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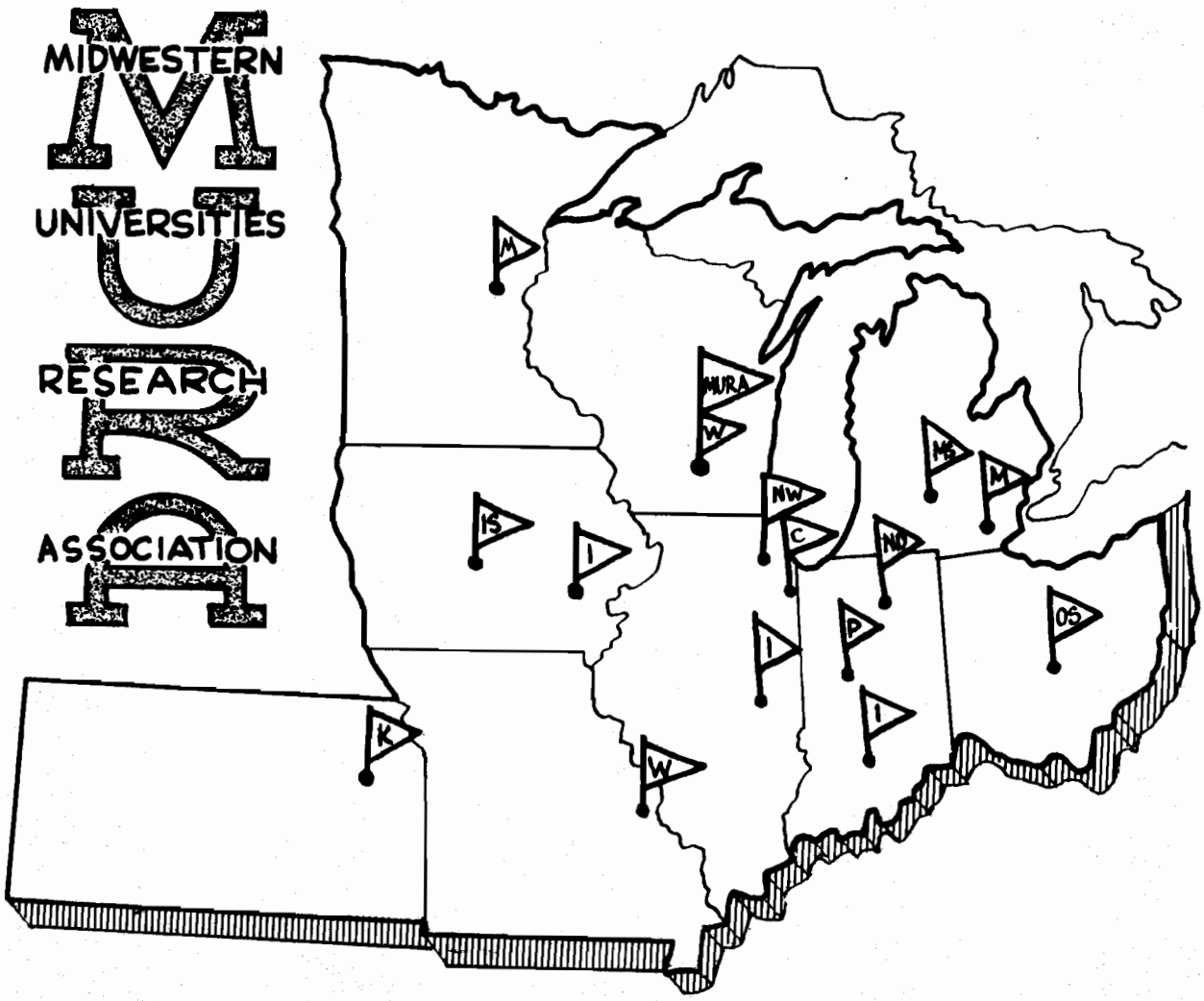

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LONGITUDINAL SPACE CHARGE EFFECTS - INFLUENCE OF ENERGY LOSS

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ABSTRACT

The longitudinal stability of a coasting beam has been discussed in previous reports. This stability analysis is here extended to include the effect of an energy loss linearly dependent upon particle energy. Consideration is restricted to the case of a beam of uniform phase space density within sharp boundaries. By suitable transformations, the equations are expressed in terms of canonical coordinates and momenta. Approximate solutions are obtained after linearization. It is found that instabilities of charge density develop above transition energy when energy spread of the beam decreases with time; the effect of the energy loss mechanism upon instability growth rate is computed.

