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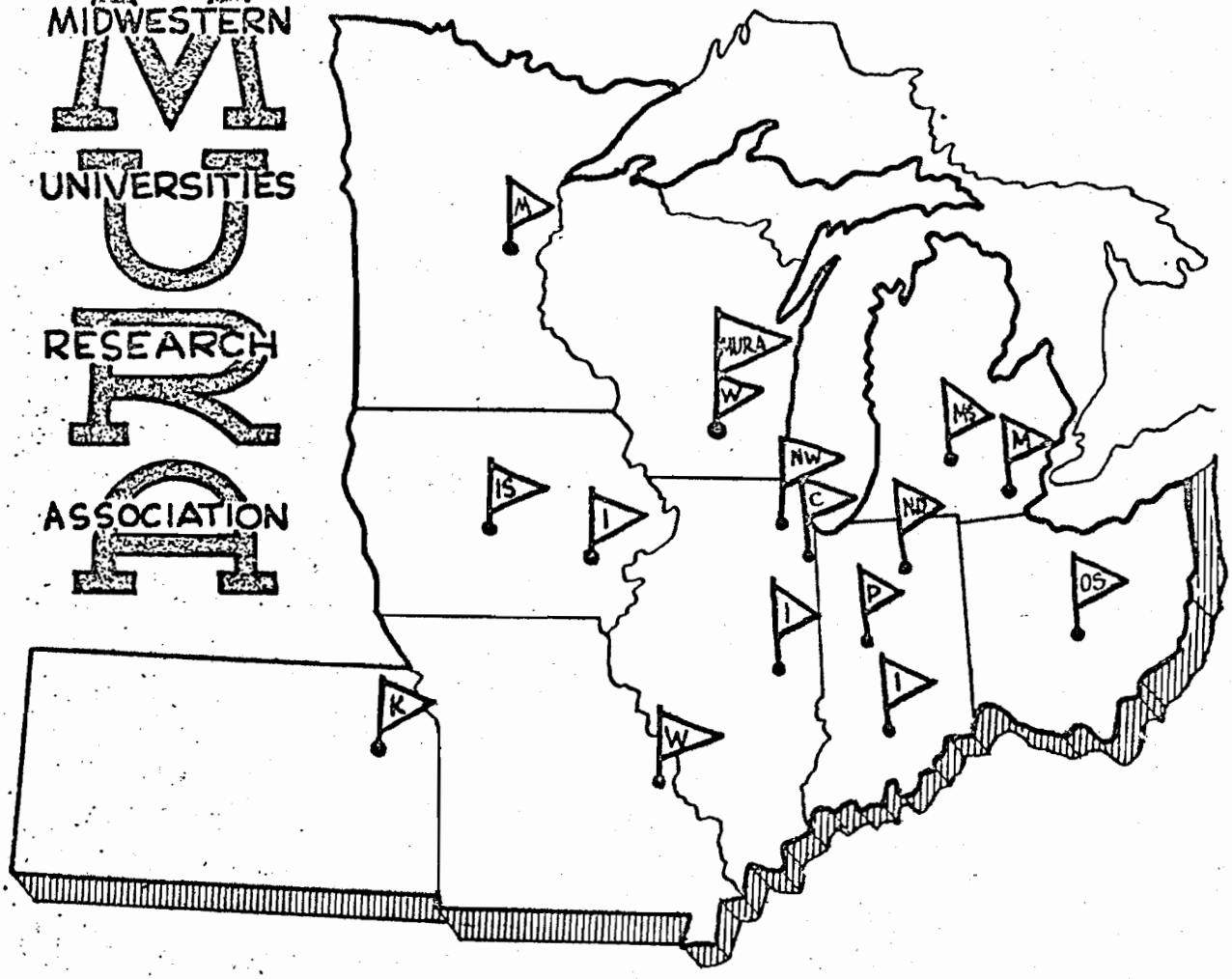

DEVIATIONS FROM ADIABATIC BEHAVIOR DURING CAPTURE OF PARTICLES INTO AN R. F. BUCKET

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DEVIATIONS FROM ADIABATIC BEHAVIOR DURING CAPTURE OF PARTICLES INTO AN R. F. BUCKET

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ABSTRACT

Capture of particles into stable synchrotron oscillations for a constant frequency accelerating voltage is studied for various voltage turn-on programs. Results are given for various linear voltage turn-on rates, and for two rates of turn-on with a non-linear voltage function V(t) designed to insure a constant degree of adiabaticity for all particles captured. A satisfactory form of the voltage turn-on function V(t) is found for which particles are captured into stable sychrotron oscillations along curves in the phase space whose areas differ from the adiabatic value by an r m s deviation of about 12%.

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INTRODUCTION

We study in this report the capture of particles into stable phase oscillations when a radio frequency voltage of constant frequency is turned on. The equations describing the energy and phase of the particle as a function of time are (MURA-106):

$$\frac{dW}{dt} = V(t) \sin \phi \quad , \tag{1}$$

$$\frac{d\phi}{dt} = 2\pi h \left[f_o - f(W) \right] \quad ,$$

where $V(t)$ is the accelerating voltage, hf_o is the radio frequency, h is the harmonic number, ϕ is the phase of the particle relative to the accelerating voltage*, $f(W)$ is the frequency of revolution of the particle, and

$$W = \int^{E} \frac{dE}{f(E)} \quad .$$

The voltage $V(t)$ is initially zero and rises to a fixed constant value V_o at which the particles which have been captured execute phase oscillations at amplitudes which we wish to determine.

Eqs. (1) are derivable from the Hamiltonian

$$H = V(t) \cos \phi + 2\pi h f_o W - 2\pi h E(W) . \tag{2}$$

The synchronous energy E_s, or equivalently W_s is defined by

$$f(W_s) = f_o \quad . \tag{3}$$

We expand the Hamiltonian (2) about W_s, keeping only second order terms in

$$W^* = W - W_s \quad ,$$

*Note that we are defining the phase $\phi = 0$ at the moment when the accelerating voltage is zero and increasing; this differs from the convention adopted in MURA-106. In the notation of MURA-106,

$$\phi = \pi - h\Theta^*$$

and omitting terms independent of W^*, ϕ :

$$H = V(t) \cos\phi - h f_s' W^{*2} , \tag{5}$$

where

$$f_s' = \left.\frac{df}{dW}\right)_s = f_s \left.\frac{df}{dE}\right)_s = \chi \frac{f_s^2}{E} , \tag{6}$$

$$\chi = \frac{E}{f} \frac{df}{dE} .$$

The Hamiltonian (5) describes the phase motion of particles near the synchronous energy with sufficient accuracy for our purpose.

