cos\frac{2s}{3} = 0. \qquad (1)$$

If one visualizes the application of the Moser procedure[5] to Eq. (1), in the spirit of previous reports in this series,[1, 2, 3] one realizes that the first step which it

*References are given in Section I at the end of this report.

would be natural to undertake would be the removal of the forcing term $-\lambda \cos \frac{2s}{3}$, from Eq. (1).* This step, which may be regarded as making a transformation of the dependent variable so as to measure displacements from the stable (forced) equilibrium orbit, appears to require, then, determination of this periodic solution (period 3π) by harmonic balance or some similar method. It may be remarked that the very steps which are then employed to determine this stable equilibrium orbit are substantially those which also can serve to give unstable equilibrium orbits and hence, to a degree, may provide the solution to the questions of major interest with respect to Eq. (1).

The elimination of the forcing term from Eq. (1) results, by this procedure, in the new differential equation containing a s-dependent (A-G) coefficient for the linear term, thus removing any simplification which it might have been supposed would result from selection of the simple non-AG coefficient for v in Eq. (1). A continuation of the analysis would then require removal of this A-G feature from the linear term, by a transformation of the dependent and independent variables through use of the function β (s), in a manner paralleling that illustrated in a previous report.[3] Following completion of such preliminary steps it should then be possible to proceed with the Moser method, as it was applied in reference 2, to obtain results which may be interpreted in terms of the original variables after application of the appropriate reverse transformations.

It can be remarked, if one may anticipate, that the preliminary steps mentioned above can typically be performed with acceptable accuracy more

---

*The writer is indebted to Dr. F. T. Cole for discussions concerning the straightforward method of applying the Moser procedure to equations of the form of Eq. (1).

satisfactorily by numerical solution of the algebraic equations, which serve to specify the coefficients of the various functions which are required, than is possible conveniently by purely algebraic means. In view of this situation it is understandably difficult to expect that one can obtain satisfactory final results in a simple closed algebraic form.

In what follows we undertake to carry through the analytical procedure outlined above for a specific example, using numerical solutions of algebraic equations where desirable but attempting also to note approximate handy formulas which may serve to indicate roughly the magnitude of the quantities with which we are concerned. As a second undertaking, we also attempt to follow, in Section H, a somewhat less logical procedure which, it is hoped, may have some merit in circumventing the inconveniences mentioned above.

C. THE FORCED MOTION
(Stable Equilibrium Orbit)

The solution of equation (1) which describes the forced motion, or stable equilibrium orbit, may be sought by harmonic balance or by application of a variational procedure similar to that employed to find the periodic (unstable) solution to the equations of references 1 et seq. We thus replace Eq. (1) by the variational statement

$$\delta \left[\langle (dv/ds)^2 \rangle - (2\,\nu/N)^2 \langle v^2 \rangle + (b/3) \langle v^3 \cos 2\,s \rangle + 2\,\lambda \langle v \cos \tfrac{2\,s}{3} \rangle \right] = 0, \tag{2}$$

in which the symbol $\langle\ \rangle$ denotes that the average value of the embraced quantity is to be taken. For the present purpose a trial function of the form

$$v = A_1 \cos 2\,s/3 + A_2 \cos 2\,s + A_3 \cos 10\,s/3 \tag{3}$$

is substituted into Eq. (2) to obtain

$$\delta \left\{ \frac{1}{2}\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_1^2 + \frac{1}{2}\left[4 - \left(\frac{2\nu}{N}\right)^2\right] A_2^2 + \frac{1}{2}\left[\frac{100}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_3^2 \right.$$
$$+ \frac{b}{24} A_1^3 + \frac{b}{4} A_1^2 A_2 + \frac{b}{4} A_1 A_2 A_3 + \frac{b}{8} A_1^2 A_3$$
$$\left. + \frac{b}{8} A_2^3 + \frac{b}{4} A_2 A_3^2 + \lambda A_1 \right\} = 0, \tag{4}$$

or

$$\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_1 + \frac{b}{8} A_1^2 + \frac{b}{2} A_1 A_2 + \frac{b}{4} A_1 A_3 + \frac{b}{4} A_2 A_3 = -\lambda \tag{5a}$$

$$\left[4 - \left(\frac{2\nu}{N}\right)^2\right] A_2 + \frac{b}{4} A_1^2 + \frac{3b}{8} A_2^2 + \frac{b}{4} A_1 A_3 + \frac{b}{4} A_3^2 = 0 \tag{5b}$$

$$\left[\frac{100}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_3 + \frac{b}{8} A_1^2 + \frac{b}{4} A_1 A_2 + \frac{b}{2} A_2 A_3 = 0 . \tag{5c}$$

Equations (5a-c) admit, of course, the solution $A_1 = A_2 = A_3 = 0$ when $\lambda = 0$, corresponding to the equilibrium orbit $v = 0$ which applies in that case; with λ not necessarily zero, the corresponding solution is such that

$$A_1 \cong - \frac{\lambda}{4/9 - (2\nu/N)^2} , \tag{6a}$$

with

$$A_2 \cong - \frac{b}{4} \frac{1}{4 - (2\nu/N)^2} A_1^2 \tag{6b}$$

and

$$A_3 \cong - \frac{b}{8} \frac{1}{100/9 - (2\nu/N)^2} A_1^2 . \tag{6c}$$

Somewhat more satisfactory results than can be obtained conveniently from Eqs. (5a-c) by algebraic means are obtainable numerically--in the particular case that

$$\nu/N = 0.3 \tag{7a}$$

$$b = 1.15 \tag{7b}$$

$$\lambda = 0.006 \tag{7c}$$

we find values of A_1, A_2, A_3 such that

$$v = -0.0831620 \cos 2s/3 - 0.0005469 \cos 2s - 0.0000937 \cos 10s/3, \tag{8a}$$

while a computer investigation[6] leads to the result

$$v = -0.083160_4 \cos 2s/3 - 0.000546_7 \cos 2s - 0.000093_7 \cos 10s/3. \tag{8b}$$

The corresponding location of the stable fixed point, for $s = 0$ (mod. 3π), is at

$$v = -.083802_6 \quad \text{from Eq. (8a)}$$

and at

$$v = -.083802_3 \quad \text{from direct computer studies.}$$

The results of the numerical solution of Eqs. (5a-c) are thus found to be in excellent agreement with the computer results, while the stable fixed point computed from the simple forms (6a-c) would be $-.07105 - .00040 - .00007 = -.07152$, or about 85% of the correct value.

D. LIMITING-AMPLITUDE SOLUTIONS
(Unstable Periodic Orbits)

1.

In addition to the solution of Eqs. (5a-c) discussed in the previous section, these equations admit a second solution--a solution with which the unstable fixed point lying on the symmetry axis of the phase plot (for $s = 0$, mod. 3π) is associated. The coefficients given by this second solution have values given roughly by

$$A_1 \cong \frac{-8\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]}{b} + \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \tag{9a}$$

(in which the first term should represent the value of A_1 for $\lambda = 0$), and

$$A_2 \cong -\frac{b}{4}\,\frac{1}{4-(2\,\nu/N)^2}\,A_1^2 \tag{9b}$$

$$A_3 \cong -\frac{b}{8}\,\frac{1}{100/9-(2\,\nu/N)^2}\,A_1^2\,, \tag{9c}$$

[as in (6b, c)].

A numerical solution of Eqs. (5a-c), for the parameters taken previously [Eqs. (7a-c)], leads to the solution (unstable periodic orbit)

$$v = -\,0.426294 \cos 2\,s/3 - 0.014466 \cos 2\,s - 0.002597 \cos 10\,s/3, \tag{10a}$$

whereas a computer investigation leads to the result

$$v = -\,0.426274 \cos 2\,s/3 - 0.014468 \cos 2\,s - 0.002598 \cos 10\,s/3$$
$$- 0.000098 \cos 14\,s/3 - 0.000010 \cos 18\,s/3 - \cdots\,. \tag{10b}$$

