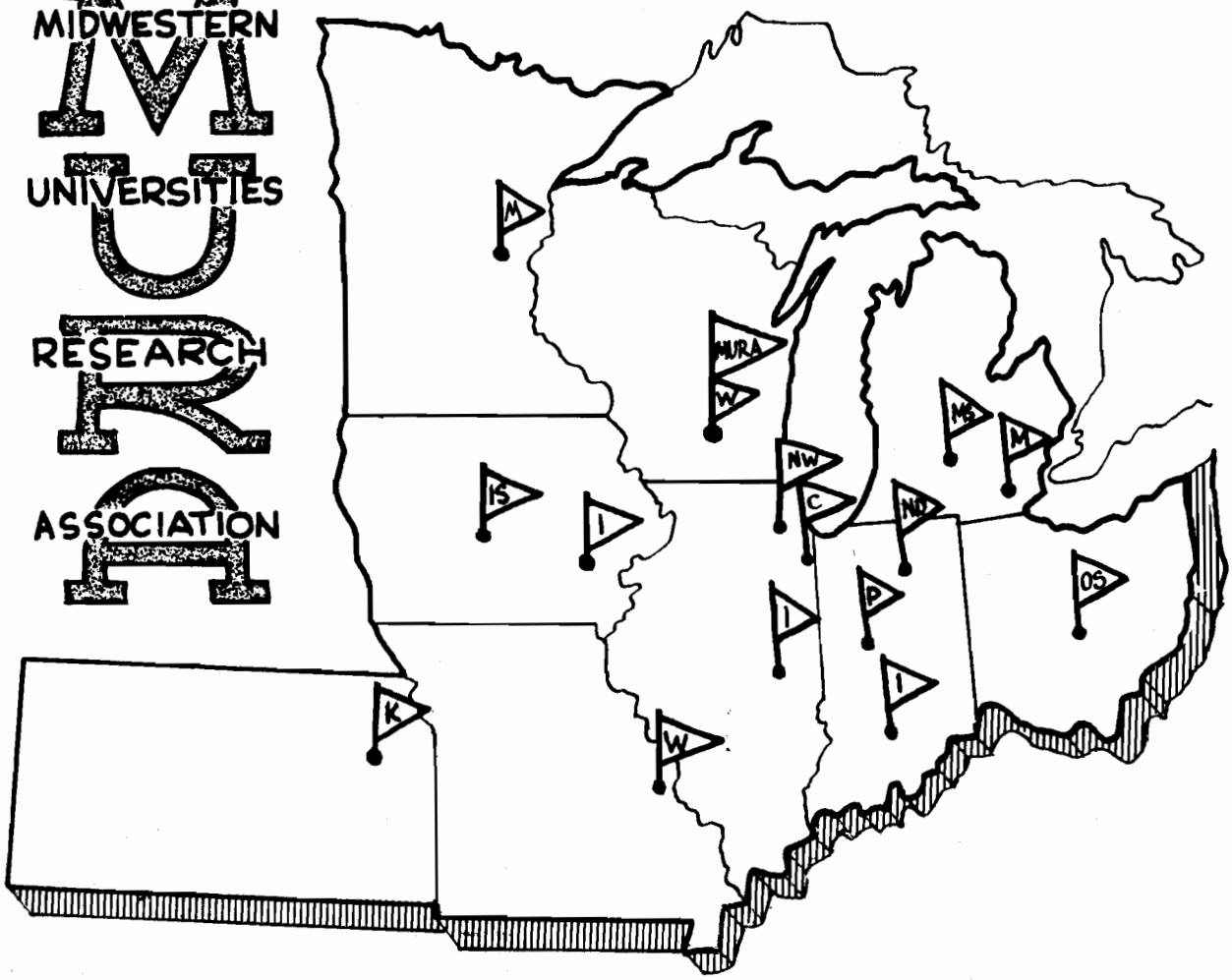
490

Fermilab Library
0 1160 0038784 9

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



CONCERNING THE $\nu/N \longrightarrow 1/3$ RESONANCE, IV A

A TRIAL FUNCTION FOR THE LIMITING-AMPLITUDE SOLUTION OF

$$\frac{d^2u}{d\phi^2} + (a + b \cos 2\phi)\, u + \frac{B_1}{2} (\sin 2\phi)\, u^2 = 0$$

L. Jackson Laslett

REPORT NUMBER 490

MURA-490
Distribution List UC 28

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*
2203 University Avenue, Madison, Wisconsin

CONCERNING THE $\nu/N \longrightarrow 1/3$ RESONANCE, IV A
A TRIAL FUNCTION FOR THE LIMITING-AMPLITUDE SOLUTION OF

$$\frac{d^2u}{d\phi^2} + (a + b \cos 2\phi)\, u + \frac{B_1}{2} (\sin 2\phi)\, u^2 = 0$$