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I. INTRODUCTION

The instability of a coasting beam of particles has been studied under the assumption that there is no energy loss mechanism,[1, 2] and under the assumption that the rate of energy loss is independent of velocity.[3] In this latter case, the energy-loss mechanism has no effect on the criterion for stability. In the present report the analysis is extended to include the effect of an energy-dependent energy-loss mechanism. Attention is restricted to energy loss which varies linearly with particle momentum. It is hoped that this assumed energy dependence is sufficient to describe 1) the energy dependence of radiation loss over small energy intervals, 2) the effect of radiation loss when the average energy is maintained by a mechanism which treats all particles equally (such as phase displacement), 3) the energy dependence of ionization loss.

It is not surprising that momentum-independent energy loss was found to have no influence on instability, since it in effect merely changes the value of the constant $\dot{w}_s$ which is not involved in the criterion for stability. But in contrast, a momentum-dependent energy loss results in a time-varying phase density in W-φ coordinates and previous analysis is not applicable. If, for example, a beam represented initially by a uniform density in W-φ phase space over a region of width Δ is subject to energy loss increasing with momentum, the width Δ will continuously decrease. Such a beam might be expected to become unstable above transition even if the initial Δ were great enough to insure stability

In Section II it is shown that for a linear momentum dependence of energy loss it is possible to transform to canonical variables in which phase density is conserved. Equations of motion in these new variables are used in Section III to obtain equations describing the behavior of a distribution of phase density uniform within sharp boundaries. The last two sections are concerned with stability in two special cases. In Section IV energy loss is taken as increasing with particle momentum, while in Section V we discuss the situation with energy loss a decreasing function of momentum.

We find that the expectation of instability when Δ (in $W-\phi$ coordinates) decreases is supported by the analysis. The situation with increasing Δ appears more difficult to deal with quantitatively and we can conclude only that a perturbation in momentum width grows; this is a kind of instability, but it does not imply instability in the usual sense of growth of charge density perturbations.

II. CANONICAL FORMULATION

In the absence of an energy loss mechanism, longitudinal motion of a particle in an accelerator is described by the canonical equations

$$\begin{aligned} \dot{W} &= 2\pi R e \mathcal{E} \\ \dot{\phi} &= 2\pi f \left(\frac{df}{dE}\right) W \end{aligned} \tag{1}$$

the notation being that of reference 2. To include energy loss effects we may put into $\mathcal{E}$ a reversely directed electric field so that

$$\mathcal{E} = -k_0 e - k_1 e (W - W_0) + \mathcal{E}_{sc} \quad . \tag{2}$$

The first two terms characterize the energy loss, while $\mathcal{E}_{sc}$ is the longitudinal electric field resulting from the other particles. There is no externally supplied rf field. The energy loss per particle per unit time* is then as a function of W

$$\frac{dE}{dt} = -2\pi R f e^2 [k_0 + k_1 (W - W_0)] \tag{3}$$

which, since

$$dW = \frac{dE}{f}$$

may be written, for small energy changes,

$$\frac{dE}{dt} = -2\pi R e^2 [f k_0 + k_1 (E - E_0)] \quad . \tag{4}$$

The last form allows one to evaluate k_0 and k_1 for any contemplated mechanism of energy loss such as radiation, scattering, or intra-beam particle scattering.[4]

It may now be observed that Eqs. (1) with $\mathcal{E}$ given by Eq. (2) are not a canonical set of equations. However, it is possible to make a non-canonical transformation from W- φ variables to new variables W_1- φ_1 which are canonical. Let

$$\begin{aligned} W_1 &= W e^{\lambda t} \\ \varphi_1 &= \varphi \end{aligned} \quad . \tag{5}$$