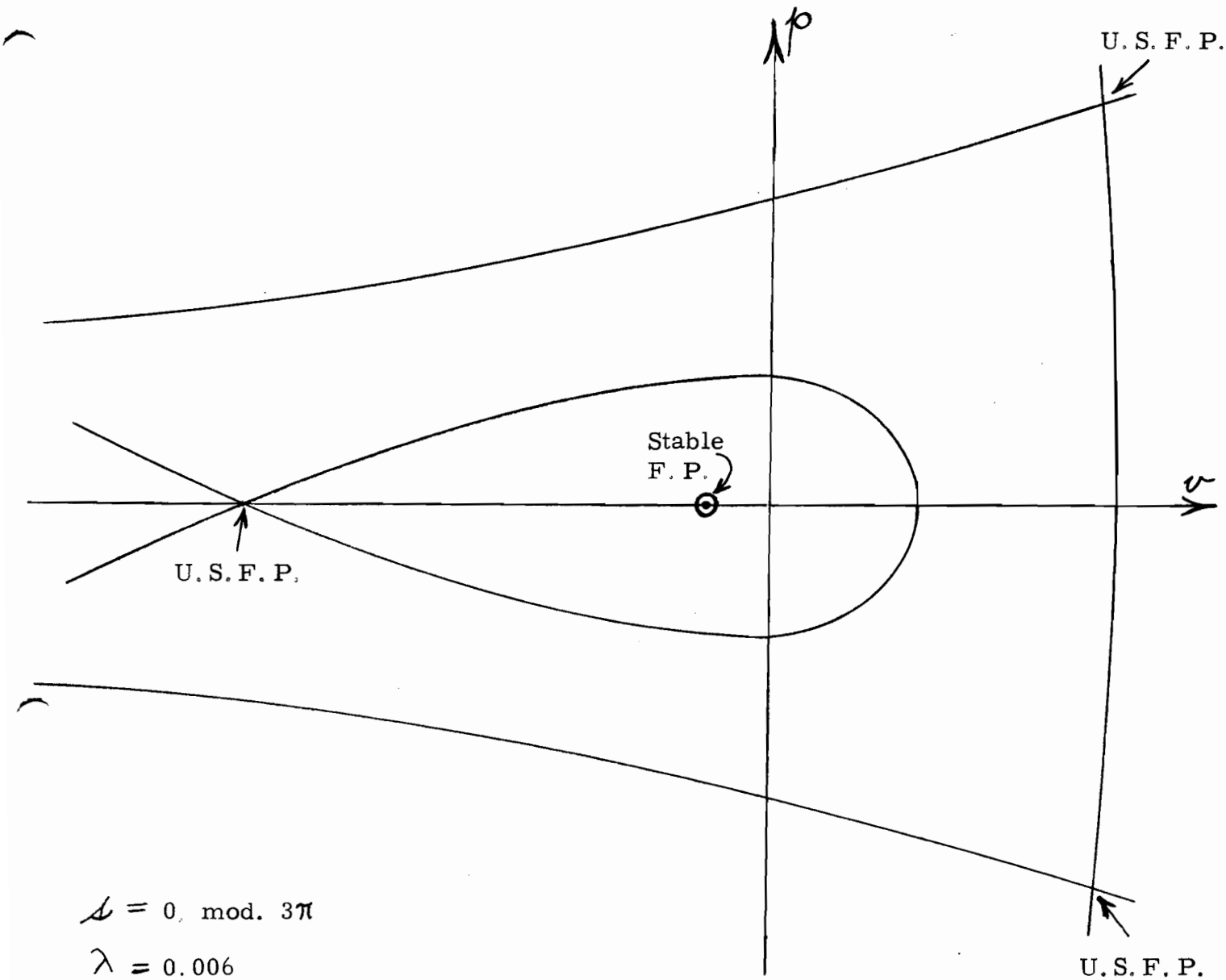
The corresponding fixed-point location (for $s = 0$, mod. 3π) is

$v = -\,0.443357$ from Eq. (10a)

and

$v = -\,0.443449$ from direct computer studies.

With a stronger perturbation (larger λ) this unstable fixed point and the stable fixed point will approach one another.



2.

To determine in this same way the locations of the other unstable fixed points--those situated above and below the symmetry axis of the $s = 0$, mod. 3π, phase plot-- a trial function more general than that shown in

Eq. (3) must be employed.* For this purpose we may employ the periodic trial function

$$v = A_1 \cos 2\,s/3 + A_2 \cos 2\,s + A_3 \cos 10\,s/3$$
$$\pm (B_1 \sin 2\,s/3 + B_2 \sin 2\,s + B_3 \sin 10\,s/3) \tag{11}$$

which, upon introduction into the variational statement (2) leads to

$$\delta \Big\{ \frac{1}{2}\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right](A_1^2 + B_1^2) + \frac{1}{2}\left[4 - \left(\frac{2\nu}{N}\right)^2\right](A_2^2 + B_2^2) + \frac{1}{2}\left[\frac{100}{9} - \left(\frac{2\nu}{N}\right)^2\right](A_3^2 + B_3^2)$$
$$+ \frac{b}{24} A_1^3 + \frac{b}{4} A_1^2 A_2 + \frac{b}{8} A_1^2 A_3 + \frac{b}{4} A_1 A_2 A_3 + \frac{b}{8} A_2^3 + \frac{b}{4} A_2 A_3^2$$
$$- \frac{b}{8} A_1 B_1^2 + \frac{b}{4} A_1 B_1 B_3 + \frac{b}{4} A_1 B_2 B_3$$
$$+ \frac{b}{4} A_2 B_1^2 - \frac{b}{4} A_2 B_1 B_3 + \frac{b}{4} A_2 B_3^2 + \frac{b}{8} A_2 B_2^2$$
$$- \frac{b}{8} A_3 B_1^2 + \frac{b}{4} A_3 B_1 B_2 + \lambda A_1 \Big\} = 0 \tag{12}$$

or

$$\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_1 + \frac{b}{8} A_1^2 + \frac{b}{2} A_1 A_2 + \frac{b}{4} A_1 A_3 + \frac{b}{4} A_2 A_3$$
$$- \frac{b}{8} B_1^2 + \frac{b}{4} B_1 B_3 + \frac{b}{4} B_2 B_3 = -\lambda \tag{13a}$$

$$\left[4 - \left(\frac{2\nu}{N}\right)^2\right] A_2 + \frac{b}{4} A_1^2 + \frac{b}{4} A_1 A_3 + \frac{3\,b}{8} A_2^2 + \frac{b}{4} A_3^2$$
$$+ \frac{b}{4} B_1^2 - \frac{b}{4} B_1 B_3 + \frac{b}{8} B_2^2 + \frac{b}{4} B_3^2 = 0 \tag{13b}$$

---

*It may be noted that, in contrast to cases discussed in previous reports (e.g., ref. 1), the basic period of the coefficients in the differential equation is $\underline{3\pi}$ when the perturbation is present and the locations of the various fixed points are no longer obtainable from a single periodic solution by substitution, in turn, of values of the independent variable differing by π from one another.

$$\left[\frac{100}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_3 + \frac{b}{8} A_1^2 + \frac{b}{4} A_1 A_2 + \frac{b}{2} A_2 A_3 - \frac{b}{8} B_1^2 + \frac{b}{4} B_1 B_2 = 0 \qquad (13c)$$

$$\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right] B_1 - \frac{b}{4} A_1 B_1 + \frac{b}{4} A_1 B_3 + \frac{b}{2} A_2 B_1 - \frac{b}{4} A_2 B_3 - \frac{b}{4} A_3 B_1 + \frac{b}{4} A_3 B_2 = 0 \qquad (13d)$$

$$\left[4 - \left(\frac{2\nu}{N}\right)^2\right] B_2 + \frac{b}{4} A_1 B_3 + \frac{b}{4} A_2 B_2 + \frac{b}{4} A_3 B_1 = 0 \qquad (13e)$$

$$\left[\frac{100}{9} - \left(\frac{2\nu}{N}\right)^2\right] B_3 + \frac{b}{4} A_1 B_1 + \frac{b}{4} A_1 B_2 - \frac{b}{4} A_2 B_1 + \frac{b}{2} A_2 B_3 = 0 . \qquad (13f)$$

Possible solutions of Eqs. (13a-f) are of course given by $B_1 = B_2 = B_3 = 0$ with A_1, A_2, A_3 then being solutions of Eqs. (5a-c); the new results which are obtained by admitting the case in which not all the coefficients B_i vanish will have, very roughly,

$$A_1 \cong \frac{4\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]}{b} \qquad (14a)$$

$$B_1 \cong \frac{4\sqrt{3}\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]}{b} \sqrt{1 + \frac{b\lambda}{6\left[\frac{4}{9} + \left(\frac{2\nu}{N}\right)^2\right]^2}} \qquad (14b)$$

$$\cong \frac{4\sqrt{3}\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]}{b} + \frac{1}{\sqrt{3}} \frac{\lambda}{\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]} . \qquad (14b')$$

A numerical solution of Eqs. (13a-f), again for the parameters specified by Eqs. (7a-c), suggests a solution

$$v = 0.244637 \cos 2s/3 - 0.022431 \cos 2s + 0.002307 \cos 10s/3$$
$$\pm (0.470329 \sin 2s/3 - 0.000021 \sin 2s - 0.003362 \sin 10s/3), \qquad (15a)$$

while a computer investigation gives the corresponding result

$$v = 0.244624 \cos 2\,s/3 - 0.022434 \cos 2\,s + 0.002309 \cos 10\,s/3$$
$$+ 0.000087 \cos 14\,s/3 - 0.000020 \cos 6\,s + \cdots$$
$$\pm (0.0470300 \sin 2\,s/3 - 0.000021 \sin 2\,s - 0.003365 \sin 10\,s/3$$
$$+ 0.000168 \sin 14\,s/3 - 0.000002 \sin 6\,s + \cdots) \,. \tag{15b}$$

The corresponding fixed-point coordinates (for $s = 0$, mod. 3π) are

$$\left.\begin{aligned} v &= 0.224513 \\ p &= \pm\, 0.3023 \end{aligned}\right\} \quad \text{from Eq. (15a)}$$

and

$$\left.\begin{aligned} v &= 0.224566 \\ p &= \pm\, 0.3030 \end{aligned}\right\} \quad \text{from direct computer studies.}$$

The methods described in this section evidently are able to give a good representation of the unstable periodic solutions for the differential equation (1). For the present, however, we shall regard this section as a diversion and proceed with the results of Section C to effect a removal of the forcing term from (1) and so permit a continuation of the analysis in the manner outlined in Section B.