When V is constant, or nearly so, the particles oscillate along curves H = constant. In particular, when $V = V_o$, we may introduce the dimensionless energy variable

$$y = \left[\frac{2\pi h |f_s'|}{V_o}\right]^{1/2} W^* . \tag{7}$$

The phase oscillations then occur along curves

$$\mp \frac{1}{2} y^2 + \cos\phi = C = \frac{H}{V_o} , \tag{8}$$

where the negative sign applies when $f_s' > 0$ (below transition) and the positive sign, when $f_s' < 0$ (above transition). Among the curves (8) is a separatrix given by

$$y = 2 \begin{matrix}\sin\\ \cos\end{matrix} \left(\frac{1}{2}\phi\right) , \quad (C = \mp 1). \tag{9}$$

The region within the separatrix ($|C| < 1$) is called a "bucket". Particles captured during the voltage turn-on are those which finally oscillate on closed curves inside the bucket. The area of the bucket in W, ϕ units is

$$A = \frac{8}{f} \left[\frac{2VE}{\pi h \chi}\right]^{1/2} . \tag{10}$$

The period of small phase oscillations near the center of the bucket, if V is constant or slowly varying, is

$$\tau_p = \frac{1}{f}\left[\frac{2\pi E}{h\chi V}\right]^{1/2} . \tag{11}$$

The period of phase oscillations becomes infinite at the separatrix.

It is shown in MURA-106 that if the voltage V is turned on sufficiently slowly, then in consequence of the adiabatic theorem a particle which when $V = 0$ lies at a distance ΔW from the synchronous energy, will after V reaches its final value oscillate along a curve of area

$$A = 4\pi \Delta W , \tag{12}$$

provided of course that this area is less than the total area of the bucket. If it is larger, the particle will move outside the separatrix on a curve such that the area between it and the $W^* = 0$ axis is $2\pi \Delta W$. It is our present purpose to study deviations from the predicted adiabatic behavior when the voltage is turned on at various finite rates.

LINEAR VOLTAGE TURN-ON

The nature of such a voltage variation was investigated by digital computation with program TTT. This program calculates $E(t)$, $\phi(t)$ for a particle with given revolution frequency $f(E)$, which receives an energy increment $V(t)\sin\phi$ each time it crosses an accelerating gap. The computations are such that when energies are expressed in terms of the variable y, areas of phase curves are expressed in terms of the final bucket area A_s, and times are in units of the final phase oscillation period τ_p, the results are independent of the parameters of the accelerator.

First observed was the manner of particle pick-up under instantaneous,

intermediate, and near-adiabatic voltage variations from zero voltage where the voltage increase was linear as a function of time. The results are illustrated in Figure 1. For instantaneous turn-on of r.f. voltage, only those particles with initial positions, W, ϕ, lying within the separatrix are captured. Furthermore, these particles oscillate on near-ellipses that intersect their initial position. In contrast, a near-adiabatic voltage turn-on captures all particles uniformly with respect to initial phase from within a range $\pm \Delta W$ centered on W_s, given by Eq. (12) where A is the area enclosed by the separatrix. Also illustrated in Figure 1 is an intermediate turn-on rate. The y and ϕ coordinates in Figure 1 are defined by Eqs. (4), (5) and (6). The dashed line in Figure 1 serves only to separate the initial positions of captured particles from uncaptured particles. The solid outline in the case of instantaneous turn-on is the separatrix.

In the limit of adiabatic voltage variation, the initial phase of a particle should be irrelevant to the curve upon which it finally oscillates. Only the initial W should determine the size of the ellipse. Figures 2, 3, and 4 illustrate the final distribution of captured particles which were initially uniformly distributed in phase and of identical initial W. Each figure represents a different initial W (or y) and illustrates the final particle distribution in terms of the ratio A/A_s of the area enclosed by their final oscillations to the area enclosed by the separatrix. Also listed is the r.m.s. deviation of A/A_s from the value corresponding to a perfectly adiabatic voltage turn-on (Eq. (12)). A broken arrow labeled "∞" indicates the adiabatic value of A/A_s. The solid curve labeled "0" indicates the theoretical distribution of particles picked up by an instantaneous turn-on.

ISO-ADIABATIC VOLTAGE TURN-ON

Analysis of the above statistics suggests a non-linear voltage variation that would ensure the same degree of adiabaticity in particle pick-up for all initial W. To this end, let us define at any time t an instantaneous bucket given by Eq. (5) with $H/V(t) = \pm 1$, of area given by formula (10), and an instantaneous period given by formula (11). Let us require that the fractional increment dA/A in bucket area in a time dt be always proportional to the fractional time increment dt/τ_p, where τ_p is the phase oscillation period at that instant:

$$\frac{dA}{A} = \alpha \frac{dt}{\tau_p} \quad . \tag{13}$$

From Eqs. (10) and (11) we have

$$\begin{aligned} A &= C_1 V^{1/2} \quad , \\ \tau_p &= C_2 V^{-1/2} \quad , \end{aligned} \tag{14}$$

so that Eq. (13) becomes

$$\frac{1}{2} V^{-1} dV = \frac{\alpha}{C_2} V^{1/2} dt. \tag{15}$$

Integrating, we obtain

$$V = \frac{V_2}{\left[\left(\frac{V_2}{V_1}\right)^{1/2} - \alpha \frac{t - t_1}{\tau_{p2}}\right]^2} \quad , \tag{16}$$

where V_1, t_1 are the initial time and initial voltage. τ_{p2} and V_2 are any chosen voltage and the corresponding phase oscillation period; we will take V_2 to be the final voltage. The degree of adiabaticity is specified by α. If $\alpha \ll 1$, the process should be adiabatic. For $\alpha \gg 1$, the process should correspond to an instantaneous turn-on. Note that formula (16) does not give $V = 0$ except at

$t = -\infty$, so that it is necessary to turn on the voltage according to some other law say, for example, to switch it on suddenly to the value V_1 at $t = t_1$. V_1 is to be taken very small, so that $A_1/4\pi = \Delta W_1$ is the distance from synchronism beyond which we wish to treat phase points adiabatically, for example, if the initial beam spread is less than ΔW_1 and we wish the beam to be captured finally into an area less than A_1. We may solve Eq. (16) for the time required to reach V_2:

$$t_2 - t_1 = \frac{\tau_{p2}}{\alpha}\left[\left(\frac{V_2}{V_1}\right)^{1/2} - 1\right]$$
$$= \frac{1}{\alpha f}\left(\frac{2\pi}{hK}\right)^{1/2}\left[\left(\frac{E}{V_1}\right)^{1/2} - \left(\frac{E}{V_2}\right)^{-1/2}\right]. \tag{17}$$

For $V_2 >> V_1$, this time depends only on the initial voltage V_1 and on the choice of α.