Then, by choosing $\lambda = 2\pi R e^2 k_1$, we obtain instead of Eqs. (1):

$$\begin{aligned} \dot{W}_1 &= 2\pi R e (-k_0 e + k_1 e W_0 + \mathcal{E}_{sc}) e^{\lambda t} \\ \dot{\varphi}_1 &= 2\pi f \left(\frac{df}{dE}\right) W_1 e^{-\lambda t} \end{aligned} \quad . \tag{6}$$

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*Exclusive of the internal energy exchanges arising from $\mathcal{E}_{sc}$

If we write the Hamiltonian

$$H_1 = \pi f\left(\frac{df}{dE}\right)e^{-\lambda t} W_1^2 - 2\pi Re\left(-k_0 e + k_1 e W_0\right) e^{\lambda t}\varphi_1 + 2\pi e U(\varphi_1) e^{\lambda t} \tag{7}$$

with $\mathcal{E}_{sc} = -\frac{1}{R}\frac{\partial U(\varphi_1)}{\partial \varphi_1}$, then

$$\begin{aligned} \dot{W}_1 &= -\frac{\partial H_1}{\partial \varphi_1} = 2\pi Re\left(-k_0 e + k_1 e W_0\right) e^{\lambda t} + 2\pi R \mathcal{E}_{sc} e^{\lambda t} \\ \dot{\varphi}_1 &= \frac{\partial H_1}{\partial W_1} = 2\pi f\left(\frac{df}{dE}\right) W_1 e^{-\lambda t} \end{aligned} \tag{8}$$

which shows that φ_1 and W_1 form a set of canonical coordinates and momenta.

It is convenient, as in reference 3 to make a canonical transformation to new canonical variables W^*, φ^* in which the stationary part of the distribution function is not explicitly time dependent. This can be accomplished by the generating function

$$S = -\left[W_1 - \frac{2\pi Re}{\lambda}\left(-k_0 e + k_1 e W_0\right) e^{\lambda t}\right]\varphi^* \tag{9}$$

which yields

$$\begin{aligned} W^* &= \frac{-\partial S}{\partial \varphi^*} = W_1 - \frac{2\pi Re}{\lambda}\left(-k_0 e + k_1 e W_0\right) e^{\lambda t} \\ \varphi_1 &= \frac{-\partial S}{\partial W_1} = \varphi^* \end{aligned} \tag{10}$$

The new Hamiltonian is now

$$H^* = H_1 + \frac{\partial S}{\partial t}$$

or

$$\begin{aligned} H^*(W_1^*\varphi_1^* t) = {} & \pi f\left(\frac{df}{dE}\right) e^{-\lambda t}\left[W^* + \frac{2\pi Re}{\lambda}\left(-k_0 e + k_1 e W_0\right) e^{\lambda t}\right]^2 \\ & - 2\pi Re\left[\left(-k_0 e + k_1 e W_0\right)\varphi^* - \frac{U(\varphi^*)}{R}\right] e^{\lambda t} \\ & + 2\pi Re\left(-k_0 e + k_1 e W_0\right)\varphi^* e^{\lambda t} \end{aligned} \tag{11}$$

which reduces to

$$H^*(W^*, \varphi^*, t) = \pi f\left(\frac{df}{dE}\right) e^{-\lambda t}\left[W^* - (k_0/k_1 - W_0)e^{\lambda t}\right]^2 + 2\pi e\, U(\varphi^*)\, e^{\lambda t} \quad . \tag{12}$$

It is easy to check by appropriate change of variables that Hamilton's equations derived from H^* correspond to Eqs. (1) and (2).

III. BOUNDARY EQUATIONS

We introduce the distribution function $\Psi(W^*, \varphi^*, t)$ which must satisfy

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial \Psi}{\partial \varphi^*}\dot{\varphi}^* - \frac{\partial \Psi}{\partial W^*}\dot{W}^* \quad . \tag{13}$$

A stationary distribution is characterized by $\frac{\partial \Psi}{\partial t} = 0$. With the further restriction to φ^*-independent distributions (which is necessary for the small amplitude analysis to be developed, but which does not necessarily exhaust all cases of interest), the equation reduces for the stationary distribution to

$$\frac{\partial \Psi}{\partial W^*}\dot{W}^* = 0$$

and since the φ^*-independence implies that

$$\dot{W}^* = -\frac{\partial H^*}{\partial \varphi^*} \; ; \quad \frac{\partial H^*}{\partial \varphi^*} = 0$$

because $U_0(\varphi^*)$ is constant, the solution is the arbitrary function $\Psi_0(W^*)$.

