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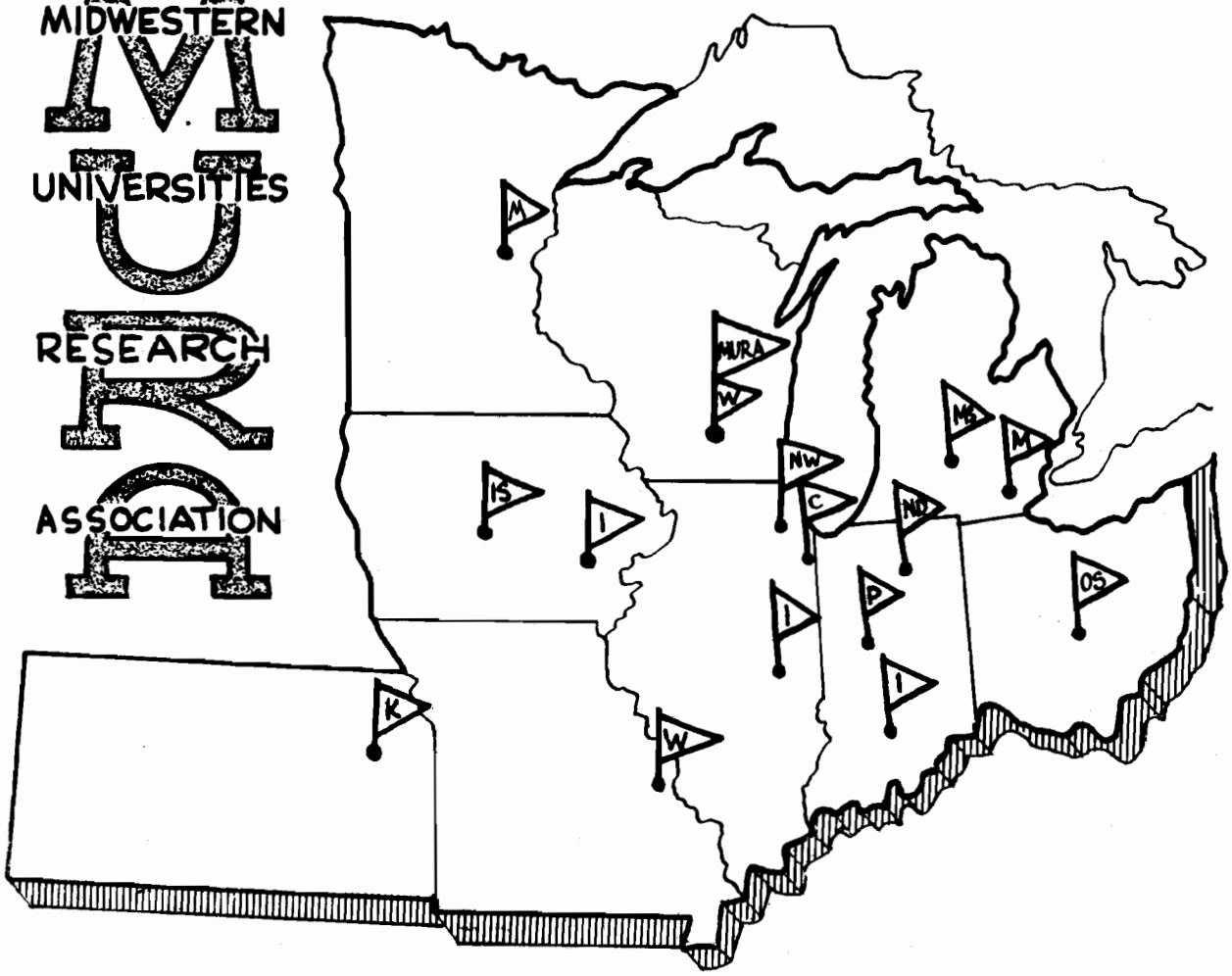

ACCELERATION ACROSS THE TRANSITION ENERGY WITHOUT INCREASE IN AMPLITUDE OF SYNCHROTRON OSCILLATIONS

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ACCELERATION ACROSS THE TRANSITION ENERGY WITHOUT INCREASE IN AMPLITUDE OF SYNCHROTRON OSCILLATIONS

N. Christofilos,** K. R. Symon and K. M. Terwilliger+

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ABSTRACT

An rf voltage and frequency regime for crossing transition energy is worked out which in linear approximation will preserve the amplitude of synchrotron oscillations in an FFAG accelerator. The initial and final voltage and stable phase may be chosen arbitrarily, as well as the energies at which the transition regime is to begin and to terminate, except that these energies must be sufficiently far from transition. The required frequency and voltage program during the transition regime are then determined.

