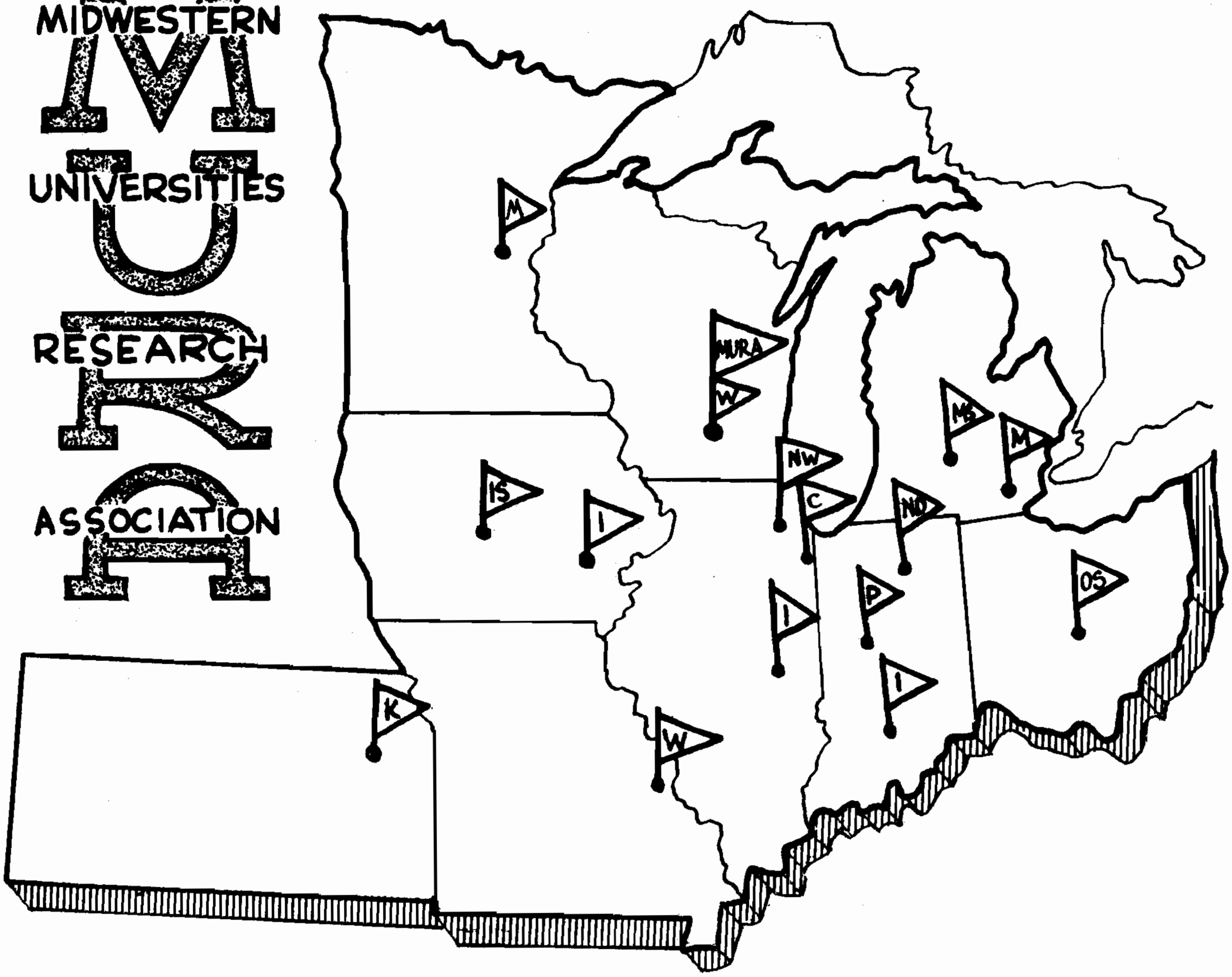
485

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION



INTERPOLATION FORMULA WITH TWO CONTINUOUS DERIVATIVES

Adam Becher

REPORT **NUMBER** 485

MURA-485
Internal

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

INTERPOLATION FORMULA WITH TWO CONTINUOUS DERIVATIVES

Adam Becher**

July 2, 1959

ABSTRACT

An interpolation formula with continuous first and second derivatives is derived. Furthermore the first and second derivatives at boundary points are the same as those given by a five-point central difference formula for derivatives.

*AEC Research and Development Report. Research supported by the Atomic Energy Commission, Contract No. AEC AT(11-1)-384.

**The Ohio State University, Columbus, Ohio.

INTRODUCTION

In the course of numerical treatment, by mesh methods, of partial differential equations it is often necessary to interpolate and interpolate-differentiate. For some physics problems there is reason to think that the mathematical approximations should replicate the true equations at least to the extent of keeping functions and their first and second derivatives continuous.