We wish to investigate the stability of the distribution $\Psi(W^*, \varphi^*, t)$ differing only slightly from $\Psi_0(W^*)$. This can easily be done for a distribution which has uniform phase density within sharp boundaries and is otherwise zero. We shall consequently specialize to a distribution of this type:

$$\Psi = \sigma^* \text{ if } W_1^* \leq W^* \leq W_2^* ; \; \Psi = 0 \text{ if } W^* < W_1^* \text{ or } W_2^* < W^* \quad . \tag{14}$$

Using the Hamiltonian of Eq. (12) and the usual approximation for $\mathcal{E}$,[5] we write Eq. 13 in the form

$$\frac{\partial \Psi}{\partial t} = -2\pi f\left(\frac{df}{dE}\right) e^{-\lambda t}\left[W^* - (k_0/k_1 - W_0)\, e^{\lambda t}\right] \frac{\partial \Psi}{\partial \varphi^*} + \frac{2\pi e^2 g}{\gamma^2 R}\, e^{\lambda t}\, \frac{\partial \Psi}{\partial W^*}\, \frac{\partial}{\partial \varphi^*} \int_{-\infty}^{\infty} \Psi\, dW^* \,. \tag{15}$$

From Eq (15), by integrating over W^* and then by multiplying by W^* and integrating over W^* we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -2\pi f\left(\frac{df}{dE}\right) e^{-\lambda t} \frac{\partial j}{\partial \varphi^*} + 2\pi f\left(\frac{df}{dE}\right)\left(k_0/k_1 - W_0\right) \frac{\partial \rho}{\partial \varphi^*} \\ \frac{\partial j}{\partial t} &= -2\pi f\left(\frac{df}{dE}\right) e^{-\lambda t} \frac{\partial m}{\partial \varphi^*} + 2\pi f\left(\frac{df}{dE}\right)\left(k_0/k_1 - W_0\right) \frac{\partial j}{\partial \varphi^*} - \frac{e^2 g N}{\gamma^2 R}\, e^{\lambda t}\, \frac{\partial \rho}{\partial \varphi^*} \end{aligned} \tag{16}$$

where

$$\rho = \int_{-\infty}^{\infty} \Psi\, dW^*, \quad j = \int_{-\infty}^{\infty} W^* \Psi\, dW^*, \quad m = \int_{-\infty}^{\infty} W^{*2} \Psi\, dW^* \,. \tag{17}$$

Evaluation of ρ, j, and m for the distribution of Eq. (14) gives, as in MURA-480,

$$\begin{aligned} \rho &= \sigma^* \Delta^* \\ j &= \sigma^* \overline{W}^* \Delta^* \\ m &= \sigma^* \overline{W}^{*2} \Delta^* + \sigma^* \Delta^{*3}/12 \end{aligned} \tag{18}$$

with

$$\Delta^* = W_2^* - W_1^* \,, \quad \overline{W}^* = \frac{W_2^* + W_1^*}{2} \,.$$

If we let

$$\begin{aligned} a &= -2\pi f\left(\frac{df}{dE}\right) \\ b &= 2\pi f\left(\frac{df}{dE}\right)\left(k_0/k_1 - W_0\right) \\ c &= -e^2 g N/\gamma^2 R \end{aligned} \tag{19}$$

then Eqs. (16) become

$$\frac{\partial \Delta^*}{\partial t} = a e^{-\lambda t} \frac{\partial}{\partial \varphi^*}(\bar{W}^* \Delta^*) + b \frac{\partial \Delta^*}{\partial \varphi^*}$$

$$\frac{\partial}{\partial t}(\bar{W}^* \Delta^*) = a e^{-\lambda t} \frac{\partial}{\partial \varphi^*}\left(\bar{W}^{*2} \Delta^* + \Delta^{*3}/12\right) + b \frac{\partial}{\partial \varphi^*}(\bar{W}^* \Delta^*) + c e^{\lambda t} \frac{\partial \Delta^*}{\partial \varphi^*} \quad . \tag{20}$$

We have not found general solutions of these equations valid at large t even in the small amplitude limit, but an approximate small amplitude solution for $\lambda \geqslant 0$ will be developed in the following section.