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ACCELERATION ACROSS THE TRANSITION ENERGY WITHOUT INCREASE IN AMPLITUDE OF SYNCHROTRON OSCILLATIONS

The equations of motion for the synchrotron oscillations of a particle in an FFAG accelerator subject to an rf accelerating voltage are derived in MURA-106:

$$\frac{dW}{dt} = V(t) \sin \phi \,,$$
$$\frac{d\phi}{dt} = 2\pi h \left[f_o^{(t)} - f(W) \right] , \tag{1}$$

where

$$W = \int \frac{dE}{f(E)} \,, \tag{2}$$

h is the harmonic number, $h f_o$ the frequency of the rf oscillator, V is the accelerating voltage, and E, f, ϕ are the energy, frequency of revolution, and phase of the particle relative to the accelerating voltage. The equations (1) are derivable from the Hamiltonian

$$H = V \cos \phi + 2\pi h \left[f_o W - E(W) \right] , \tag{3}$$

so that W, ϕ are canonically conjugate variables.

The character of the phase oscillations of an accelerated particle have been discussed in MURA-106 when the energy is well above or below the transition energy. For a given voltage and rate of frequency modulation, there is a stable phase ϕ_s at which a particle can remain in synchronism with the rf, its energy increasing so as to correspond with the synchronous value $W_s(t)$. Around the synchronous phase point W_s, ϕ_s, there is a region of stable phase oscillations. According to the adiabatic theorem, a particle initially oscillating on a curve around W_s, ϕ_s of area A will at a

later time be found on a curve of area A around the new stable phase point, provided that the parameters (V, $f'_s = df/dW$, $\dot{W}_s = \dot{f}_o/f'_s$) which characterize the dynamics change sufficiently slowly. So long as the adiabatic theorem applies, the phase density of a group of particles being accelerated will be preserved. Each particle continues to oscillate at constant amplitude, if we measure amplitude of synchrotron oscillations by the area A of the W^*, ϕ phase plane encircled by the particle, viewed from a synchronous coordinate system ($W^* = W - W_s$, ϕ) moving with the synchronous phase point. It is shown in MURA-423 that the adiabatic condition is not satisfied near the transition energy. If the rf schedule is such that $\Gamma = \sin \phi_s$ is constant, or nearly so, then the condition for applicability of the adiabatic theorem is

$$\frac{f_t - f_o}{f_t} \gg \left[\left(\frac{V}{h E_t}\right)^2 \frac{\pi^2}{k} \frac{\Gamma^4}{1 - \Gamma^2}\right]^{1/3} , \qquad (4)$$

where the subscript 't' refers to the transition energy. When the synchronous frequency f_o is closer to transition than this, the amplitude of synchrotron oscillations does not necessarily remain constant. In general, the amplitude increases and the phase density decreases due to mixing of empty phase space with phase regions occupied by particles as the transition energy is crossed. Particles initially executing stable phase oscillations may be lost from the stable rf bucket as the transition energy is crossed. Computations with the TTT program, and experiments with the spiral sector model agree in suggesting as typical a loss of 50 per cent of the particles in a bucket crossing transition.

It is the purpose of this study to find an rf schedule which will accelerate particles over transition without increase in amplitude of synchrotron oscillations, at least for particles near enough to the synchronous energy so that linearized equations of motion can be used. Our procedure is suggested by a method used by Christofilos[1] in crossing transition in an AGS.

Let the functions $W_s(t)$, $\phi_s(t)$ describe the history of a standard particle which is in stable synchronism with the rf voltage below the transition energy and which is to be accelerated over the transition energy in such a way as to be in stable synchronism with the rf voltage above the transition energy. By "above" and "below" transition, we mean points where Eq. (4) is satisfied. If $W_s(t)$, $\phi_s(t)$ are given, Eqs. (1) determine uniquely the rf voltage and frequency schedule $V(t)$, $f_o(t)$ in order that $W_s(t)$, $\phi_s(t)$ be a solution of Eqs. (1):

$$V = \frac{\dot{W}_s}{\sin \phi_s} ,$$
$$h f_o = h f_s + \frac{\dot{\phi}_s}{2\pi} . \tag{5}$$

We assume that below transition, when Eq. (4) holds,

$$V_s(t) = V_o,$$
$$\sin \phi_s(t) = \Gamma_o , \tag{6}$$

where V_o, Γ_o are constants characterizing the acceleration process up to a time t_o at which the voltage-frequency regime to cross transition begins. At a time t_1 after transition when

$$V_s(t_1) = V_1 ,$$
$$\sin \phi_s(t_1) = \Gamma_1 , \tag{7}$$

[1]N. Christofilos. Proposed Air-Core Strong-Focusing Synchrotron, UCRL-4310-08 CVL-1, Oct. 12, 1957, Appendix I.

the transition regime terminates and the particles are again to find themselves in a normal bucket at their original amplitudes.