E. REMOVAL OF FORCING TERM AND DETERMINATION OF β (s)

1.

If we denote by v_s the stable periodic orbit resulting from the forcing term $-\lambda \cos \frac{2\,s}{3}$ in Eq. (1), with v_s taken as well given by expressions presented in Section C [e.g., Eq. (3), with coefficients as illustrated in Eqs. (8a, b)], we may write

$$v = v_s + q \tag{16}$$

and transform Eq. (1) to the form

$$\frac{d^2q}{ds^2} + \left[\left(\frac{2\nu}{N}\right)^2 - b\,(\cos 2\,s)\,v_s\right] q - \frac{b}{2}\,(\cos 2\,s)\,q^2 = 0 \tag{17a}$$

or, making use of (3),

$$\frac{d^2q}{ds^2} + \left[\left(\frac{2\nu}{N}\right)^2 - \frac{bA_2}{2} - b\,\frac{A_1 + A_3}{2}\cos\frac{4\,s}{3} - b\,\frac{A_1}{2}\cos\frac{8\,s}{3} - b\,\frac{A_2}{2}\cos 4\,s - b\frac{A_3}{2}\cos\frac{16\,s}{3}\right] q$$

$$- \frac{b}{2}\,(\cos 2\,s)\,q^2 = 0 \tag{17b}$$

in which the terms of primary importance in the coefficient of q would normally be

$$\left(\frac{2\nu}{N}\right)^2 - \frac{bA_2}{2}\,, \quad -\frac{b\,A_1}{2}\cos\frac{4\,s}{3}\,, \quad \text{and} \quad -\frac{b\,A_1}{2}\cos\frac{8\,s}{3}\,.$$

With the coefficients of v_s found in Section C by numerical methods [cf. Eq. (8a)], for the parameters specified by Eqs. (7a-c), the differential equation (17b) for q becomes

$$\frac{d^2q}{ds^2} + \Big[\,0.3603\,145 + 0.0478\,720\cos\frac{4\,s}{3} + 0.0478\,182\cos\frac{8\,s}{3}$$

$$+ 0.0003\,145\cos 4\,s + 0.0000\,539\cos\frac{16\,s}{3}\Big]\,q$$

$$- 0.575\,(\cos 2\,s)\,q^2 = 0. \tag{18}$$

2.

It is of some interest to estimate the small-amplitude oscillation frequency ν_λ for Eq. (18), and it is necessary for what follows to describe the variation of the function β which characterizes the solutions of the linearized equation. To this end it is convenient to introduce a change of scale for the independent variable,

$$\tau = \frac{2}{3}\,s \tag{19}$$

and consider the linearized equation

$$\frac{d^2q}{d\tau^2} + (a + b\cos 2\tau + c\cos 4\tau)\, q = 0 \tag{20}$$

with, in the case corresponding to Eq. (18),

$$a = 0.810\,708 \tag{21a}$$

$$b = 0.107\,712 \tag{21b}$$

$$c = 0.107\,591 \tag{21c}$$

(the coefficients of the higher-order terms, $\cos 6\tau$ and $\cos 8\tau$, being ignored).

(i) It is tempting to attempt to estimate the oscillation frequency for Eq. (20) by means of the "smooth approximation"--since the value of ν_λ for Eq. (18) is not very far from $\frac{2}{3}$ and hence the corresponding value, ν', for Eq. (20) not far from unity (σ' near π), however, this method would be inappropriate. A possible, relatively quick, estimate may be obtained by reference to available ILLIAC tables,[7] from which one finds

$$\cos \nu'\pi \cong \cos \sqrt{a}\,\pi - 0.36\, b^2 - 0.022\, c^2 \tag{22}$$

for b and c small, $\sqrt{a}$ in the neighborhood of 0.9, and with ν' denoting $\frac{3}{2}\nu_\lambda$ in the present application. With the particular coefficients of interest here [Eqs. (21a-c)], the expression (22) gives $\nu' = 0.9051$, or $\nu_\lambda = 0.6034$, in complete agreement with the value found by direct computation[4, 6a] for $\lambda = 0.006$. Alternatively, a somewhat less arbitrary estimate may be made in connection with an examination of the range of variation of β, to be discussed below.

(ii) The differential equation (20) is of the form

$$\frac{d^2q}{d\tau^2} + (a + b\cos N\tau + c\cos 2N\tau)\, q = 0, \tag{20'}$$

with N = 2. As has been noted previously,[8] a rather accurate solution may be found by use of the trial function

$$q = g_o \cos \nu'\tau + f_1 \cos (N - \nu')\tau + g_1 \cos (N + \nu')\tau + f_2 \cos (2N - \nu')\tau + g_2 \cos (2N + \nu')\tau \tag{23}$$

and use of harmonic balance.* There results in this way the algebraic equations

$$a - \nu'^2 + \frac{b}{2}(f_1 + g_1) + \frac{c}{2}(f_2 + g_2) = 0 \tag{24a}$$

$$\left[a - (N - \nu')^2\right] f_1 + \frac{b}{2}(1 + f_2) + \frac{c}{2} g_1 = 0 \tag{24b}$$

$$\left[a - (N + \nu')^2\right] g_1 + \frac{b}{2}(1 + g_2) + \frac{c}{2} f_1 = 0 \tag{24c}$$

$$\left[a - (2N - \nu')^2\right] f_2 + \frac{b}{2} f_1 + \frac{c}{2} = 0 \tag{24d}$$

$$\left[a - (2N + \nu')^2\right] g_2 + \frac{b}{2} g_1 + \frac{c}{2} = 0 . \tag{24e}$$

Guided by prior knowledge of at least an approximate value of ν', a numerical solution of Eqs. (24a-e) is readily obtained, leading in the present case [coefficients given by Eqs. (21a-c)] to

$$f_1 = 0.1408\ 59 \tag{25a}$$

$$g_1 = 0.0080\ 69 \tag{25b}$$

$$f_2 = 0.0070\ 01 \tag{25c}$$

$$g_2 = 0.0023\ 3\dot{3} \tag{25d}$$

and $$\nu' \doteq 0.9051 \qquad (\nu_\lambda = 0.6034) . \tag{25e}$$

The <u>extreme</u> values of $\nu'\beta(\tau)$, and hence of the quantity $\nu_\lambda\beta(s)$ for Eq. (18), are given by[8]

$$\left[\nu_\lambda\beta\right] = \frac{1 \pm (f_1 + g_1) + (f_2 + g_2)}{1 \mp \left[\left(\frac{N}{\nu'} - 1\right) f_1 - \left(\frac{N}{\nu'} + 1\right) g_1\right] - \left[\left(\frac{2N}{\nu'} - 1\right) f_2 - \left(\frac{2N}{\nu'} + 1\right) g_2\right]} , \tag{26}$$

*We here omit, for simplicity, the phase shift (denoted by ϵ in ref. 8) which permits one to form in this way a <u>general</u> solution.

the upper and lower signs referring respectively to $s = 0$ ($\tau = 0$) and $s = 3\pi/4$ ($\tau = \pi/2$)--the range of values for $\nu_\lambda \beta$ suggested by the numerical values of f_1, etc. given in Eqs. (25a-e) is, then,

$$0.759 \leqslant \nu_\lambda \beta \leqslant 1.372 . \tag{27a}$$

These limits, (27a), are within a few tenths of a percent of the computational values:

$$0.7578 \leqslant \nu_\lambda \beta \leqslant 1.3755 . \tag{27b}$$

It appears to be quite tedious to derive $\nu' \beta(\tau)$ <u>as a function of τ</u> from the solution $q(\tau)$ as expressed by Eq. (23)--on the supposition that the variation is a pure cosine function, however, one might write roughly

$$\nu_\lambda \beta \cong 1.066 + 0.306 \cos 4 s/3. \tag{28a}$$

A corresponding very approximate formula, based on taking $f_1 \cong \frac{b}{8(1 - \nu')}$ and ignoring $g_1, \cdots$, might be written

$$\nu_\lambda \beta \cong 1 + \frac{b}{4(1 - \nu')} \cos 4 s/3 \quad \text{or} \quad 1 + \frac{b}{2(1 - a)} \cos 4 s/3 , \tag{28b}$$

which, in the present example, would lead to

$$\nu_\lambda \beta \cong 1 + 0.284 \cos 4 s/3 . \tag{28b'}$$

A more satisfactory evaluation of the functional dependence of may be sought by reference to the differential equation which is satisfied by β :*

$$\frac{\beta}{2} \frac{d^2\beta}{d\tau^2} - \frac{1}{4}\left(\frac{d\beta}{d\tau}\right)^2 + (a + b \cos 2\tau + c \cos 4\tau)\beta^2 = 1 . \tag{29}$$

A functional dependence

$$\beta = A + B \cos 2\tau + C \cos 4\tau \tag{30}$$

*<u>Cf.</u> Eq. (8) of reference 3.