IV. LOSS INCREASING WITH MOMENTUM, I.E., $\lambda > 0$

In this section we consider the case in which there is increasing energy loss with increasing particle momentum, i. e., in which $\lambda > 0$. We may proceed by linearizing the equations about a distribution for which $\bar{W}^* = 0$ and $\Delta^* = \Delta^*_0$. Thus we are considering small perturbations about an initially uniform beam. We must examine the validity of the linearization carefully, which we shall do later after the linear solution is obtained. In the linear approximation, Eqs. (20) become

$$\frac{\partial \Delta^*}{\partial t} = a \Delta^*_0 e^{-\lambda t} \frac{\partial \bar{W}^*}{\partial \varphi^*} + b \frac{\partial \Delta^*}{\partial \varphi^*} \tag{21a}$$

$$\Delta^*_0 \frac{\partial \bar{W}^*}{\partial t} = a e^{-\lambda t} \Delta^{*2}_0/4 \frac{\partial \Delta^*}{\partial \varphi^*} + b \Delta^*_0 \frac{\partial \bar{W}^*}{\partial \varphi^*} + c e^{\lambda t} \frac{\partial \Delta^*}{\partial \varphi^*} \quad . \tag{21b}$$

These two equations can be combined to give a single second-order equation in Δ^* only. Equation (21b) becomes, after rearrangement and differentiation with respect to φ^*

$$\Delta_0^*\left(\frac{\partial^2 \overline{W}^*}{\partial t\,\partial \varphi^*} - b\,\frac{\partial^2 \overline{W}^*}{\partial \varphi^{*2}}\right) = \left(a e^{-\lambda t}\Delta_0^{*2}/4 + c e^{\lambda t}\right)\frac{\partial^2 \Delta^*}{\partial \varphi^{*2}} \quad . \tag{22}$$

Relations for the W^* derivatives in terms of Δ^* derivatives are obtained from Eq. (21a) by differentiation with respect to t and with respect to φ^* to give

$$\begin{aligned} \Delta_0^*\,\frac{\partial^2 \overline{W}^*}{\partial t\,\partial \varphi^*} &= e^{\lambda t}/a\left(\frac{\partial^2 \Delta^*}{\partial t^2} - b\,\frac{\partial^2 \Delta^*}{\partial t\,\partial \varphi^*}\right) + \lambda e^{\lambda t}/a\left(\frac{\partial \Delta^*}{\partial t} - b\,\frac{\partial \Delta^*}{\partial \varphi^*}\right) \\ \Delta_0^*\,\frac{\partial^2 \overline{W}^*}{\partial \varphi^{*2}} &= e^{\lambda t}/a\left(\frac{\partial^2 \Delta^*}{\partial t\,\partial \varphi^*} - b\,\frac{\partial^2 \Delta^*}{\partial \varphi^{*2}}\right) \quad . \end{aligned} \tag{23}$$

Elimination of W^* derivatives from Eq. (22) by use of Eqs. (23) and division by $e^{\lambda t}/a$ leaves

$$\frac{\partial^2 \Delta^*}{\partial t^2} - 2b\,\frac{\partial^2 \Delta^*}{\partial t\,\partial \varphi^*} + \lambda\left(\frac{\partial \Delta^*}{\partial t} - b\,\frac{\partial \Delta^*}{\partial \varphi^*}\right) + b^2\,\frac{\partial^2 \Delta^*}{\partial \varphi^{*2}} = \left(a^2 e^{-2\lambda t}\Delta_0^{*2}/4 + ac\right)\frac{\partial^2 \Delta^*}{\partial \varphi^{*2}} \, . \tag{24}$$