Digital Computer experiments with voltage rise according to formula (16) were carried out with results indicated in Figures 5 to 11. Two values of α were studied, $\alpha = 10$ and $\alpha = 1$. For $\alpha = 10$, the pickup is clearly not adiabatic. For $\alpha = 1$, however, the results are acceptably close to adiabatic behavior. It therefore appears safe to design pickup programs with $\alpha = 1$ with assurance that the particles will be picked up with phase oscillation amplitudes reasonably close to the values predicted from the adiabatic theorem.

The r.m.s. deviations $\sigma_A = \left[\overline{(\Delta A)^2}\right]^{1/2}$ of the final phase oscillation area A from the adiabatic value $\bar{A}$ are tabulated in Table 1 for various initial values of y. It is of interest to compare σ_A both with the final bucket area A_s and with the adiabatic value $\bar{A}$. It can be noted that for $\alpha = 10$, the ratio σ_A/A_s is nearly independent of y, whereas for $\alpha = 1$, it is the ratio $\sigma_A/\bar{A}$ which is nearly independent of y. We may perhaps explain this by noting

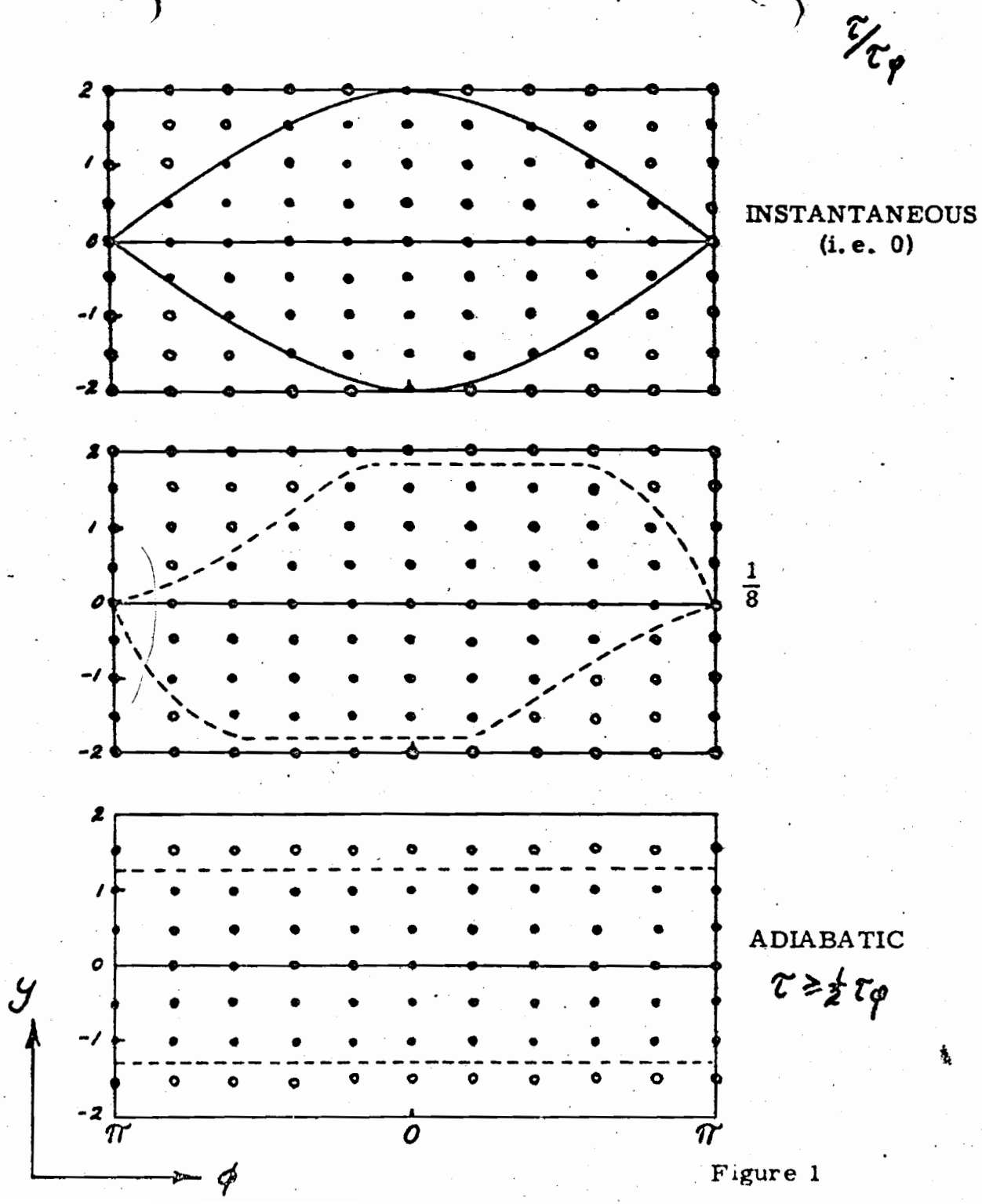
that there are three stages in the pickup of any particle. During the first stage, the particle is outside the bucket and moves on a curve such that in the adiabatic limit the area between this curve and the line $W = W_s$ remains constant. During the second stage the particle crosses the separatrix. This stage can never be adiabatic because the period of phase oscillations is infinite on the separatrix. During the third stage, the particle oscillates inside the bucket on a curve whose area is constant in the adiabatic limit. Now, if the parameters change very slowly so that the motion during stages 1 and 3 is essentially adiabatic, then the deviation from adiabaticity occurs only during the second stage. During the second stage the bucket area is near the adiabatic area $\bar{A}$ for the given particle, and since the voltage-time program (16) is arranged so that the relative voltage and time scales are the same for particles of any value of $\bar{A}$, we should expect the relative deviation $\sigma_A/\bar{A}$ to be the same for all values of y, as it appears to be for $\alpha = 1$. The smaller the value of α, the shorter stage two becomes in the sense that the particles approach closer to the separatrix during stages 1 and 3 before the motion becomes non-adiabatic. Hence, we would expect $\sigma_A/\bar{A}$ to become smaller as α is reduced. For $\alpha = 1$, $\sigma_A/\bar{A}$ is about 12%, which is quite acceptable for most purposes. For $\alpha >> 1$, we expect that the motion will be non-adiabatic even during stages 1 and 3. It is therefore plausible that for $\alpha >> 1$ most of the deviation from the adiabatic value of A will occur at the end of stage 3 when the area is largest. Put another way, for $\alpha >> 1$, the bucket may be regarded as switched on suddenly to its final voltage. Then we might expect σ_A/A_s to be roughly independent of initial y, as it appears to be for $\alpha = 10$.