may be inserted into Eq. (29) and the coefficients adjusted by harmonic balance to obtain the set of simultaneous equations

$$a A^2 - \frac{6 - 2a - c}{4} B^2 - (6 - \frac{a}{2}) C^2 + b A B + \frac{b}{2} B C + c A C = 1 \quad (31a)$$

$$- 2 (1 - a - \frac{c}{2}) A B - (7 - a - c) B C + b (A^2 + \frac{3}{4} B^2 + \frac{1}{2} C^2 + A C) = 0 \quad (31b)$$

$$- 2 (4 - a) A C - \frac{1 - a - c}{2} B^2 + b (A B + B C) + c A^2 + \frac{3}{4} c C^2 = 0 . \quad (31c)$$

For the parameters a, b, c as given by Eqs. (21a-c), a numerical solution of Eqs. (31a-c) leads to

$$A = 1.1536 \quad (32a)$$

$$B = 0.3365 \quad (32b)$$

$$C = 0.024\dot{7} ; \quad (32c)$$

substitution of these values into the expression (30) and multiplication by ν' (= 0.9051)* leads to the result

$$\nu_\lambda \beta = 1.044 + 0.305 \cos 4 s/3 + 0.022 \cos 8 s/3 . \quad (33a)$$

The results of a computer analysis of this case leads to

$$\nu_\lambda \beta = 1.045\,01 + 0.307\,35 \cos 4 s/3 + 0.021\,56 \cos 8 s/3 + 0.001\,51 \cos 4 s + 0.000\,06_6 \cos 16 s/3 , \quad (33b)$$

with which the numerical result (33a) is in reasonable agreement.

F. ELIMINATION OF THE A-G COEFFICIENT FROM THE LINEAR TERM AND CONTINUATION OF THE MOSER METHOD

1.

For continuation of the analysis of Eq. (18), it is convenient to introduce the independent variable

*By use of the values (32a-c) in connection with Eq. (30), a value of ν' could be estimated from this solution for β by forming $\nu' = \langle 1/\beta \rangle$.

$$t = \tau/2 = s/3 \tag{34}$$

to obtain

$$\frac{d^2q}{dt^2} + 4\,(a + b\cos 4\,t + c\cos 8\,t + \cdots)\,q - 5.175\,(\cos 6\,t)\,q^2 = 0. \tag{35}$$

As in an example presented previously [Sect. B of ref. 3], the transformation[9]

$$Q = q/\sqrt{\nu\beta} \tag{36a}$$

$$T = \int_0^t \frac{dt}{\nu\beta} \tag{36b}$$

enables one to eliminate the A-G aspect of the coefficient of the linear term in Eq. (34), to obtain:

$$\frac{d^2Q}{dT^2} + \nu''^2\,Q - 5.175\,(\nu\beta)^{5/2}\,(\cos 6\,t)\,Q^2 = 0\,, \tag{37}$$

in which $\nu'' = 2\,\nu' = 2\,(0.9051) = 1.8102$. The variables t and T become equal at $t = 0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi$, etc. The quantity $(\nu\beta)^{5/2}\cos 6\,t$, if expressed[6b, 10] in terms of T (Fig. 1), permits Eq. (37) to be written

$$\frac{d^2Q}{dT^2} + 3.2768\,Q - \Big[1.03504\cos 2\,T + 5.41441\cos 6\,T + 3.05511\cos 10\,T + 1.26600\cos 14\,T + 0.46114\cos 18\,T + 0.15573\cos 22\,T + 0.04940\cos 26\,T + 0.01441\cos 30\,T + \cdots\Big]\,Q^2 = 0\,. \tag{38}$$

It may be helpful to note that, with ν'' near 2, the oscillations will have a phase change of about 2π in one period of the term $1.03504\cos 2\,T$ (as for an integral resonance) and a phase change near $2\pi/3$ in one period of the (larger) term $5.41441\cos 6\,T$ (third-integral resonance). Accordingly, as we shall indicate in the work to follow, in undertaking to remove by the Moser method[5] the T-dependence from the Hamiltonian associated with Eq. (38) special attention must be given both to terms stemming from the cos 2 T term above and to those stemming from cos 6 T, in order to avoid potentially-resonant denominators.

Solutions for the unstable equilibrium orbits associated with Eq. (38) could, of course, be sought by harmonic balance, although this procedure would be of value only as a check of the preceding work since the original equation [Eq. (1)] was already treated satisfactorily by this method in earlier sections (Sects. C and D). Thus one solution of Eq. (38) may be sought in the form

$$Q = C_1 \cos 2\,T + C_2 \cos 6\,T + C_3 \cos 10\,T\,,$$

in which, approximately,

$$A_1 = -\,0.29519,\ A_2 = -\,0.01036,\ \text{and}\ A_3 = -\,0.00303;$$

accordingly the corresponding fixed point for Q (at $T = 0$) is at $Q = -\,0.30858$, $q = \sqrt{2\beta}\,Q = 1.17282 \times (-\,0.30858) = -\,0.36191$, and $v = v_s + q = -\,0.08380 - 0.36191 = -\,0.44571$, which is in error by about one-half of one percent of the computer fixed point. As a further check, a direct computational determination of the unstable fixed points for Eq. (38) was made, retaining just the first four cosine terms in the coefficient of Q^2; the values of (Q, P) found in this way were $(-\,0.307_{29}, 0)$ and $(0.263, \pm 1.064)$, which correspond to values of (v, p) which are $(-\,0.444_{20}, 0)$ and $(0.224_{65}, \pm 0.3024_{05})$ and thus are in good agreement with the results $(-\,0.44345, 0)$ and $(0.2246, \pm 0.3030)$ reported previously (Sect. D) from direct computer studies of Eq. (1).

In the subsection to follow we continue with the Moser procedure, which is of greater versatility than the harmonic-balance methods of Sects. C and D, applying the Moser method to Eq. (38) and then deducing in particular the fixed points in this way.

2.

The Hamiltonian associated with Eq. (38) is

$$H = \frac{1}{2} P^2 + \frac{1}{2} \nu''^2 Q^2 - \frac{1}{3} (b_1 \cos 2\,T + b_2 \cos 6\,T + \cdots + b_j \cos 2\,(2\,j - 1)\,T + \cdots) Q^3, \tag{39}$$

where P denotes dQ/dT, $\nu'' = 1.8102$, $b_1 = 1.03504$, $b_2 = 5.41441$, etc.

As in previous reports,[1-3] we now employ the generating function

$$G_o\,(Q, \gamma_o) = (\nu''/2)\, Q^2 \operatorname{ctn} \gamma_o \tag{40}$$

to effect the transformation

$$P = \partial G_o / \partial Q = \nu'' Q \operatorname{ctn} \gamma_o \tag{41a}$$

$$J_o = - \partial G_o / \partial \gamma_o = (\nu''/2)\, Q^2 \csc^2 \gamma_o\,; \tag{41b}$$

thus

$$\operatorname{ctn} \gamma_o = \frac{1}{\nu''} \frac{P}{Q} \tag{42a}$$

$$J_o = \frac{1}{2\,\nu''} P^2 + \frac{\nu''}{2} Q^2 \tag{42b}$$

$$Q = (2/\nu'')^{1/2} J_o^{1/2} \sin \gamma_o \tag{42c}$$

$$P = (2\,\nu'')^{1/2} J_o^{1/2} \cos \gamma_o\,, \tag{42d}$$

and the new Hamiltonian is

$$\begin{aligned} K_o &= H + \partial G_o / \partial T \\ &= H \\ &= \nu'' J_o - \frac{1}{3} \left(\frac{2}{\nu''}\right)^{3/2} J_o^{3/2} \sin^3 \gamma_o \sum_{j=1} b_j \cos 2\,(2\,j - 1)\,T \\ &= \nu'' J_o - \frac{1}{24} \left(\frac{2}{\nu''}\right)^{3/2} J_o^{3/2} \sum_{j=1} b_j \left\{ \begin{matrix} 3\sin\left[\gamma_o + 2(2j-1)T\right] + 3\sin\left[\gamma_o - 2(2j-1)T\right] \\ -\sin\left[3\gamma_o + 2(2j-1)T\right] - \sin\left[3\gamma_o - 2(2j-1)T\right] \end{matrix} \right\}. \end{aligned} \tag{43}$$