Most of the interpolation formulae given in the literature[1] are based on the assumption that the investigated function may be approximated by a polynomial between the mesh points on which the function is tabulated. Ordinarily, it is assumed that the higher the degree of the polynomial used in the interpolation, the better is the investigated function approximated. If one uses such a formula to interpolate a function between a sequence of equidistant points, which we shall call for convenience junction points, one obtains an expression which is continuous and which has continuous derivatives <u>between</u> those junction points, but which in general has discontinuous derivatives at junction points. In some problems, however, it is often desirable to have an interpolation formula which, in addition to being continuous and having continuous derivative between the function points, has also continuous derivatives at the junction points. Such a formula, having a continuous first derivative at the junction points has been given by Dr. A. M. Sessler[2]. In the present report we have expanded this formula by making the second derivative at the junctions also continuous, and by making the first and the second derivatives at the junctions equal to the first and second derivatives given by a five-point central difference formula for derivatives.[3]

II. Derivation of the Interpolation Formula.

We represent the investigated function between the points i and i + 1 by the expression

$$u = u_i + \mu (a_{-2}u_{i-2} + a_{-1}u_{i-1} + a_o u_i + a_1 u_{i+1} + a_2 u_{i+2} + a_3 u_{i+3})$$

$$+ \mu^2 (b_{-2}u_{i-2} + b_{-1}u_{i-1} + \dots\dots\dots + b_3 u_{i+3})$$

$$+ \mu^3 (c_{-2}u_{i-2} + c_{-1}u_{i-1} + \dots\dots\dots + c_3 u_{i+3})$$

$$+ \mu^4 (d_{-2}u_{i-2} + d_{-1}u_{i-1} + \dots\dots\dots + d_3 u_{i+3})$$

$$+ \mu^5 (e_{-2}u_{i-2} + e_{-1}u_{i-1} + \dots\dots\dots + e_3 u_{i+3})$$

where the distance between two mesh points has been taken as unity, and where μ runs between zero and one. We further require that

$$u(i) = u_i$$

$$u(i + 1) = u_{i+1},$$

and that the derivatives at the junction points be equal to the derivatives at these points as given by the five-point central difference formula for derivatives[3], that is:

$$u'(i) = \frac{1}{12} u_{i-2} - \frac{2}{3} u_{i-1} + \frac{2}{3} u_{i+1} - \frac{1}{12} u_{i+2}$$

$$u''(i) = -\frac{1}{12} u_{i-2} + \frac{4}{3} u_{i-1} - \frac{5}{2} u_i + \frac{3}{4} u_{i+1} - \frac{1}{12} u_{i+2}$$

and similar expressions for the point (i + 1).

As a result of these requirements one obtains the following set of equations

$$a_{-2} u_{i-2} + b_{-2} u_{i-2} + c_{-2} u_{i-2} + \dots\dots + e_{-2} u_{i-2} = 0$$

$$a_{-1} u_{i-1} + b_{-1} u_{i-1} + \dots\dots\dots\dots\dots + e_{-1} u_{i-1} = 0$$

$$a_o u_i + b_o u_i + \dots\dots\dots + e_o u_i = -u_i$$

$$a_1 u_{i+1} + b_1 u_{i+1} + \dots\dots\dots + e_1 u_{i+1} = u_{i+1}$$

$$a_2 u_{i+2} + b_2 u_{i+2} + \dots\dots\dots + e_2 u_{i+2} = 0$$

$$a_3 u_{i+3} + b_3 u_{i+3} + \dots\dots\dots + e_3 u_{i+3} = 0$$

$$a_{-2} u_{i-2} + 2b_{-2} u_{i-2} + 3c_{-2} u_{i-2} + \dots + 5c_{-2} u_{i-2} = 0$$

$$a_{-1} u_{i-1} + 2b_{-1} u_{i-1} + \dots\dots\dots + 5e_{-1} u_{i-1} = \frac{1}{12} u_{i-1}$$

$$a_o u_i + 2b_o u_i + \dots\dots\dots + 5e_o u_i = -\frac{2}{3} u_i$$

$$a_1 u_{i+1} + 2b_1 u_{i+1} + \dots\dots\dots + 5e_1 u_{i+1} = 0$$

$$a_2 u_{i+2} + 2b_2 u_{i+2} + \dots\dots\dots + 5e_2 u_{i+2} = \frac{2}{3} u_{i+2}$$

$$a_3 u_{i+3} + 2b_3 u_{i+3} + \dots\dots\dots + 5e_3 u_{i+3} = -\frac{1}{12} u_{i+3}$$

$$2b_{-2} u_{i-2} + 6e_{-2} u_{i-2} + 12\, d_{-2} u_{i-2} + 20e_{-2} u_{i-2} = 0$$

$$2b_{-1} u_{i-1} + 6c_{-1} u_{i-1} + \dots\dots\dots + 20e_{-1} u_{i-1} = -1/12\, u_{i-1}$$