It can be seen that if $\lambda > 0$ the first term on the right-hand side decreases very rapidly with increasing time, and thus can be neglected. This decrease corresponds to the initial energy spread of the beam (characterized in the original coordinates by $\Delta_0 = \Delta_0^* e^{-\lambda t}$) being damped out by the energy loss mechanism which is energy dependent (k_1). The resulting Eq. (24) may now be solved easily since it is a linear partial differential equation with constant coefficients. Let

$$\Delta^* = A e^{i(n\varphi^* - \Omega t)} \tag{25}$$

which upon substitution into Eq. (24) without the $e^{-2\lambda t}$ term yields

$$\Omega^2 + (2bn + i\lambda)\Omega + i\lambda bn + (b^2 - ac)n^2 = 0 \quad . \tag{26}$$

An appropriate criterion for stability is that the charge density per unit angle in the accelerator be stable. The charge density is given by

$$\rho = \int \Psi(W, \varphi, t)\, dW \quad . \tag{27}$$

For uniform phase density this becomes, in W^* variables,

$$\rho^* = \sigma^* \Delta^* \quad . \tag{28}$$

Charge density is the same as in the laboratory coordinates; σ transforms inversely to Δ :

$$\rho = \left(\Delta^* e^{-\lambda t}\right)\left(\sigma^* e^{\lambda t}\right) = \rho^* \quad . \tag{29}$$

Thus since σ^* is constant, the criterion for stability is that Δ^* not grow, which is assured if

$$Im\, \Omega < 0 \quad . \tag{30}$$

Solving Eq. (26) for Ω we have:

$$\Omega = -bn - i\lambda/2 \pm 1/2 \sqrt{4acn^2 - \lambda^2} \quad . \tag{31}$$

Below transition energy, $\frac{df}{dE}$ is positive and therefore $4\,a\,c\,n^2$ is positive; the criterion for stability is satisfied for all positive λ.

The more interesting case is above transition when $\frac{df}{dE}$ is negative and the root is pure imaginary; it follows that the value of Ω_1 obtained for the positive root has an imaginary part greater than zero for all λ and the

criterion for stability is not satisfied. Growth rate is then obtained from Eq. (31). The growing term has initially an exponent, for a perturbation of wave number n, computed from

$$Im\,\Omega_1 = -\lambda/2 + 1/2\sqrt{4|ac|n^2+\lambda^2}$$

which becomes

$$Im\,\Omega_1 = -\lambda/2 + 1/2\left(\frac{8\pi e^2 g N}{\gamma^2 R} f\left|\frac{df}{dE}\right| n^2 + \lambda^2\right)^{1/2} \tag{32}$$

where from the definition of λ and Eq. (4),

$$\lambda = \frac{d}{dE}\left(-\frac{dE}{dt}\right) \quad . \tag{33}$$

It will be observed that as $\lambda \to 0$ this reduces to the result obtained earlier[1, 2] for a monoenergetic beam. When λ becomes large, the rate of growth is reduced, but no finite λ leads to stabilization. In contrast to the situation without energy loss, positive λ leads to different exponents for the damped and antidamped components of the perturbation $(\Omega_1 \neq \Omega_2)$.

It should be noted[6] that the self-field approximation breaks down when n is so large that the wavelength is of the order of beam (or in practice, gap) size. The work of reference 2 then implies that one can obtain an estimate of the maximum growth rate by taking $n \approx \frac{2\pi R}{G}$.

We must now re-examine the validity of the linearization which was made at the beginning of this section. Below transition, we have stability. Consequently, the initially uniform beam will remain uniform. In this case the linearization is clearly valid. Mathematically, the dominant dependence of Δ^* on time (ignoring oscillatory terms) is given by

$$\Delta^* \sim e^{\Omega_1 t} \tag{34}$$

where by Eq. (31)

$$-\lambda/2 \leqslant \Omega_1 < 0 \quad . \tag{35}$$

From Eq. (21a), one can deduce the dominant dependence of $\overline{W}^*$ to be

$$\overline{W}^* \sim e^{(\lambda + \Omega_1)t} \quad . \tag{36}$$

By inserting these into the non-linear Eqs. (20), one can easily check that the terms dropped from Eqs. (20) to obtain the linear Eqs. (21) are actually small in the limit of large t. Thus the stable case in linear approximation is truly a stable case.