As a second generating function we next employ

$$G_1(\gamma_0, J_1) = J_1 \cdot \gamma_0 - \frac{1}{24}\left(\frac{2}{\nu''}\right)^{3/2} J_1^{3/2} \sum_{j=1} b_j \left\{ \begin{array}{l} 3\,\dfrac{\cos[\gamma_0 + 2(2j-1)T]}{\nu'' + 2(2j-1)} + 3(1-\delta_j^1)\,\dfrac{\cos[\gamma_0 - 2(2j-1)T]}{\nu'' - 2(2j-1)} \\ -\dfrac{\cos[3\gamma_0 + 2(2j-1)T]}{3\nu'' + 2(2j-1)} - (1-\delta_j^2)\,\dfrac{\cos[3\gamma_0 - 2(2j-1)T]}{3\nu'' - 2(2j-1)} \end{array} \right\}, \quad (44)$$

in which the Kronecker delta, δ_j^1 or δ_j^2, serves to eliminate terms which, with j = 1 or j = 2, would lead to terms with potentially-resonant denominators. The transformation equations which result from the generating function $G_1(\gamma_0, J_1)$ are

$$J_0 = \partial G_1 / \partial \gamma_0$$

$$= J_1 + \frac{1}{8}\left(\frac{2}{\nu''}\right)^{3/2} J_1^{3/2} \sum_{j=1} b_j \left\{ \begin{array}{l} \dfrac{\sin[\gamma_0 + 2(2j-1)T]}{\nu'' + 2(2j-1)} + (1-\delta_j^1)\,\dfrac{\sin[\gamma_0 - 2(2j-1)T]}{\nu'' - 2(2j-1)} \\ -\dfrac{\sin[3\gamma_0 + 2(2j-1)T]}{3\nu'' + 2(2j-1)} - (1-\delta_j^2)\,\dfrac{\sin[3\gamma_0 - 2(2j-1)T]}{3\nu'' - 2(2j-1)} \end{array} \right\} \quad (45a)$$

$$\gamma_1 = \partial G_1 / \partial J_1$$

$$= \gamma_0 - \frac{1}{16}\left(\frac{2}{\nu''}\right)^{3/2} J_1^{1/2} \sum_{j=1} b_j \left\{ \begin{array}{l} 3\,\dfrac{\cos[\gamma_0 + 2(2j-1)T]}{\nu'' + 2(2j-1)} + 3(1-\delta_j^1)\,\dfrac{\cos[\gamma_0 - 2(2j-1)T]}{\nu'' - 2(2j-1)} \\ -\dfrac{\cos[3\gamma_0 + 2(2j-1)T]}{3\nu'' + 2(2j-1)} - (1-\delta_j^2)\,\dfrac{\cos[3\gamma_0 - 2(2j-1)T]}{3\nu'' - 2(2j-1)} \end{array} \right\}, \quad (45b)$$

with the new Hamiltonian

$$K_1 = K_0 + \partial G_1 / \partial T$$

$$= \nu'' J_1 + \frac{1}{4}\left(\frac{2}{\nu''}\right)^{1/2} J_1^{3/2} \sum_{j=1} b_j \left\{ \begin{array}{l} \dfrac{\sin[\gamma_0 + 2(2j-1)T]}{\nu'' + 2(2j-1)} + (1-\delta_j^1)\,\dfrac{\sin[\gamma_0 - 2(2j-1)T]}{\nu'' - 2(2j-1)} \\ -\dfrac{\sin[3\gamma_0 + 2(2j-1)T]}{3\nu'' + 2(2j-1)} - (1-\delta_j^2)\,\dfrac{\sin[3\gamma_0 - 2(2j-1)T]}{3\nu'' - 2(2j-1)} \end{array} \right\}$$

$$-\frac{1}{24}\left(\frac{2}{\nu''}\right)^{3/2}\left[J_1^{3/2}+\frac{3}{16}\left(\frac{2}{\nu''}\right)^{3/2}J_1^2\sum_{j=1}b_j\left\{\begin{array}{l}\dfrac{\sin[\gamma_0+2(2j-1)T]}{\nu''+2(2j-1)}+(1-\delta_j^1)\dfrac{\sin[\gamma_0-2(2j-1)T]}{\nu''-2(2j-1)}\\[2ex] -\dfrac{\sin[3\gamma_0+2(2j-1)T]}{3\nu''+2(2j-1)}-(1-\delta_j^2)\dfrac{\sin[3\gamma_0-2(2j-1)T]}{3\nu''-2(2j-1)}\end{array}\right\}\right]$$

$$\times\sum_{j=1}b_j\left\{\begin{array}{l}3\sin[\gamma_0+2(2j-1)T]+3\sin[\gamma_0-2(2j-1)T]\\ -\sin[3\gamma_0+2(2j-1)T]-\sin[3\gamma_0-2(2j-1)T]\end{array}\right\}$$

$$+\frac{1}{24}\left(\frac{2}{\nu''}\right)^{3/2}J_1^{3/2}\sum_{j=1}2(2j-1)b_j\left\{\begin{array}{l}3\dfrac{\sin[\gamma_0+2(2j-1)T]}{\nu''+2(2j-1)}-3(1-\delta_j^1)\dfrac{\sin[\gamma_0-2(2j-1)T]}{\nu''-2(2j-1)}\\[2ex] -\dfrac{\sin[3\gamma_0+2(2j-1)T]}{3\nu''+2(2j-1)}+(1-\delta_j^2)\dfrac{\sin[3\gamma_0-2(2j-1)T]}{3\nu''-2(2j-1)}\end{array}\right\}$$

$$=\nu''J_1+\frac{1}{24}\left(\frac{2}{\nu''}\right)^{3/2}J_1^{3/2}\left[b_2\sin 3(\gamma_0-2T)-3b_1\sin(\gamma_0-2T)\right]$$

$$-\frac{1}{16\nu''^3}J_1^2\sum_{j=1}b_j\left\{\begin{array}{l}\dfrac{\sin[\gamma_0+2(2j-1)T]}{\nu''+2(2j-1)}+(1-\delta_j^1)\dfrac{\sin[\gamma_0-2(2j-1)T]}{\nu''-2(2j-1)}\\[2ex] -\dfrac{\sin[3\gamma_0+2(2j-1)T]}{3\nu''+2(2j-1)}-(1-\delta_j^2)\dfrac{\sin[3\gamma_0-2(2j-1)T]}{3\nu''-2(2j-1)}\end{array}\right\}$$

$$\times\sum_{j=1}b_j\left\{\begin{array}{l}3\sin[\gamma_0+2(2j-1)T]+3\sin[\gamma_0-2(2j-1)T]\\ -\sin[3\gamma_0+2(2j-1)T]-\sin[3\gamma_0-2(2j-1)T]\end{array}\right\}. \qquad (46)$$

To continue the work beyond this point, K_1, as expressed by Eq. (46), should be written in terms of γ_1 and a final transformation then made[1-3] to new variables, $\gamma_2 = \gamma_1 - 2\,T$ and $J_2 = J_1$, with the aim of obtaining a new Hamiltonian which is substantially independent of T. It would be the intention to keep in the J_2^2 term, which is in a sense regarded as a correction term, only terms which are constant or possibly functions of γ_2 (i.e., circular functions with arguments which are multiples of $\gamma_2 = \gamma_1 - 2\,T$ and hence are T-independent). Since by Eq. (45b), the difference between γ_1 and γ_0 is of order $J_1^{1/2}$, γ_0 may simply be replaced by γ_1 in the J_1^2 terms of Eq. (46). The distinction between γ_1 and γ_0 in the term involving $J_1^{3/2}\left[b_2 \sin 3\,(\gamma_0 - 2\,T) - 3\,b_1 \sin(\gamma_0 - 2\,T)\right]$ does not appear to introduce into the J_1^2 term any terms of the form which we elect to retain. Considerable complexity arises, however, in evaluating in this same sense the product of the two sums which appear in the J_1^2 term of Eq. (46), since numerous cross products occur which involve circular functions with arguments that are multiples of $\gamma_2 = \gamma_1 - 2\,T$.

The J_1^2 term of Eq. (46) includes, then firstly the constant terms $\dfrac{\alpha' b_2^2}{192\,\nu''^3}\,J_1^2$, where α' denotes

$$\alpha' = -\frac{1 + (3\,b_1/b_2)^2}{\frac{\nu''}{2} + 1} + 36\,\nu'' \sum_{j=1} \left(\frac{b_j}{b_2}\right)^2 \left\{ \frac{1 - \delta_j'}{\left[2\,(2\,j - 1)\right]^2 - \nu''^2} + \frac{1 - \delta_j^2}{\left[2\,(2\,j - 1)\right]^2 - (3\,\nu'')^2} \right. \tag{47a}$$

$$= +\,1.75516 \quad \text{(in our example).} \tag{47b}$$

There are, in addition, cross terms which involve circular functions of arguments that are multiples of $\gamma_2 = \gamma_1 - 2\,T$, of which we write those depending on b_2 in combination with b_1 or b_3 as $\frac{b_2^2}{192\,\nu''^3} J_1^2\, F(\gamma_2)$, with

$$F(\gamma_2) = -6\left\{\frac{b_3}{b_2}\left[\frac{1}{6-\nu''} + \frac{3}{10-3\nu''} - \frac{3}{6+\nu''} + \frac{3}{10-\nu''} - \frac{3}{6+3\nu''} - \frac{1}{10+\nu''}\right]\cos 2\gamma_2\right.$$

$$-\frac{b_1}{b_2}\left[\frac{3}{2+\nu''} - \frac{3}{6-\nu''} + \frac{3}{2+3\nu''} + \frac{1}{6+\nu''}\right]\cos 2\gamma_2$$

$$\left.+\frac{b_1}{b_2}\left[\frac{3}{3\nu''-2} - \frac{1}{6-\nu''}\right]\cos 4\gamma_2\right\} \tag{48a}$$

$$= -1.10355\cos 2\gamma_2 - 0.72926\cos 4\gamma_2 \quad \text{(in our example).} \tag{48b}$$