$$2\, b_o u_i + 6c_o u_i + \dots\dots\dots + 20e_o u_i = 4/3\, u_i$$

$$2\, b_1 u_{i+1} + 6c_1 u_{i+1} + \dots\dots\dots + 20e_i u_{i+1} = -5/2\, u_{i+1}$$

$$2\, b_2 u_{i+2} + 6c_2 u_{i+2} + \dots\dots\dots + 20e_2 u_{i+2} = 4/3\, u_{i+2}$$

$$2\, b_3 u_{i+3} + 6c_3 u_{i+3} + \dots\dots\dots + 20e_3 u_{i+3} = -1/12\, u_{i+3}$$

$$a_{-2} = 1/12$$

$$a_{-1} = -2/3$$

$$a_o = 0$$

$a_1 = 2/3$

$a_2 = -1/12$

$a_3 = 0$

$2\ b_{-2} = -1/12$

$2\ b_{-1} = 4/3$

$2\ b_o = -5/2$

$2\ b_1 = 4/3$

$2\ b_2 = -1/12$

$2\ b_3 = 0$

From the solution of this system of 30 equations one obtains:

$$
\begin{aligned}
u = u_i &+ \mu\,(1/12\ u_{i-2} - 2/3\ u_{i-1} + 2/3\ u_{i+1} - 1/12\ u_{i+2}) \\
&+ \mu^2(-1/24\ u_{i-2} + 2/3\ u_{i-1} - 5/4\ u_i + 2/3\ u_{i+1} - 1/24\ u_{i+2}) \\
&+ \mu^3(-3/8\ u_{i-2} + 39/24\ u_{i-1} - 35/12\ u_i + 33/12\ u_{i+1} \\
&\qquad - 11/8\ u_{i+2} + 7/24\ u_{i+3}) \\
&+ \mu^4(13/24\ u_{i-2} - 8/3\ u_{i-1} + 21/4\ u_i - 31/6\ u_{i+1} \\
&\qquad + 61/24\ u_{i+2} - 1/2\ u_{i+3}) \\
&+ \mu^5(-5/24\ u_{i-2} + 25/24\ u_{i-1} - 25/12\ u_i + 25/12\ u_{i+1} \\
&\qquad - 25/24\ u_{i+2} + 5/24\ u_{i+3})
\end{aligned}
$$

$$u' = 1/12\ u_{i-2} - 2/3\ u_{i-1} + 2/3\ u_{i+1} - 1/12\ u_{i+2}$$

$$+\mu\left(-1/12\ u_{i-2} + 4/3\ u_{i-1} - 5/2\ u_i + 4/3\ u_{i+1} - 1/12\ u_{i+2}\right)$$

$$+\mu^2\left(-9/8\ u_{i-2} + 39/8\ u_{i-1} - 35/4\ u_1 + 33/4\ u_{i+1} - 33/8\ u_{i+2} + 7/8\ u_{i+3}\right)$$

$$+\mu^3\left(13/6\ u_{i-2} - 32/3\ u_{i-1} + 21\ u_i - 62/3\ u_{i+1} + 61/6\ u_{i+2} - 2\ u_{i+3}\right)$$

$$+\mu^4\left(-25/24\ u_{i-2} + 125/24\ u_{i-1} - 125/12\ u_i + 125/12\ u_{i+1} - 125/24\ u_{i+2} + 25/24\ u_{i+3}\right)$$

$$u'' = -1/12\ u_{i-2} + 4/3\ u_{i-1} - 5/2\ u_i + 4/3\ u_{i+1} - 1/12\ u_{i+2}$$

$$+\mu\left(-9/4\ u_{i-2} + 39/4\ u_{i-1} - 35/2\ u_i + 33/2\ u_{i+1} - 33/4\ u_{i+2} + 7/4\ u_{i+3}\right)$$

$$+\mu^2\left(13/2\ u_{i-2} - 32\ u_{i-1} + 63 u_i - 62\ u_{i+1} + 61/2\ u_{i+2} - 64 u_{i+3}\right)$$

$$+\mu^3\left(-25/6\ u_{i-2} + 125/6\ u_{i-1} - 125/3\ u_i + 125/3\ u_{i+1} - 125/6\ u_{i+2} + 25/6\ u_{i+3}\right)$$

ACKNOWLEDGMENT

The author would like to thank Dr. A. M. Sessler for having suggested this problem and for helpful discussions during the course of the investigation.

FOOTNOTES

1. Booth A., "Numerical Methods." Butterworth Scientific Publications, 1955. Steffensen, J. F., "Interpolation," Chelsea Publishing Company, 1950. Scarborough, J. "Numerical Mathematical Analysis." John Hopkins Press, 1950. Milne, "Numerical Calculus," Princeton University Press, 1949.
2. The formula of Dr. A. M. Sessler is quoted by J. Laslett in MURA-99.
3. Booth (ibid), page 29.