TABLE 1

Deviation from Adiabatic Behavior for Iso-Adiabatic Voltage Turn-On

($\sigma_A = \left[\overline{(A-\bar{A})^2}\right]^{1/2}$, where $\bar{A}$ is the adiabatic value.)

	$\alpha = 10$		$\alpha = 1$		
y	σ_A/A_s	$\sigma_A/\bar{A}$	σ_A/A_s	$\sigma_A/\bar{A}$	$\bar{A}/A_s$
0	0.233	- - -	0.006	- - -	0
0.05	0.310	7.95	0.005	0.128	0.039
0.10	0.339	4.34	0.007	0.090	0.078
0.15	0.366	3.13	0.020	0.171	0.117
0.20	0.382	2.45	0.018	0.115	0.156
0.50	0.264	0.68	0.042	0.108	0.39
1.0	0.262	0.336	0.095	0.122	0.78
Average	0.308 ± 0.02	- - -	- - -	0.122 ± 0.01	- - -



PARTICLE CAPTURE
vs.
TURN-ON RATE

Where τ_φ is the phase oscillation period

- ● particles picked up
- ○ particles not picked up

Figure 1

DISTRIBUTION IN AREA OF CAPTURED PARTICLES

$Y_0 = D$

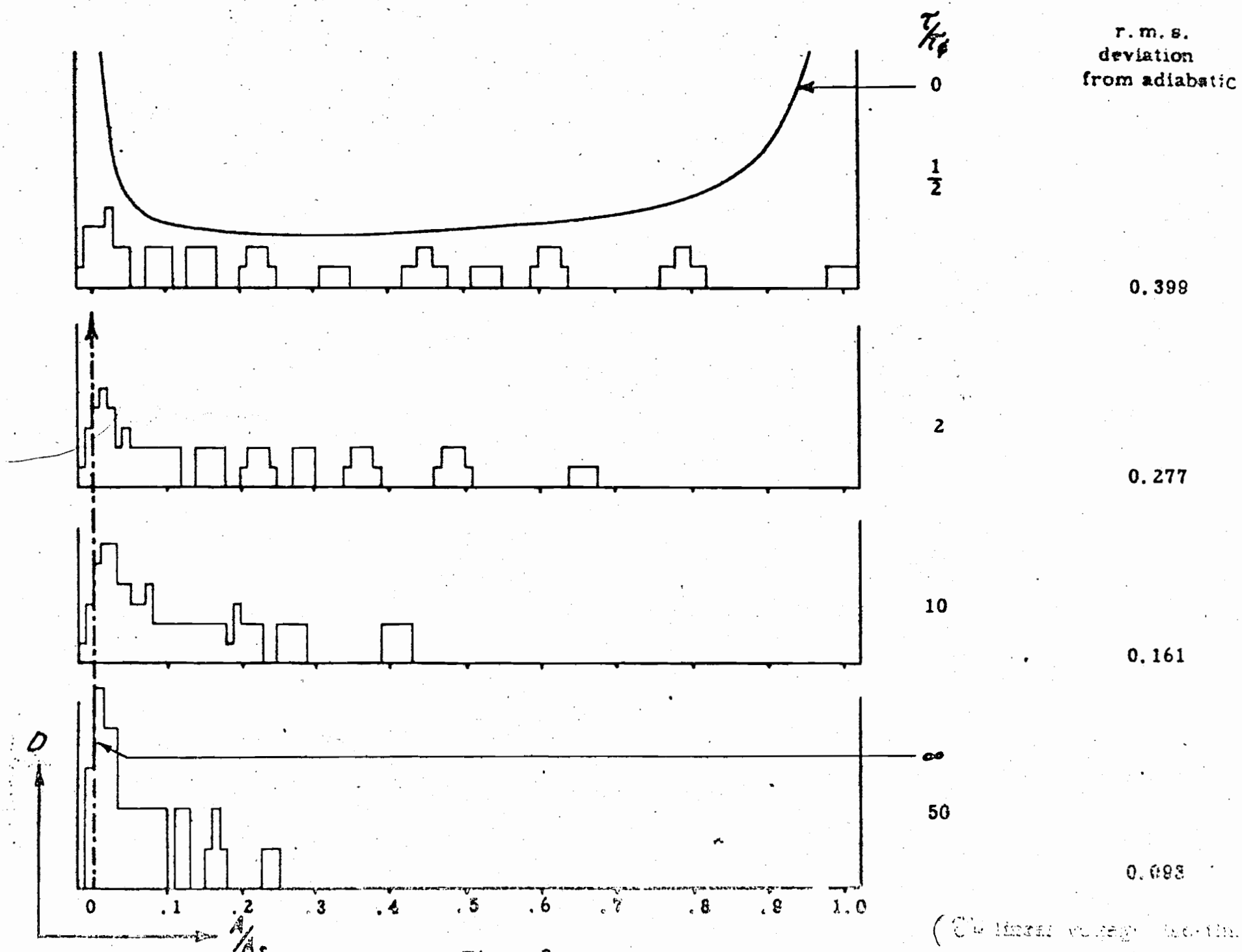


Figure 3

DISTRIBUTION IN AREA OF CAPTURED PARTICLES $\gamma_s = 0.5$

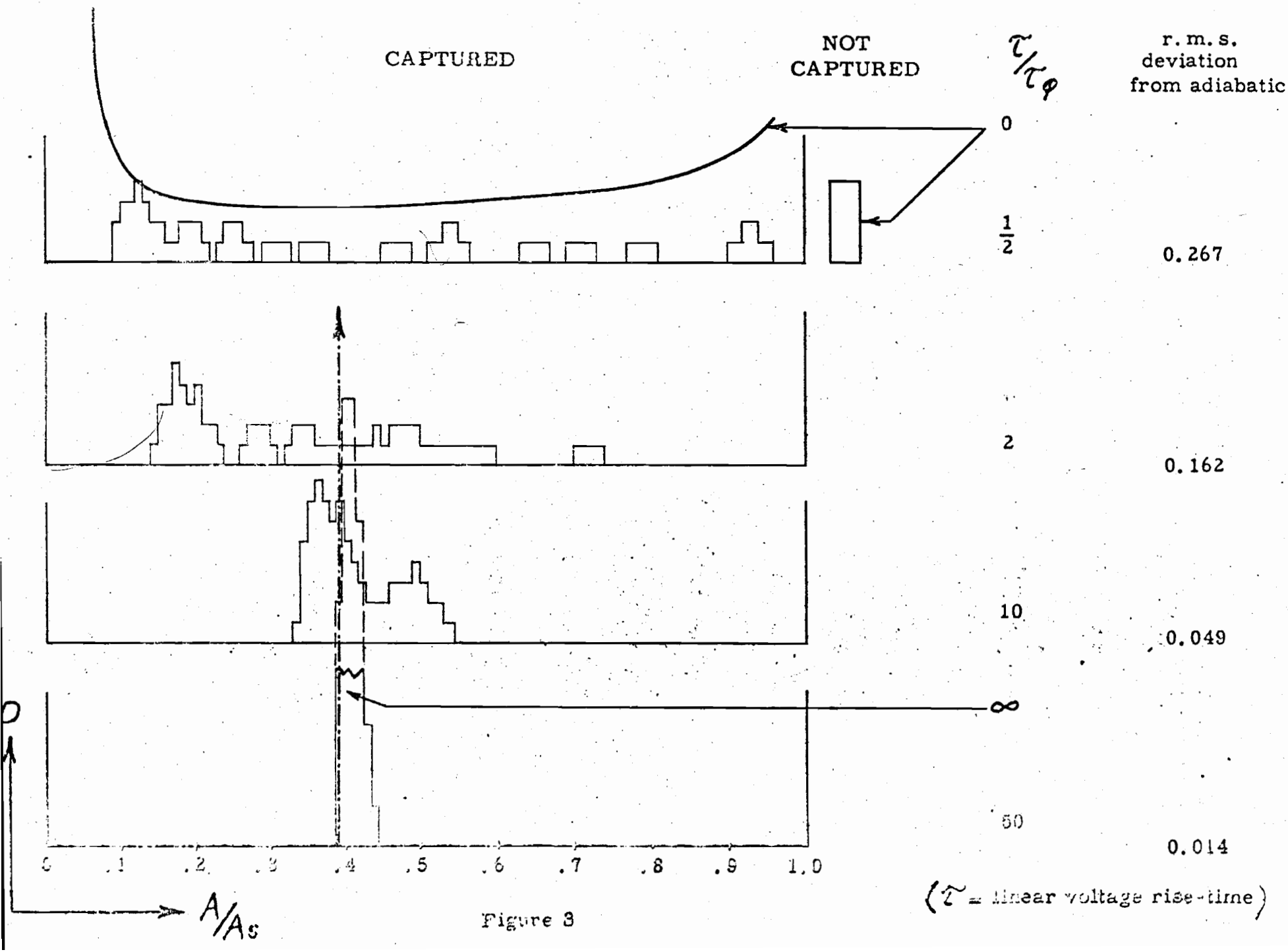
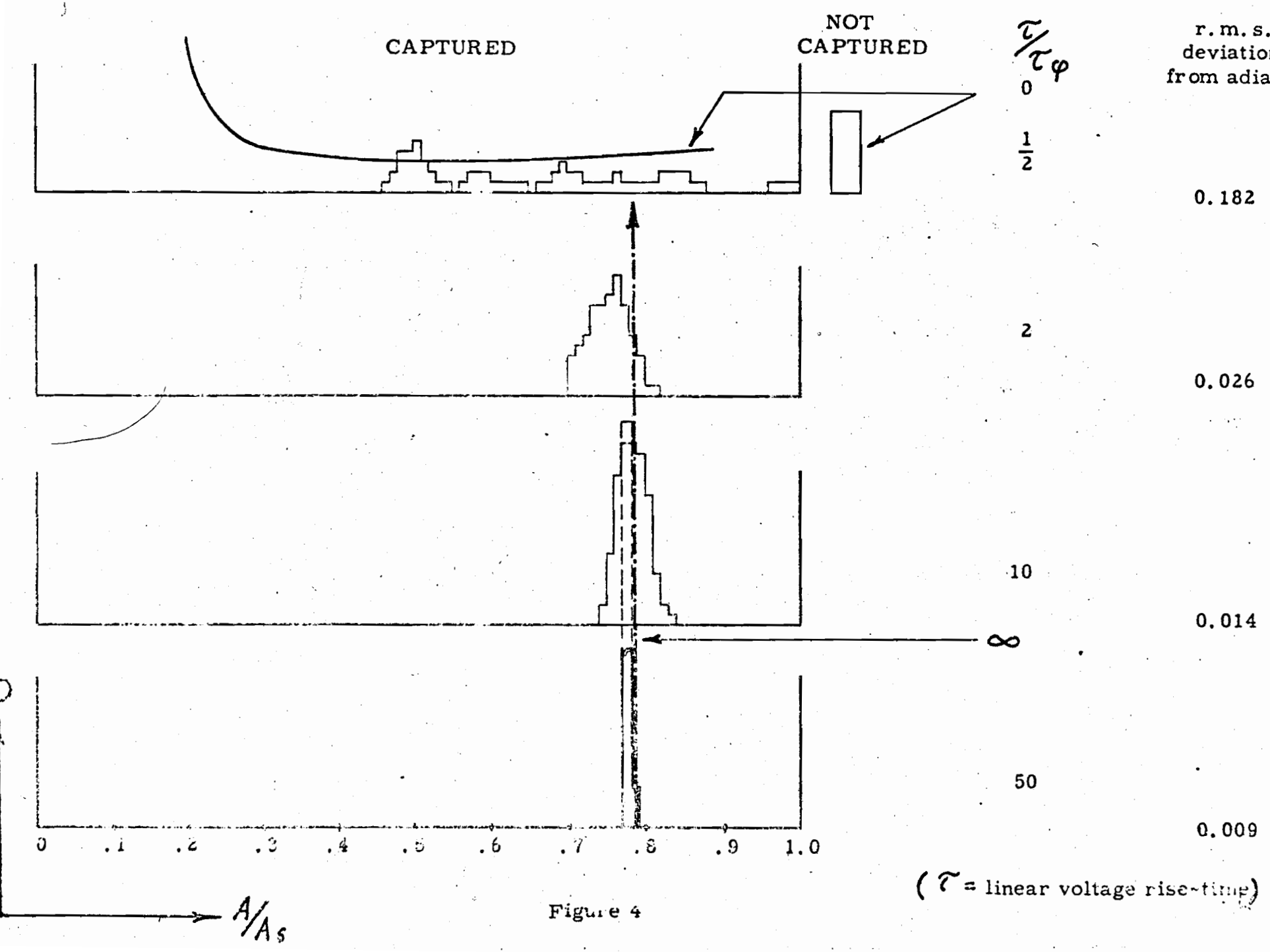


Figure 3

(τ = linear voltage rise-time)

DISTRIBUTION IN AREA OF CAPTURED PARTICLES $Y_0 = 1.0$



(τ = linear voltage rise-time)