In case 2, the situation is more complicated since the instability leads soon to amplitudes which are sufficiently large to make the non-linear terms important. This is no different from the situation in the absence of radiation energy loss ($\lambda = 0$), where the influence of non-linear terms has so far escaped analysis. It does remain to be shown, however, that the linear approximation has at least some degree of validity. What has been done is that in Eq. (24) the term

$$a^2 e^{-2\lambda t} \Delta_0^{*2} / 4 \; \frac{\partial^2 \Delta^*}{\partial \varphi^{*2}} \tag{37}$$

has been neglected. This corresponds to neglecting the term

$$a e^{-\lambda t} / 12 \; \frac{\partial \Delta^{*3}}{\partial \varphi^*} \tag{38}$$

in Eq. (20). The question then arises whether neglecting this term does not invalidate the linearization in which other terms were neglected as small by comparison with this one.

Suppose the beam has initially an energy spread sufficiently large to maintain stability. Then the neglect of the above term will be initially incorrect (although the linearization will be valid). In time, however, the energy spread will damp out so that the beam becomes unstable. Now the above analysis is correct if the beam rapidly reaches negligible energy spread. In the intermediate case, the rate of growth of instabilities will be somewhat overestimated by Eq. (32). It is the authors' opinion that this overestimate is not too severe, that it is an error of about the same magnitude as those introduced through neglect of the non-linear terms in the usual analysis. It is certainly clear that there is no possibility of stabilization and that the $\lambda > 0$ above-transition situation is always unstable; the time for the growth of the instability should be given at least in order of magnitude by Eq. (32). To this one might add the amount of time required for the initial spread to damp to negligible spread.

All of the discussion of this section has presupposed a single energy-loss mechanism tending to reduce energy spread. In practice there may be other mechanisms in operation, such as external rf, gas scattering, or quantum effects, which have the opposite effect, and there may well be an equilibrium reached with energy spread sufficient to insure stability.

V. LOSS DECREASING WITH MOMENTUM, I.E., $\lambda < 0$

In this section we are concerned with the situation in which there is decreasing energy loss with increasing particle momentum. In this case, any initial energy spread will increase in time. Thus the energy loss mechanism

would appear to be a stabilizing influence with respect to the longitudinal instability above transition energy. On the other hand, any perturbation in energy spread will grow simply because the beam energy spread is growing, and in fact in such a way that the larger the spread the larger the growth.

Since we anticipate from previous studies that longitudinal instability is absent below transition irrespective of beam energy spread, we have no reason to suppose that growth of energy spread from an energy loss mechanism will have any interesting consequences. The growth is in itself an instability in the sense that energy spread and variations in energy spread are magnified.

Above transition energy, it is difficult to assess the effect of growing energy spread except in certain more or less trivial cases. Thus, in an intense monoenergetic beam, the usual instability will certainly grow too rapidly to be influenced significantly by any likely rate of increase of energy spread. At the other extreme, a beam with enough initial spread to be stable will surely remain stable. Intermediate cases with initial instability removed by increasing energy spread probably exist but we have not attempted to determine their behavior.

On the whole, the influence of increasing energy spread upon collective instabilities appears to be much less important, taken alone, than the certain effect of decreasing energy spread in producing instabilities above transition. (We have noted earlier that energy spreading mechanisms taken in combination with others tending to reduced spread may be important in determining the

eventual equilibrium distribution and the consequent stability.) It seems therefore doubtful that further study of the increasing spread case in the hope of securing more quantitative results would be justifiable at this time.

FOOTNOTES

1. C. E. Nielsen and A. M. Sessler, MURA-441 (1959).
2. C. E. Nielsen and A. M. Sessler, MURA-480 (1959).
3. A. M. Sessler, MURA-464 (Internal) (1959).
4. A. Becher, MURA-458 (Internal) (1959).
5. C. E. Nielsen and A. M. Sessler, Rev. Sci. Instr. 30, 80 (1959), Eq. (16).
6. See reference 2, Appendix I.