Let us define relative coordinates

$$W^* = W - W_s(t),$$
$$\phi^* = \phi - \phi_s(t). \tag{8}$$

We expand the equations of motion (1) about the standard solution W_s, ϕ_s, keeping only linear terms in W^*, ϕ^*:

$$\frac{dW^*}{dt} = V \cos \phi_s \, \phi^*,$$
$$\frac{d\phi^*}{dt} = -2\pi h f_s' W^*. \tag{9}$$

All subscripts 's' refer to the standard particle. These equations are derivable from the Hamiltonian

$$H^* = -\pi h f_s' W^{*2} - \frac{1}{2} V \cos \phi_s \, \phi^{*2}, \tag{10}$$

where the coefficients are known functions of t. Below and above transition, V and $\cos \phi_s$ are to be held constant or nearly so, and f_s' is slowly varying if Eq. (4) is satisfied. H^* is therefore approximately independent of t and particles near W_s, ϕ_s execute phase oscillations around ellipses in W^*, ϕ^*-space given by Eq. (10) with H^* = constant. The ratio of major to minor axes is

$$\beta = \frac{W^*_{max}}{\phi^*_{max}} = \left[\frac{V \cos \phi_s}{2\pi h f_s'}\right]^{1/2}. \tag{11}$$

It is clear that $\cos \phi_s$ must have the same sign as f_s', i.e., positive below transition and negative above transition. This, together with Eq. (6) defines ϕ_s uniquely for a given Γ. During acceleration below transition,

the parameter β may change slowly, but the adiabatic theorem guarantees that a given particle remains on an ellipse of constant area

$$A = \pi W^*_{max} \phi^*_{max} . \tag{12}$$

The phase oscillation frequency is readily obtained from Eqs. (9) and is

$$f_p = \left[\frac{h f'_s}{2\pi} V \cos \phi_s \right]^{1/2} . \tag{13}$$

Since $f'_s = 0$ at transition, this expression gives an infinite period at the transition energy, which accounts for the failure of the adiabatic theorem.

A plausible possible solution for the phase oscillations as transition is crossed is

$$\phi^* = a \sin (\alpha t^2 + \theta) , \tag{14}$$

which represents a phase oscillation of constant amplitude and an angular frequency (α t) which goes through zero at transition, if we let t = 0 at transition. Formula (14) is a solution for arbitrary a, θ, of the equation

$$\frac{d^2 \phi^*}{dt^2} - \frac{1}{t} \frac{d \phi^*}{dt} + 4 \alpha^2 t^2 \phi^* = 0. \tag{15}$$

If we eliminate W^* between eqs. (9), we obtain

$$\frac{d^2 \phi^*}{dt^2} - \frac{f''_s \dot{W}_s}{f'_s} \frac{d \phi^*}{dt} + 2 \pi h f'_s V \cos \phi_s \phi^* = 0 , \tag{16}$$

where primes refer to differentiation with respect to W. Let us require that Eqs. (15) and (16) be the same. We then have

$$\frac{1}{t} = \frac{f''_s \dot{W}_s}{f'_s} = \frac{d}{dt} (\ln f'_s) , \tag{17}$$

so that

$$f'_s = - b t , \tag{18}$$

where b is an arbitrary constant, and

$$4\alpha^2 t^2 = 2\pi h f'_s V \cos\phi_s = -2\pi h b V t \cos\phi_s , \quad (19)$$

so that

$$V \cos\phi_s = -\frac{2\alpha^2}{\pi h b} t . \quad (20)$$

Now from Eqs. (9), (17), (18), we obtain

$$V \sin\phi_s = \frac{b}{-f''_s} , \quad (21)$$

(note that $f''_s < 0$ near transition), and hence

$$\cot\phi_s(t) = \frac{2\alpha^2 f''_s}{h b^2} t , \quad (22)$$

$$V(t) = \frac{b}{-f''_s}\left[1 + \left(\frac{2\alpha^2 f''_s}{\pi h b^2} t\right)^2\right]^{1/2} . \quad (23)$$

If $W_s(t)$, $\phi_s(t)$, and hence $h f_o(t)$, $V(t)$, are chosen according to Eqs. (18), (22), (23), then the solution of the phase oscillation equations during the transition regime will be formula (4), and the amplitude of synchrotron oscillations will remain constant across the transition energy. In particular, if we take f'' as constant:

$$f'' = -K , \quad (24)$$

then Eqs. (18) and (5) give

$$W_s = W_t + \frac{bt}{K} , \quad (25)$$

$$f_s = f_t - \frac{b^2}{2K} t^2 , \quad (26)$$

$$hf_o = hf_t - \frac{hb^2}{2K} t^2 + \frac{1}{2\pi} \frac{\frac{2\alpha^2 K}{\pi h b^2}}{1 + \left(\frac{2\alpha^2 K}{\pi h b^2}\right)^2 t^2} . \quad (27)$$

The third term represents a small increment in oscillator frequency above the synchronous value in order to advance the phase ϕ_s from its initial value in the first quadrant to its final value in the second quadrant.