Figure 4

DISTRIBUTION IN AREA OF CAPTURED PARTICLES

$Y = 0.0$

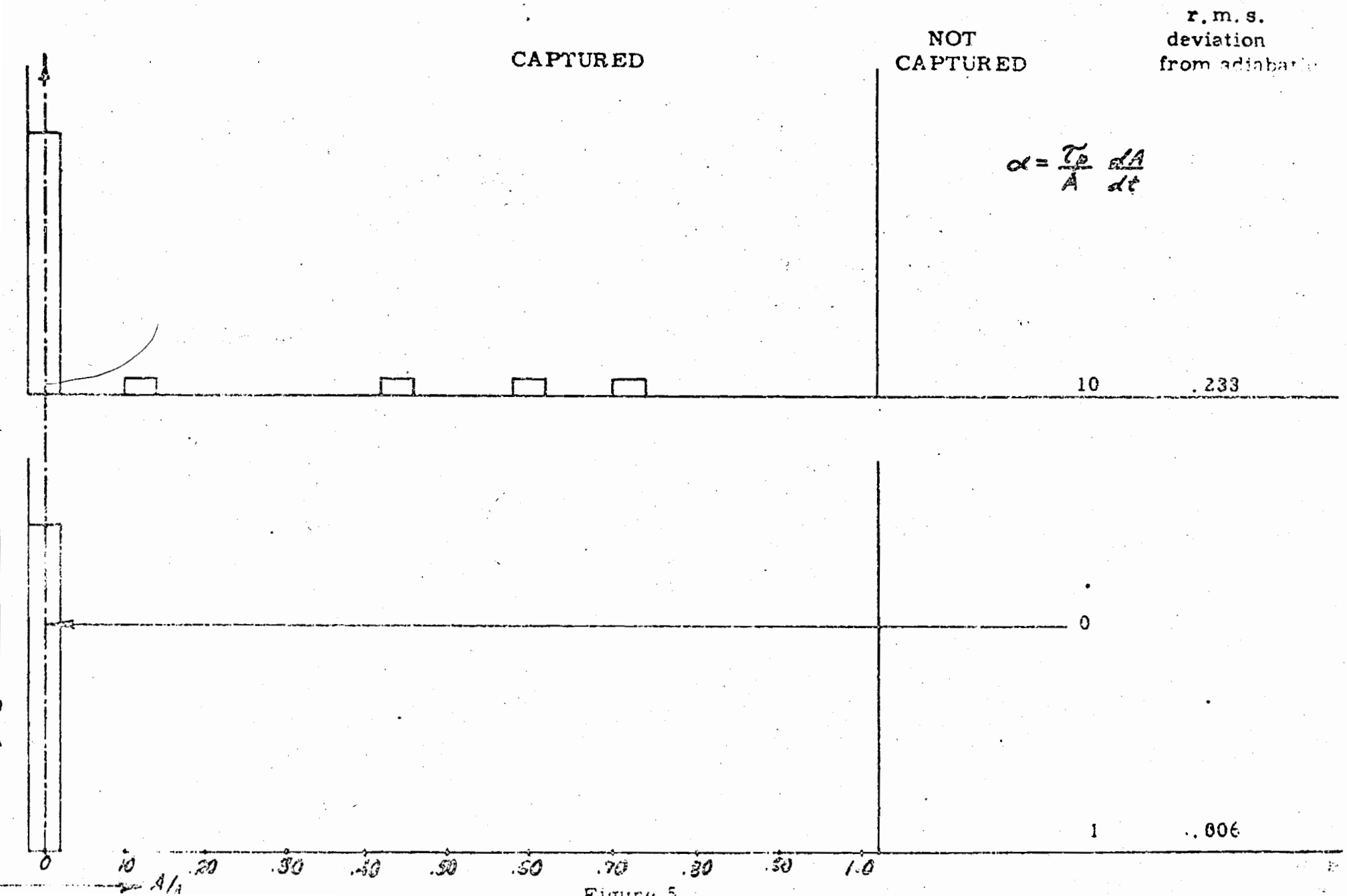


FIGURE 5

$Y = 0.05$

DISTRIBUTION IN AREA OF CAPTURED PARTICLES