We accordingly take

$$K_1 = \nu'' J_1 + \frac{1}{24}\left(\frac{2}{\nu''}\right)^{3/2} J_1^{3/2}\left[b_2 \sin 3(\gamma_1 - 2\,T) - 3\,b_1 \sin(\gamma_1 - 2\,T)\right]$$

$$+ \frac{b_2^2\left[\alpha' + F(\gamma_1 - 2\,T)\right]}{192\,\nu''^3} J_1^2 \quad . \tag{49}$$

For the final transformation we now, of course, employ the simple generating function

$$G_2(\gamma_1, J_2) = J_2 \cdot (\gamma_1 - 2\,T), \tag{50}$$

so that

$$J_1 = \partial G_2/\partial\gamma_1 = J_2 \tag{51a}$$

$$\gamma_2 = \partial G_2/\partial J_2 = \gamma_1 - 2\,T \tag{51b}$$

and

$$
\begin{aligned}
K_2 &= K_1 + \partial G_2 / \partial T \\
&= K_1 - 2 J_1 \\
&= (\nu'' - 2) J_2 + \frac{1}{24}\left(\frac{2}{\nu''}\right)^{3/2} J_2^{3/2} (b_2 \sin 3\gamma_2 - 3 b_1 \sin \gamma_2) \quad (52a) \\
&\qquad + \frac{b_2^2 \left[\alpha' + F(\gamma_2)\right]}{192\, \nu''^3} J_2^2 .
\end{aligned}
$$

Since K_2, as expressed by Eq. (52a), is so written as to be T-independent, we take K_2 to be a constant of the motion. In our present example this invariant is

$$
\begin{aligned}
K_2 = &- 0.1898\, J_2 + 0.048389\, (5.41441 \sin 3\gamma_2 - 3.10512 \sin \gamma_2)\, J_2^{3/2} \\
&+ \left[0.045179 - 0.028406 \cos 2\gamma_2 - 0.018772 \cos 4\gamma_2\right] J_2^2 . \qquad (52b)
\end{aligned}
$$

G. THE FIXED POINTS
[In Particular For T = 0]

1.

The fixed points associated with the Hamiltonian K_2 of Eq. (52a) are given by points which simultaneously satisfy

$$\partial K_2 / \partial \gamma_2 = 0 \qquad \underline{\text{and}} \qquad \partial K_2 / \partial J_2 = 0 , \qquad (53a, b)$$

so that K_2 is stationary. If it were not for the presence of the function $F(\gamma_2)$, the first condition would be met when

$$\cos \gamma_2 = 0 \qquad \underline{\text{or}} \text{ when} \qquad 4 \cos^2 \gamma_2 - 3 = \frac{b_1}{b_2} . \qquad (54a, b)$$

The two roots in addition to the root $\gamma_2 = 270^\circ$ appear to be shifted by about 5/3 degree by inclusion of the function $F(\gamma_2)$ in the calculation, and the value of $J_2^{1/2}$ which corresponds to these latter roots increased by about 3 percent. Estimates of these solutions to Eqs. (53a, b) are given in Table I,*

*The roots chosen here are selected so that, with $\nu'' < 2$, $J_2^{1/2}$ will be positive. At T = 0 the values of γ_1 will be identical with γ_2 [Eq. (51b)].

together with the associated values of K_2, which are now necessarily <u>not all the same.</u>

TABLE I

Values of γ_2, J_2 for which the Hamiltonian K_2 can be Stationary $(b_1/b_2 = 1.03504/5.41441)^2$

Root	γ_2	$3\gamma_2$	$J_2^{1/2}$	J_2	K_2
1	-90°	-270°	0.291 84	0.0851 68	-0.0055 21
2, 3	$28^\circ.41$ $151^\circ.59$	$85^\circ.23$ $454^\circ.77$	0.579 58	0.335 915	-0.0226 50

It will be recalled that $J_1 = J_2$ and, for $T = 0$, $\gamma_1 = \gamma_2$. In the following subsection we make the inverse transformations necessary to express these results in terms of the original variables, specifically for $T = 0$ $(s = 0)$.

2.

For the assumed value of T, namely $T = 0$ in the present case, the values of γ_2 $(= \gamma_1)$ and J_2 $(= J_1)$ may be transformed to corresponding values of γ_0, J_0 by means of Eqs. (45a, b). This transformation is least laborious in the case designated as "Root 1" in Table I, since, for that case, $\gamma_0 = \gamma_1$ $(= 270^\circ)$. Once the desired values of γ_0, J_0 are obtained, Q and P $(= dQ/dT)$ follow immediately from Eqs. (42c, d). Since, at $T = 0$, $\nu\beta = 1.3755$ and $d(\nu\beta)/dt = 0$, one next may evaluate

$$q = \sqrt{\nu\beta}\, Q = 1.17282\, Q \tag{55a}$$

$$dq/dt = (1/\sqrt{2\beta})\, dQ/dT = P/\sqrt{2\beta} = P/1.17282.^{*} \tag{55b}$$

Finally, of course,

$$v = v_s + q = -0.08380_2 + q \qquad [\text{from Eq. (16)}] \tag{56a}$$

and

$$p \equiv \frac{dv}{ds} = \frac{1}{3}\frac{dq}{dt} \quad , \tag{56b}$$

since $t = s/3$ [Eq. (34)] and $dv_s/ds = 0$ at $s = 0$. In this way we estimate the values listed in Table II.

TABLE II

Values Leading to Fixed-Point Coordinates
($T = 0$, $s = 0$)

Root	γ_0	$J_0^{1/2}$	Q	P	q	dq/dt	v	p=dv/ds
1	-90°	0.29265	-0.30761	0	-0.36077	0	-0.44457	0
2, 3	$25^\circ.11$ $154^\circ.89$	0.5665	0.2527	± 0.9761	0.2964	± 0.8323	0.2126	± 0.2774

The true values for the coordinates v, p of the unstable fixed points, as given by the computer, are (Sect. D)

$$v = -0.44345, \qquad p = 0\,,$$

$$v = 0.2245, \qquad p = \pm\, 0.3030\,;$$

it is seen, accordingly, that the present "analytic" method gives (as Root 1) the location of the unstable fixed point which lies on the v-axis with an accuracy considerably better than 1 %, but that the values of v and p for the other unstable fixed points are respectively smaller than the correct values by $5\frac{1}{2}$ and

*Cf. Eqs. (17a, b) of ref. 3.

$8\frac{1}{2}$ %. These analytic results were not materially affected by a refinement of the function $F(\gamma_2)$ [Eq. (48b)], which enters in Eqs. (52a, b), through inclusion of terms involving $b_3 b_4$ in the coefficient of $\cos 2\gamma_2$ and terms involving $b_1 b_3$ and $b_2 b_4$ in the coefficient of $\cos 4\gamma_2$.

H. ALTERNATIVE, SIMPLIFIED, ANALYTIC METHOD

The analytic method of the previous sections, in which it was attempted to follow the Moser procedure in an orderly fashion, clearly involved considerable complexity in the details of the calculations. It was necessary, firstly, to undertake some numerical work in order to estimate adequately the stable solution for the forced motion. Subsequently, once the forcing term was removed from the equation of motion, additional labor was required because the new differential equation then contained an A-G coefficient for the linear term. Because of these complications, it would seem difficult to arrive at useful formulas by following the methodology on which our numerical work was based and, accordingly, it is of interest to explore a somewhat less straightforward, but simplified, analytical procedure.

In this second method the effect of the forcing term will only be eliminated immediately by subtraction of that part which would result from consideration of the linear terms of Eq. (1). In the subsequent work, terms of order $\lambda J_1^{3/2}$ and $\lambda^2 J_1$ in the Hamiltonian will be neglected, in comparison to the $J_1^{3/2}$ term and the constant part of J_1^2 term, since λ may in a sense be regarded as a perturbation. Information concerning the stable equilibrium orbit, as well as pertaining to the unstable equilibrium orbits and other features of the motion, should then result from the analysis.

1.