L. Jackson Laslett**

June 17, 1959

ABSTRACT

For comparison with the results given in an earlier report, use of a trial function for the limiting-amplitude solution of the equation given in the title is illustrated for an example in which

$$a = 0.1262875 \qquad b = 1.15$$
$$\nu/N = 0.2997 \qquad B_1 = 1.$$

The trial function employed sine functions of argument $2\phi/3$, $4\phi/3$, 2ϕ, $8\phi/3$, and $10\phi/3$. The coefficient found for the dominant term appeared to be within one-tenth of a per cent of the computer result and the spatial fixed-point coordinate (for the unstable fixed points situated to the right and left of the symmetry axis at $\phi = 0$, mod. π) within 0.2 per cent; the corresponding fixed-point momentum is found to be somewhat less accurate, due to the enhanced contributions of error from the higher-frequency terms, the error being roughly 3% in this example.

*AEC Research and Development Report. Research supported by the Atomic Energy Commission, Contract No. AEC AT (11-1)-384.

**Department of Physics and Institute for Atomic Research, Iowa State College, Ames, Iowa.

A. INTRODUCTION

In a previous report[1*] solutions to the differential equation

$$\frac{d^2u}{d\phi^2} + (a + b \cos 2\phi)\, u + \frac{B_1}{2} (\sin 2\phi)\, u^2 = 0 \tag{1}$$

were studied by the Moser method, after use of a suitable transformation to eliminate the alternating-gradient (A-G) character of the linear term. The limiting-amplitude solution was examined in this way for a particular example and the results compared with corresponding computer information.

Recently an interest has been expressed[2] in the use of a variational or harmonic-balance method to estimate the limiting-amplitude solution of eqn. (1), in a way which would parallel closely the application of this method in other papers[3] of this series and in earlier reports.[4] In the present report we apply this method to eqn. (1) and illustrate the results for the example which was previously employed in reference 1.

B. THE VARIATIONAL METHOD

As in earlier work,[3, 4] the differential equation is replaced by a variational statement for purposes of determining the (periodic) unstable equilibrium orbit. In the case of eqn. (1), this statement is

$$\delta \left[\langle (u')^2 \rangle - a \langle u^2 \rangle - b \langle u^2 \cos 2\phi \rangle - (B_1/3) \langle u^3 \sin 2\phi \rangle \right] = 0, \tag{2}$$

the prime denoting differentiation with respect to ϕ and the symbol $\langle\ \rangle$ denoting that the function embraced is to be averaged over one or more periods. The coefficient B_1 of eqn. (1) may, of course, be made unity by suitable scaling of the dependent variable u.

---

*References are given in Section D.

In selecting an adequate, but reasonably simple, trial function we note first that the dominant term in a development of the periodic solution controlled by the $\nu/N \longrightarrow 1/3$ resonance would be expected to be of the form $A_1 \sin 2\phi/3$, the sine function being selected because of the predictable symmetry of the phase plots (at $\phi = 0$, mod. π) about the vertical axis. Because of the nature of the coefficient of the linear term in eqn. (1), this dominant term should be supplemented[5] by terms of argument $4\phi/3$ and $8\phi/3$, while the non-linear term suggests[3] supplementary terms of argument 2ϕ and $10\phi/3$. We select, therefore, the five-term trial function

$$u = A_1 \sin 2\phi/3 + A_2 \sin 4\phi/3 + A_3 \sin 2\phi + A_4 \sin 8\phi/3 + A_5 \sin 10\phi/3. \tag{3}$$

Substitution of the trial function (3) into the variational statement (2) leads to

$$\begin{aligned}
\delta \Big\{ & (1/2)\left[(2/3)^2 - a\right] A_1^2 + (1/2)\left[(4/3)^2 - a\right] A_2^2 + (1/2)\left[(2)^2 - a\right] A_3^2 \\
& + (1/2)\left[(8/3)^2 - a\right] A_4^2 + (1/2)\left[(10/3)^2 - a\right] A_5^2 \\
& + (b/2) A_1 A_2 - (b/2) A_1 A_4 - (b/2) A_2 A_5 \\
& + (1/24) A_1^3 - (1/4) A_1^2 A_3 + (1/8) A_1^2 A_5 \\
& - (1/8) A_1 A_2^2 - (1/4) A_1 A_2 A_4 - (1/4) A_1 A_3 A_5 \\
& - (1/4) A_2^2 A_3 - (1/4) A_2 A_3 A_4 - (1/4) A_2 A_4 A_5 \\
& - (1/8) A_3^3 - (1/4) A_3 A_4^2 - (1/4) A_3 A_5^2 - (1/8) A_4^2 A_5 \Big\} = 0,
\end{aligned} \tag{4}$$

where, for simplicity, we have set $B_1 = 1$.