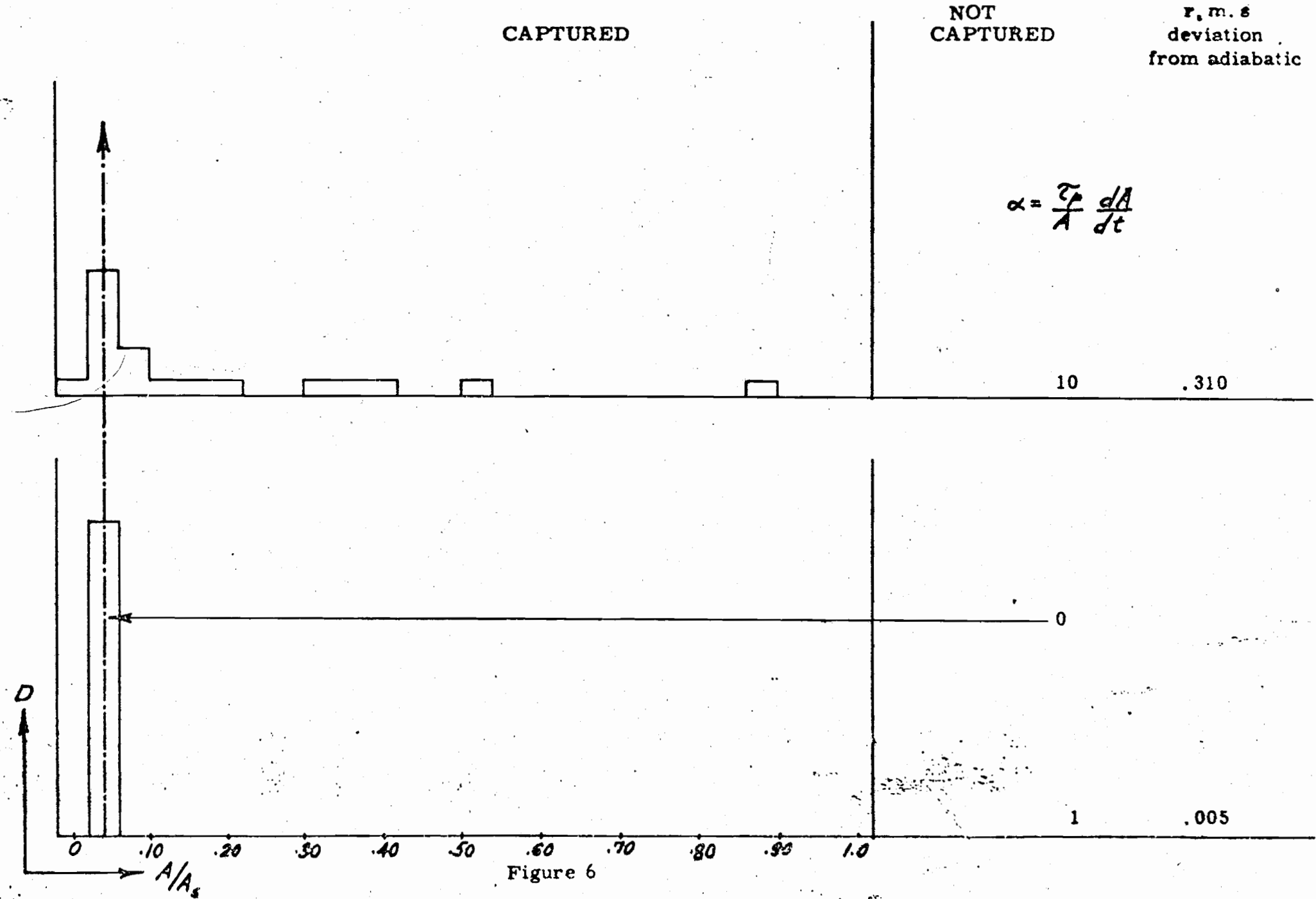


Figure 6

$Y = 0.10$

DISTRIBUTION IN AREA OF CAPTURED PARTICLES

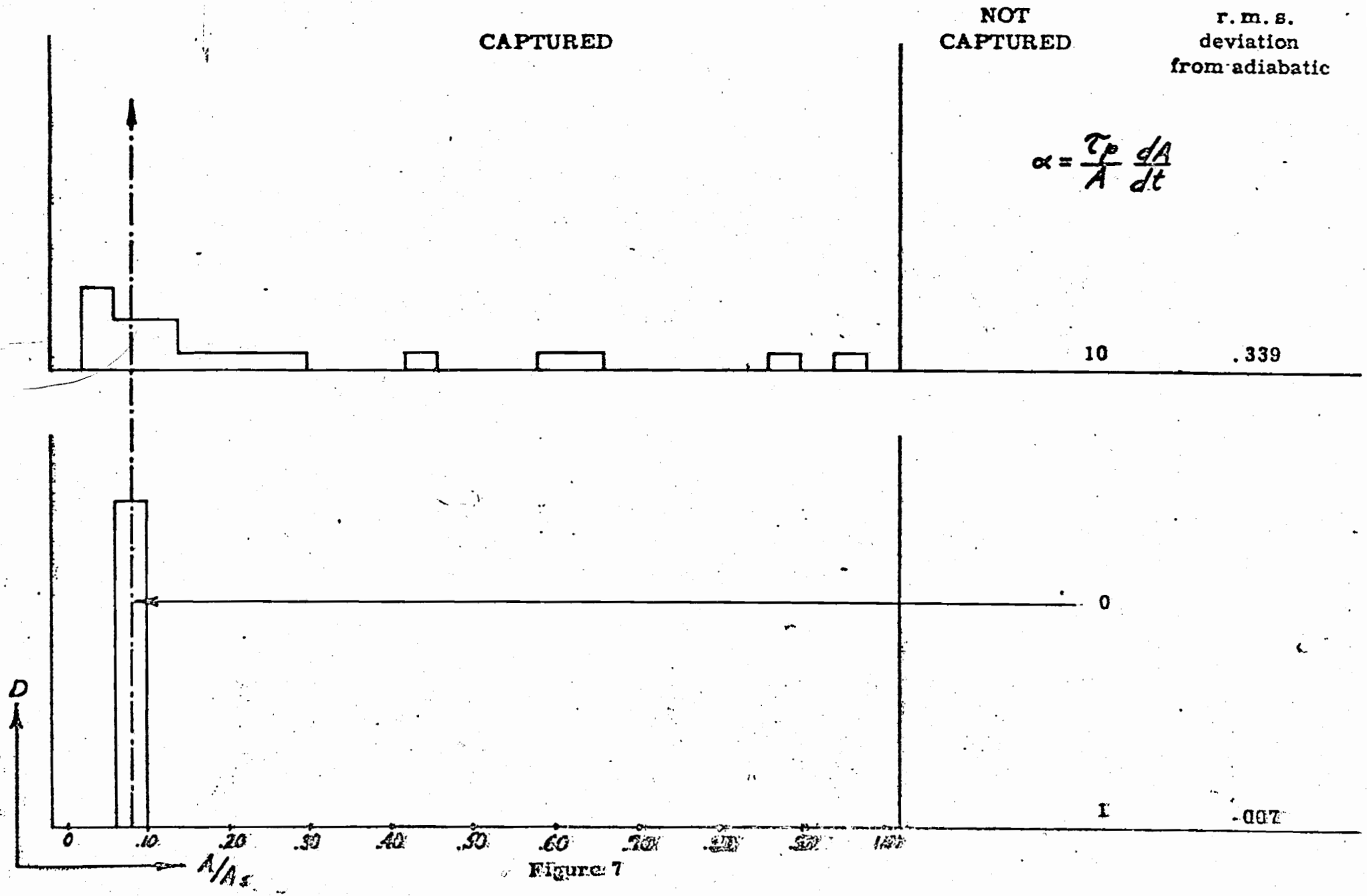


Figure 7

$Y = 0.15$