It remains to insure that the transfer into and out of the transition regime does not disturb the amplitude of phase oscillations. To do this we must insure that the ellipses of constant amplitude for the transition regime coincide at the times t_o, t_1, with the ellipses for the normal buckets. During the transition regime the particles move on a phase ellipse given by Eqs. (14), (9) and (18):

$$\phi^* = a \sin(\alpha t^2 + \theta) ,$$
$$W^* = \frac{a\alpha}{\pi h b} \cos(\alpha t^2 + \theta) . \tag{28}$$

The area and axis ratio are

$$A = \frac{\alpha a^2}{h b} , \tag{29}$$

$$\frac{W^*_{max}}{\phi^*_{max}} = \frac{\alpha}{\pi h b} . \tag{30}$$

Let us use a subscript 'o' to refer to quantities at time t_o just before the transition regime is switched on. Then we require that the phase ϕ_s given by Eq. (22) coincide with ϕ_{so}:

$$\cot \phi_{so} = \frac{2 \alpha^2 f''_{so}}{\pi h b^2} t_o , \tag{31}$$

and that the axis ratios given by Eqs. (30) and (11) coincide:

$$\frac{V_o \cos \phi_{so}}{2 \pi h f'_{so}} = \frac{\alpha^2}{\pi^2 h^2 b^2} . \tag{32}$$

Given the value of W_{so} (hence f_{so}, f'_{so}, f''_{so}), these equations, together with (18), determine the quantities α, b, t_o :

$$t_o = \frac{f'_{so}}{f''_{so} V_o \sin \phi_{so}} , \tag{33}$$

$$b = - f''_{so} V_o \sin \phi_{so} ,$$

$$\alpha = \left[\frac{\pi}{2} h \frac{f''^2_{so}}{f'_{so}} V_o^3 \cos \phi_{so} \sin^2 \phi_{so}\right]^{1/2} .$$

It is of interest to calculate $V(t_o)$ from Eq. (23):

$$V(t_o) = V_o , \tag{34}$$

so that there is no discontinuity in voltage as the transition regime is switched on. According to Eq. (5), there is a small discontinuity in oscillator frequency of amount

$$\begin{aligned} \Delta(h f_o) &= \frac{1}{2\pi} \dot{\phi}_s (t_o) \\ &= - \frac{f''_{so}}{2\pi f'_{so}} V_o \cos \phi_{so} \sin^2 \phi_{so} . \end{aligned} \tag{35}$$

This discontinuity is smaller the further W_{so} is from transition and the smaller is $\sin \phi_{so}$. It is of the order of $1/(4\pi \Delta t)$ where Δt is the time to reach transition at $V_o \sin \phi_{so}$ volts per turn.

At time t_1 we reverse the problem and match ϕ_{s1} and the axis ratios in switching to normal buckets above transition. We then have for a given W_{s1} ,

$$t_1 = \frac{f'_{s1}}{f''_{so} V_o \sin \phi_{so}} , \tag{36}$$

$$\cot \phi_{s1} = \frac{f''_{s1}}{f''_{so}} \frac{f'_{s1}}{f'_{so}} \cot \phi_{so} , \tag{37}$$

$$V_1 = V_o \left[\left(\frac{f'_{s1}}{f'_{so}} \cos \phi_{so} \right)^2 + \left(\frac{f''_{so}}{f''_{s1}} \sin \phi_{so} \right)^2 \right]^{1/2} . \tag{38}$$

We note again that

$$V(t_1) = V_1 , \tag{39}$$

so that the voltage is continuous at t_1, and that the frequency has a discontinuity of amount

$$\begin{aligned} \Delta(h f_o) &= - \frac{1}{2\pi} \dot{\phi}_s (t_1) \\ &= \frac{f''_{s1}}{2\pi f'_{so}} V_o \cos \phi_{so} \sin^2 \phi_{s1} . \end{aligned} \tag{40}$$

Note that if the curve $f(W)$ is symmetric about W_t, and we choose for W_{so}, W_{s1} symmetrical points so that $f'_{s1} = - f'_{so}$, $f''_{s1}, = f''_{so}$, then $t_o = - t_1$, $\phi_{s1} = \pi - \phi_{so}$, $V_1 = V_o$, and we have a complete symmetry on either side of transition.