By making the appropriate differentiations of eqn. (4), one then obtains the simultaneous non-linear algebraic equations

$$\left[(2/3)^2 - a\right] A_1 + \frac{b}{2} A_2 - \frac{b}{2} A_4 + \frac{1}{8} A_1^2 - \frac{1}{2} A_1 A_3 + \frac{1}{4} A_1 A_5 - \frac{1}{8} A_2^2 - \frac{1}{4} A_2 A_4 - \frac{1}{4} A_3 A_5 = 0$$

$$\left[(4/3)^2 - a\right] A_2 + \frac{b}{2} A_1 - \frac{b}{2} A_5 - \frac{1}{4} A_1 A_2 - \frac{1}{4} A_1 A_4 - \frac{1}{2} A_2 A_3 - \frac{1}{4} A_3 A_4 - \frac{1}{4} A_4 A_5 = 0$$

$$\left[(2)^2 - a\right] A_3 - \frac{1}{4} A_1^2 - \frac{1}{4} A_1 A_5 - \frac{1}{4} A_2^2 - \frac{1}{4} A_2 A_4 - \frac{3}{8} A_3^2 - \frac{1}{4} A_4^2 - \frac{1}{4} A_5^2 = 0$$

$$\left[(8/3)^2 - a\right] A_4 - \frac{b}{2} A_1 - \frac{1}{4} A_1 A_2 - \frac{1}{4} A_2 A_3 - \frac{1}{4} A_2 A_5 - \frac{1}{2} A_3 A_4 - \frac{1}{4} A_4 A_5 = 0$$

$$\left[(10/3)^2 - a\right] A_5 - \frac{b}{2} A_2 + \frac{1}{8} A_1^2 - \frac{1}{4} A_1 A_3 - \frac{1}{4} A_2 A_4 - \frac{1}{2} A_3 A_5 - \frac{1}{8} A_4^2 = 0, \qquad (5)$$

which serve to determine the coefficients $A_1, \cdots A_5$.

C. NUMERICAL EXAMPLE

In the specific case taken as an example in reference 1, for which

$$a = 0.1262875 \qquad b = 1.15$$

$$\nu/N = 0.2997 \qquad B_1 = 1 ,$$

an approximate numerical solution of eqns. (5) leads to coefficients such that the trial solution assumes the form:

$$u = -0.5520 \sin 2\phi/3 + 0.1840 \sin 4\phi/3 + 0.0213 \sin 2\phi - 0.0497 \sin 8\phi/3 + 0.0057 \sin 10\phi/3 . \qquad (6)$$

This result may be compared with the Fourier analysis of the limiting-amplitude solution given by direct computational integration[1, 6] of eqn. (1), namely

$$u = -0.55231 \sin 2\phi/3 + 0.18429 \sin 4\phi/3 + 0.02167 \sin 2\phi - 0.04919 \sin 8\phi/3 + 0.00575 \sin 10\phi/3 + 0.00283 \sin 4\phi - 0.00140 \sin 14\phi/3 + \cdots . \qquad (7)$$

From comparison of eqns. (6) and (7) it is noted that the coefficients given in (6) agree through three decimal places with the computational result

and the coefficient of the dominant sin $2\phi/3$ term is within one-tenth of one per cent of the value found computationally. From eqn. (6) the spatial fixed-point coordinate (for the unstable fixed points situated to the right and left of the symmetry axis at $\phi = 0$, mod. π) is obtainable within 0.2 per cent [cf. Table II of reference 1]. The corresponding fixed-point momenta are found to be somewhat less accurate, due to the enhanced contributions of error from the higher-frequency terms--including those omitted from eqn. (6)--the error being of the order of 3% in this example.

In summary, it appears that the use of a trial function of the form given in eqn. (3) permits one to obtain a reasonably accurate representation of the periodic solution to eqn. (1) with rather better accuracy and somewhat less complexity than by employing the methods outlined in reference 1. These latter more general methods, however, do of course permit additional features of solutions to equation (1) to be estimated analytically.

D. REFERENCES

1. L. Jackson Laslett, MURA-463 (June 3, 1959).

2. F. T. Cole, private communication (June 16, 1959).

3. L. Jackson Laslett, MURA-452 (April 13, 1959).

   L. Jackson Laslett, MURA-459 (May 20, 1959).

4. Cf. L. J. Laslett and A. M. Sessler, MURA-248 (1957).

5. Cf. L. Jackson Laslett and A. M. Sessler, MURA-252 (1957).

6. Integration of the differential equation (1) was accomplished with the MURA IBM 704 computer, by aid of the DUCK-ANSWER computational program [J. N. Snyder (IBM Program 75), MURA-237 (1957)]. Fourier analysis of periodic solutions is aided by use of the DUCKNALL program [John McNall (IBM Program 219), MURA-438 (1958)], which is based on the earlier FORANAL program [J. N. Snyder (IBM Program 52), MURA-228 (1957)].