DISTRIBUTION IN AREA OF CAPTURED PARTICLES

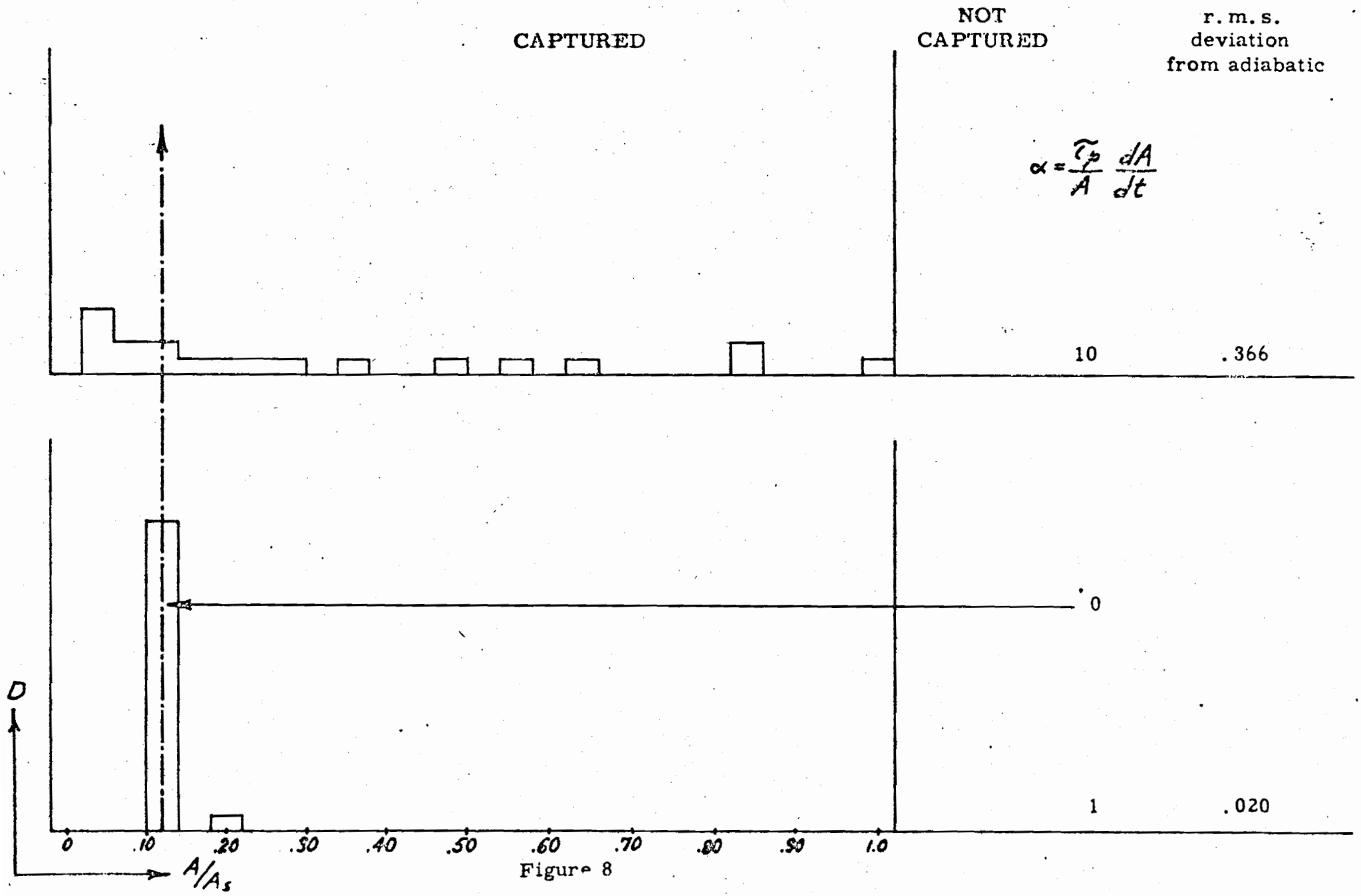


Figure 8

DISTRIBUTION IN AREA OF CAPTURED PARTICLES

$Y = 0.20$

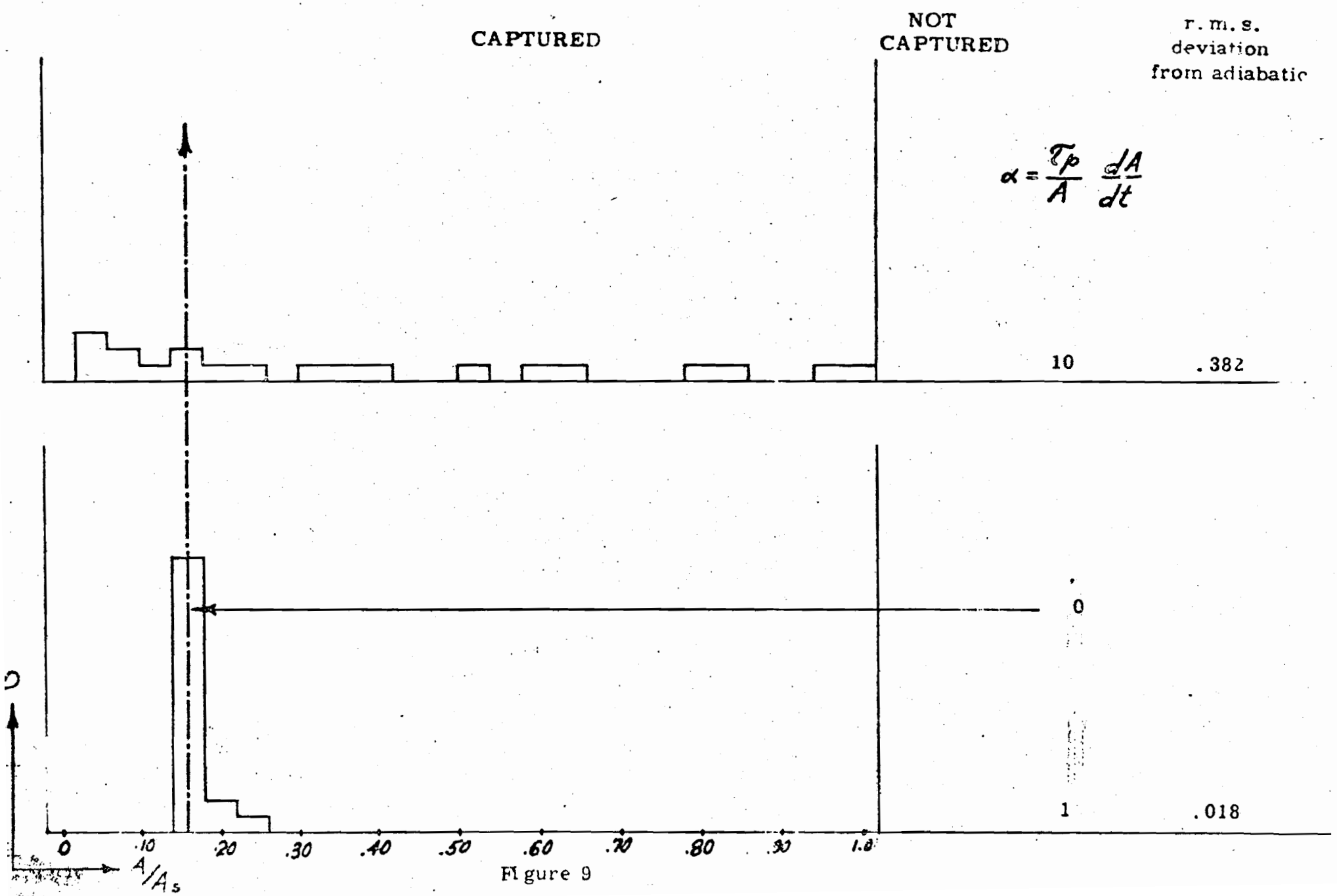


Figure 9