We commence, therefore, with the differential equation (1), for which the Hamiltonian has the form

$$h = \frac{1}{2} p^2 + \frac{1}{2}\left(\frac{2\nu}{N}\right)^2 v^2 - (b/6)(\cos 2s) v^3 - \lambda (\cos \frac{2s}{3}) v . \tag{57}$$

For the initial transformation, to quantities akin to angle-action variables, we employ the generating function

$$F_o(v, \gamma_o) = \frac{\nu}{N}\left[v + \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \cos \frac{2s}{3}\right]^2 \operatorname{ctn} \gamma_o + \frac{2}{3} \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} (\sin \frac{2s}{3}) v + f(s) , \tag{58}$$

where f (s) would be selected to obviate the need to include in the new Hamiltonian terms which only involve the independent variable and hence play no significant rôle. The resultant transformation is

$$p = \partial F_o / \partial v = \frac{2\nu}{N}\left[v + \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \cos \frac{2s}{3}\right] \operatorname{ctn} \gamma_o + \frac{2}{3} \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \sin \frac{2s}{3} \tag{59a}$$

$$J_o = - \partial F_o / \partial \gamma_o = \frac{\nu}{N}\left[v + \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \cos \frac{2s}{3}\right]^2 \csc^2 \gamma_o ; \tag{59b}$$

so that

$$\operatorname{ctn} \gamma_o = \frac{N}{2\nu} \frac{p - \frac{2}{3} \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \sin \frac{2s}{3}}{v + \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \cos \frac{2s}{3}} \tag{60a}$$

$$J_o = \frac{N}{4\nu}\left[p - \frac{2}{3} \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \sin \frac{2s}{3}\right]^2 + \frac{\nu}{N}\left[v + \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \cos \frac{2s}{3}\right]^2 , \tag{60b}$$

$$v = \left(\frac{N}{\nu}\right)^{1/2} J_o^{1/2} \sin\gamma_o - \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \cos\frac{2s}{3} \tag{60c}$$

$$p = 2\left(\frac{\nu}{N}\right)^{1/2} J_o^{1/2} \cos\gamma_o + \frac{2}{3} \frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} \sin\frac{2s}{3}, \tag{60d}$$

and the new Hamiltonian is found to be (after some intermediate algebraic work)

$$\begin{aligned} H_o &= h + \partial F_o/\partial s \\ &= 2\frac{\nu}{N} J_o + \frac{b}{48}\left(\frac{N}{\nu}\right)^{3/2} J_o^{3/2} \begin{bmatrix} \sin(3\gamma_o + 2s) + \sin(3\gamma_o - 2s) \\ - 3\sin(\gamma_o + 2s) - 3\sin(\gamma_o - 2s) \end{bmatrix} \\ &\quad + \frac{b}{16}\frac{N}{\nu}\frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} J_o \begin{bmatrix} 2\cos\frac{8s}{3} + 2\cos\frac{4s}{3} \\ - \cos(2\gamma_o + \frac{8s}{3}) - \cos(2\gamma_o - \frac{8s}{3}) \\ - \cos(2\gamma_o + \frac{4s}{3}) - \cos(2\gamma_o - \frac{4s}{3}) \end{bmatrix} \\ &\quad + \frac{b}{16}\left(\frac{N}{\nu}\right)^{1/2} \frac{\lambda^2}{\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]^2} J_o^{1/2} \begin{bmatrix} - 2\sin(\gamma_o + 2s) - 2\sin(\gamma_o - 2s) \\ - \sin(\gamma_o + \frac{10s}{3}) - \sin(\gamma_o - \frac{10s}{3}) \\ - \sin(\gamma_o + \frac{2s}{3}) - \sin(\gamma_o - \frac{2s}{3}) \end{bmatrix}. \end{aligned} \tag{61}$$

[The nature of the transformation and its effectiveness in removing completely the coupling term from the <u>linearized</u> differential equation (1) may be evident from Eqs. (60a) and (61). The general character of the Hamiltonian H_o of (61) is seen to correspond to that given for K_o by Eq. (50) of ref. 1 (p. 21), noting that, in ref. 1, $b = 1$ and that for our present $-\cos 2s$ the function $\sin 2t$ is employed in the v^2 term of the differential equation.]

Paralleling previous work,[1] we now make the next transformation by use of the generating function

$$F_1(\gamma_0, J_1) = J_1 \cdot \gamma_0 + \frac{b}{96}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \left[\frac{\cos(3\gamma_0 + 2s)}{1 + 3\nu/N} - 3\frac{\cos(\gamma_0 + 2s)}{1 + \nu/N} + 3\frac{\cos(\gamma_0 - 2s)}{1 - \nu/N} \right]; \tag{62}$$

so that

$$J_0 = \partial F_1/\partial\gamma_0 = J_1 + \frac{b}{32}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \left[-\frac{\sin(3\gamma_0 + 2s)}{1 + 3\nu/N} + \frac{\sin(\gamma_0 + 2s)}{1 + \nu/N} - \frac{\sin(\gamma_0 - 2s)}{1 - \nu/N} \right] \tag{63a}$$

$$\gamma_1 = \partial F_1/\partial J_1 = \gamma_0 + \frac{b}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2} \left[\frac{\cos(3\gamma_0 + 2s)}{1 + 3\nu/N} - 3\frac{\cos(\gamma_0 + 2s)}{1 + \nu/N} + 3\frac{\cos(\gamma_0 - 2s)}{1 - \nu/N} \right], \tag{63b}$$

and

$$\begin{aligned} H_1 &= H_0 + \partial F_1/\partial s \\ &= H_0 + \frac{b}{48}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \left[-\frac{\sin(3\gamma_0 + 2s)}{1 + 3\nu/N} + 3\frac{\sin(\gamma_0 + 2s)}{1 + \nu/N} + 3\frac{\sin(\gamma_0 - 2s)}{1 - \nu/N} \right] \\ &\cong 2\frac{\nu}{N} J_1 + \frac{b}{48}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \sin(3\gamma_1 - 2s) \\ &\quad + \frac{b^2}{2048}\left(\frac{N}{\nu}\right)^3 J_1^2 \left[\frac{6\,\nu/N}{1 - \nu^2/N^2} - \frac{1}{1 + 3\nu/N} \right] \\ &\quad - \frac{b}{16}\frac{N}{\nu}\frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} J_1 \cos\left(2\gamma_1 - \frac{4s}{3}\right) \\ &\quad - \frac{b}{16}\left(\frac{N}{\nu}\right)^{1/2} \frac{\lambda^2}{\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]^2} J_1^{1/2} \sin\left(\gamma_1 - \frac{2s}{3}\right), \end{aligned} \tag{64}$$

in which the last result follows after some algebraic simplification, and as a result of neglecting terms of the order $\lambda J_1^{3/2}$, $\lambda^2 J_1$, and terms which do not involve circular functions which are multiples of $(\gamma_1 - 2s/3)$-- [cf. Eq. (54) of ref. 1 (p. 22)].

For the final transformation we employ, as in previous work, the generating function

$$F_2(\gamma_1, J_2) = J_2 \cdot \left(\gamma_1 - \frac{2s}{3}\right); \tag{65}$$

so that

$$J_1 = \partial F_2/\partial \gamma_1 = J_2 \tag{66a}$$

$$\gamma_2 = \partial F_2/\partial J_2 = \gamma_1 - \frac{2s}{3} \tag{66b}$$

and

$$\begin{aligned} H_2 &= H_1 + \partial F_2/\partial s \\ &= H_1 - \frac{2}{3} J_2 \\ &= 2\left(\frac{\nu}{N} - \frac{1}{3}\right) J_2 + \frac{b}{48}\left(\frac{N}{\nu}\right)^{3/2} J_2^{3/2} \sin 3\gamma_2 + \frac{b^2\alpha}{2048}\left(\frac{N}{\nu}\right)^3 J_2^2 \\ &\quad - \frac{b}{16}\frac{N}{\nu}\frac{\lambda}{\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2} J_2 \cos 2\gamma_2 \\ &\quad - \frac{b}{16}\left(\frac{N}{\nu}\right)^{1/2} \frac{\lambda^2}{\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right]^2} J_2^{1/2} \sin \gamma_2 \end{aligned} \tag{67}$$

[cf. Eq. (57) of ref. 1 (p. 23)], where

$$\alpha = \frac{6\,\nu/N}{1 - \nu^2/N^2} - \frac{1}{1 + 3\,\nu/N}$$

[cf. Eq. (25) of ref. 1 (pp. 13, 23)]. The Hamiltonian, H_2, in the form written, is independent of s and will be taken as a constant of the motion.

2.

To obtain the fixed points, in particular, we may take the Hamiltonian H_2 to be stationary, as given by setting the partial derivatives $\partial H_2/\partial \gamma_2$ and $\partial H_2/\partial J_2$ each equal to zero; specifically,

$$\frac{b}{16}\left(\frac{N}{\nu}\right)^{3/2} J_2^{3/2} \cos 3\gamma_2 + \frac{b}{8}\left(\frac{N}{\nu}\right)\frac{\lambda}{\frac{4}{9}-\left(\frac{2\nu}{N}\right)^2} J_2 \sin 2\gamma_2 - \frac{b}{16}\left(\frac{N}{\nu}\right)^{1/2}\frac{\lambda^2}{\left[\frac{4}{9}-\left(\frac{2\nu}{N}\right)^2\right]^2} J_2^{1/2} \cos\gamma_2 = 0 \tag{69a}$$

and

$$2\left(\frac{\nu}{N}-\frac{1}{3}\right) + \frac{b}{32}\left(\frac{N}{\nu}\right)^{3/2} J_2^{1/2} \sin 3\gamma_2 + \frac{b^2\alpha}{1024}\left(\frac{N}{\nu}\right)^3 J_2$$

$$-\frac{b}{16}\frac{N}{\nu}\frac{\lambda}{\frac{4}{9}-\left(\frac{2\nu}{N}\right)^2}\cos 2\gamma_2 - \frac{b}{32}\left(\frac{N}{\nu}\right)^{1/2}\frac{\lambda^2}{\left[\frac{4}{9}-\left(\frac{2\nu}{N}\right)^2\right]^2} J_2^{-1/2} \sin\gamma_2 = 0. \tag{69b}$$

Two roots of interest for Eq. (69a, b), corresponding to the stable and unstable fixed points which lie on the v-axis of the phase plot for $s = 0$ (mod. 3π), are obtained by taking $\gamma_2 = -90^\circ$ and J_2 as satisfying

$$2\left(\frac{\nu}{N}-\frac{1}{3}\right) + \frac{b}{32}\left(\frac{N}{\nu}\right)^{3/2} J_2^{1/2} + \frac{b^2\alpha}{1024}\left(\frac{N}{\nu}\right)^3 J_2$$

$$+\frac{b}{16}\frac{N}{\nu}\frac{\lambda}{\frac{4}{9}-\left(\frac{2\nu}{N}\right)^2} + \frac{b}{32}\left(\frac{N}{\nu}\right)^{1/2}\frac{\lambda^2}{\left[\frac{4}{9}-\left(\frac{2\nu}{N}\right)^2\right]^2} J_2^{-1/2} = 0\,, \tag{70}$$

in which the term of order J_2 is comparatively small--for small λ one root, in fact, may be estimated roughly by consideration of just the constant term and that involving $J_2^{-1/2}$, while the other is given roughly by use of just the constant term and that which involves $J_2^{1/2}$. Numerically, for

$$\nu/N = 0.3, \qquad b = 1.15, \qquad \text{and} \qquad \lambda = 0.006,$$

Eq. (70) becomes

$$0.06944\, J_2 + 0.21871\, J_2^{1/2} - 0.04964 + 0.00033124\, J_2^{-1/2} = 0\,, \qquad (70')$$

with roots for $J_2^{1/2}$ given by 0.006881_5 and $0.2061_{\dot{5}}$. The remaining roots of interest similarly involve simultaneous solution of Eqs. (69a, b) for values of γ_2 near 30° or the supplementary angle 150°. To obtain the corresponding v, p coordinates of the fixed points, at $s = 0$ (mod. 3π), one must next transform γ_2 ($= \gamma_1$ for $s = 0$) and J_2 ($= J_1$) to γ_o, J_o and thence to v, p by use of Eqs. (63a, b) and (60c, d).

It may be noted in passing that the two roots of Eq. (70) become coincident for a critical value of λ given approximately by

$$\lambda_c \cong \frac{64}{b}\left(\frac{1}{3} - \frac{\nu}{N}\right)^2\left(\frac{1}{3} + \frac{\nu}{N}\right)\frac{\nu}{N} \qquad (71)$$
$$= 0.01175\,,$$

for $\nu/N = 0.3$ and $b = 1.15$; a more accurate numerical estimate, again based on Eq. (70), gives

$$\lambda_c = 0.01168\,,$$

with

$$J_2^{1/2} = 0.0740_4\,,$$

from which one finds, by Eq. (63a),

$$J_o^{1/2} = 0.07412$$

and, by Eqs. (60c, d),

$$v_c = -\left(\frac{10}{3}\right)^{1/2}(0.07412) - \frac{0.01168}{\frac{4}{9} - \frac{36}{100}}$$

$$= -0.13532 - 0.13832$$

$$= -0.2736_4\,,$$

$$p_c = 0\,.$$

The correct values for this confluent situation, as obtained by direct digital computation,[4] are

$$\lambda_c = 0.01136 ,$$

$$v_c = -0.2650 ,$$

$$p_c = 0 ;$$

so that our estimates for λ_c and v_c are evidently some 3% larger, in relative magnitude, than the correct values.[11]

Returning to our example with $\lambda = 0.006$, the fixed-point coordinates found by the present method of analysis are as summarized in Table III.

TABLE III

Estimated Fixed-Point Coordinates at $s = 0$, mod. 3π

$\nu/N = 0.3$ $\quad b = 1.15$ $\quad \lambda = 0.006$

Root	Calculated Values						Computer Results		Relative Error	
	γ_2	J_2	γ_0	J_0	v	p	v	p	ϵ_v	ϵ_p
Stable	-90°	0.006881_5	-90°	0.006882_2	-0.08361_8	0	-0.083802	0	-0.22%	--
1	-90°	0.2061_5	-90°	0.2067_6	-0.4485_5	0	-0.44345	0	+1.15%	--
2, 3	$34.^\circ20_6$ $145.^\circ79_4$	0.312_{69}	$31.^\circ24$ $148.^\circ76$	0.303_3	0.216_1	$\pm 0.284_1$	0.2246	± 0.3030	-4 %	-6%

The results obtained by the present simplified method not only are far more easily obtainable but appear to be of as good accuracy as those previously summarized in Table II (Sect. G).

I. REFERENCES AND NOTES

1. L. Jackson Laslett, MURA-452 (April 13, 1959).

2. L. Jackson Laslett, MURA-459 (May 20, 1959).

3. L. Jackson Laslett, MURA-463 (June 3, 1959).

4. In the main series of computer studies the differential equation was regarded as of the form

$$\frac{d^2v}{d\tau^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{b}{2}(\sin 2\tau)\, v^2 - \lambda \cos\left(\frac{2\tau}{3} + \frac{\pi}{6}\right) = 0 \quad \text{and}$$

attention was directed in particular to phase plots made at $\tau = 2.75\pi$, mod 3π. The translation of independent variable given by setting $s = \tau - 2.75\pi$ transforms this form of the equation to

$$\frac{d^2v}{ds^2} + \left(\frac{2\nu}{N}\right)^2 v - \frac{b}{2}(\cos 2s)\, v^2 - \lambda \cos \frac{2s}{3} = 0,$$

with the corresponding phase plots pertaining to $s = 0$, mod. 3π. It is this latter equation which we have introduced as Eq. (1) of the text. In the computational work ν/N was taken as 0.3, $b = 1.15$, and λ covered positive values extending through the critical value $\lambda_c = 0.01136$. Some simplification is possible, of course, by introduction of the scaled quantities $V = b\,v$ and $\mu = b\lambda$, in terms of which the differential equation becomes

$$\frac{d^2V}{ds^2} + \left(\frac{2\nu}{N}\right)^2 V - \frac{1}{2}(\cos 2s)\, V^2 - \mu \cos \frac{2s}{3} = 0.$$

5. Jürgen Moser, Nach. Gött. Akad. (Math.-Phys. Kl.) **IIa**, #6, 87-120 (1955).

6. The computational work was done with the MURA IBM 704 computer, by aid of the following programs:

(a) J. N. Snyder, DUCK-ANSWER (IBM Program 75), MURA-237 (1957);
(b) J. N. Snyder, FORANAL (IBM Program 52), MURA-228 (1957); and
(c) John McNall, DUCKNALL (IBM Program 219), MURA-438 (1958).

7. L. Jackson Laslett, MURA Notes (20 April 1955) [MURA Tables, Book 1B, Table 5].

8. L. Jackson Laslett and A. M. Sessler, MURA-252 (1957).

9. Cf. E. D. Courant and H. S. Snyder, Annals of Physics 3, No. 1, 1-48 (January, 1958)--Sect. 4a, esp. Eqs. (4.4) and (4.5), p. 18.

10. The computer solution for $\nu\beta$ was employed in forming this transformation, since the data were available in tabular form convenient for proceeding with the calculation. Values of $(\nu\beta)^{5/2} \cos 6\,t$, at equally spaced values of T, were then prepared and presented to the FORANAL program[6b] for Fourier analysis.

11. An alternative method for obtaining a formula similar to Eq. (71), to afford an approximate estimate of λ, has also been reported [L. Jackson Laslett and K. R. Symon, "Computational Results Pertaining to Use of a Time-Dependent Magnetic Field Perturbation to Implement Injection or Extraction in a FFAG Synchrotron," Paper submitted for inclusion in the Proceedings of the Second CERN Symposium on High Energy Accelerators, Genève, 1959]. In this method we note from Eqs. (6a) [Sect. C] and (9a) [Sect. D] that, for small λ, the locations of the two fixed points with which we are concerned may be estimated by

$$v_1(\lambda) \cong -\frac{\lambda}{4/9 - (2\nu/N)^2} \tag{72a}$$

$$v_2(\lambda) \cong v_2(\lambda = 0) + \frac{\lambda}{4/9 - (2\nu/N)^2} \quad . \tag{72b}$$

A parabolic fit, tangent to the lines (11a, b) at $\lambda = 0$, may be obtained by writing

$$\lambda \cong \frac{v \cdot [v - v_2(\lambda = 0)]}{v_2(\lambda = 0)} \cdot [4/9 - (2\nu/N)^2] , \tag{73}$$

for which the maximum value of λ,

$$\lambda_c = \frac{1}{4} \cdot [4/9 - (2\nu/N)^2] \cdot [-v_2(\lambda = 0)] , \tag{74a}$$

is attained at

$$v_c = \frac{1}{2} v_2 (\lambda = 0) . \tag{74b}$$

With $2\nu/N = 0.6$ and $-v_2(\lambda = 0) = 0.5238$ [from computational results cited in ref. 1, after division by $b = 1.15$], Eqs. (74a, b) suggest

$$\lambda_c = 0.01106$$

$$v_c = -0.2619 ,$$

which may be compared with the computational results cited in the text, namely

$$\lambda_c = 0.01136$$

$$v_c = -0.2650 .$$