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IDENTIFICATION

Fixed Point Runge-Kutta, MU RKY4
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PURPOSE

To solve a set of N simultaneous, first order, ordinary differential equations in fixed point arithmetic.

RESTRICTIONS

No internal checks are made on scaling and overflow, since it is left to the user to ensure that all numbers stay in the range $-(1 - 2^{-35})$ to $+(1 - 2^{-35})$.

METHOD

Given the set of N differential equations

$$y'_i = f_i\,(y_0, y_1, y_2, \ldots y_{N-1}), \quad (i = 0, 1, 2, \ldots, N-1)$$

the process used in the integration is defined by the following equations:

$$k_{ij} = 2^M h f_i\,(y_{0j}, y_{1j}, \ldots, y_{(N-1)j})$$

$$r_{i,\,j+1} = (A_{j+1} + 1)\,(k_{i,j} - B_j q_{i,j})$$

$$y_{i,\,j+1} = y_{i,j} + 2^{-M} r_{i,\,j+1}$$

$$q_{i,\,j+1} = q_{i,j} + 3r_{i,\,j+1} + (C_j - 1)\,k_{i,j}$$

(Cf. ILLIAC Subroutine Library - Program 114 - F1)

with the following table of values

j	A_{j+1}	B_j	C_j
0	$-1/2$	2	$1/2$
1	$-(1/2)^{1/2}$	1	$(1/2)^{1/2}$
2	$(1/2)^{1/2}$	1	$-(1/2)^{1/2}$
3	$-5/6$	2	$1/2$

where the subscript i indicates the variable, the subscript j indicates the four parts of the integration step, h is the integration step size, and M is a scale parameter. This process is sometimes known as the Gill-Kutta method.*

The k_{ij}'s are the scaled derivative with scale factor $2^M h$. They are evaluated by a closed subroutine known as the AUXILIARY SUBROUTINE, which must be provided by the user. During each pass through MU RKY4 four entries are made into the AUXILIARY SUBROUTINE to obtain the k_{0j}'s, the k_{1j}'s, the k_{2j}'s and the k_{3j}'s. MU RKY4 itself does the arithmetic indicated in the r_{ij}, y_{ij}, and q_{ij} equations above.

Accuracy: This is a fourth order process, hence the truncation error in one step is of the order of h^5. An approximation of the error is obtained from the expression

$$1/15 \quad y_h - y_{h/2}$$

where y_h is the value of y obtained from using an interval of length h and $y_{h/2}$ is the value of y obtained from using an interval of length h/2.

It is important to note that with this subroutine rapid accumulation of roundoff errors is suppressed by retaining the quantities $2^M q_i$ between integration steps. This has the effect of increasing the register length by M binary places. (See reference cited in footnote.)

Minimization of roundoff errors is achieved by making $2^M h$ as near to 1 as possible.

This routine is a faster, more accurate version of MU RKY1, which it obsoletes.

USAGE

Calling Sequence:

UA SAP		MURASS	
Loc.	Instruction	Loc.	Instruction
α	TSX RKY 4, 4	α	B TSX A 4
$\alpha + 1$	HTR M	$\alpha + 1$	BHTR M
$\alpha + 2$	return	$\alpha + 2$	return

where M is the scale parameter appearing in the scale factor $2^M h$.

*S. Gill: "A Process for the Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine," Proceedings of the Cambridge Philosophical Society, vol. 47 (1951), pp. 96-108.

The locations of the AUXILIARY SUBROUTINE and four storage banks must be defined by the user in the assembly of MU RKY4. In addition the symbolic address NURKY4 which, as an integer, is equal to the number of equations, must be defined. The table below lists the symbols which must be defined and their meaning.

SYMBOL		MEANING
UA SAP	MURASS	
NURKY4	N	The numerical value of this symbol is the number of equations, N.
TURKY4	T	Locations TURKY4, TURKY4 + 1, and TURKY4 + 2 are used as temporary storage.
WRKY4	W	Location of the AUXILIARY SUBROUTINE.
XRKY4	X	y_o, y_1,..., y_{N-1} are stored in locations XRKY4 - N, XRKY4 - N + 1,..., XRKY4 - 1, respectively.
YRKY4	Y	k_o, k_1,...k_{N-1} are stored in locations YRKY4 - N, YRKY4 - N + 1,..., YRKY4 - 1, respectively.
ZRKY4	Z	$2^M q_o$, $2^M q_1$...$2^M q_{N-1}$ are stored in locations ZRKY4 - N, ZRKY4 - N + 1,..., ZRKY4 - 1, respectively.

Space Required:

	UA SAP	MURASS
48 words program at	RKY4	A
3 words temporary at	TURKY4	T

and 3 N words for the storage banks described above, and space for the AUXILIARY SUBROUTINE.

Relocatable Cards: The nominal addresses in the relocatable program deck have been defined as follows:

SYMBOLIC ADDRESS	NOMINAL ADDRESS
RKY4	0
NURKY4	1000_8
TURKY4	2000_8
WRKY4	3000_8
XRKY4	4000_8
YRKY4	5000_8
ZRKY4	6000_8

Note that the number of equations, NURKY4, is relocatable, thus enabling the user to specify any desired value for the number of equations when relocating the deck.

CODING INFORMATION

Before the very first entry to this program the user must:

(1) Insert the initial conditions in the y_i bank.

(2) Clear the $2^M q_i$ bank to zero (i.e. clear to zero the locations ZRKY4 - N, ZRKY4 - N + 1,...ZRKY4 - 1).

The AUXILIARY SUBROUTINE must be written as a closed subroutine which will compute $k_0, k_1, \ldots k_{N-1}$ and place them in locations YRKY4 - N, YRKY4 - N + 1,...YRKY4 - 1, respectively. The contents of the temporary storage bank at TURKY4 must not be altered by the AUXILIARY SUBROUTINE. However, the TURKY4 bank can be used for temporary storage between Runge - Kutta steps, while MU RKY4 does not have control.

There are two ways in which the step size may be altered. One may change M and thus effectively alter h by powers of 2. For example if one makes the change $M \rightarrow M + 1$ with no change in the scale factor $2^M h$ we have effectively a step size of h/2 since $2^M h = 2^{M+1} h/2$. When this is done the quantities $2^M q_i$ must be altered accordingly. Thus in making the above change $M \rightarrow M + 1$, the user must replace $2^M q_i$ by $2 \cdot 2^M q_i$ $(i = 0, 1 \ldots N - 1)$. Alternatively one may change h itself. In this case it is not necessary to alter the $2^M q_i$'s. But it must be remembered that as the scale factor $2^M h$ is reduced , by a reduction of h, the roundoff errors are increased.

If the independent variable x occurs in the functions f_i or if it is required during an integration as an index, then it must be obtained by integrating the equation $x' = 1$. The independent variable is then treated as an additional dependent variable, for which the AUXILIARY SUBROUTINE has to provide the quantity $2^M h x' = 2^M h$. However, this

latter quantity may be planted at the beginning of the integration in the appropriate location and left there so that the auxiliary subroutine is relieved of the task. If the independent variable does not appear in any of the f_i's but is merely wanted for indication purposes, it is quicker to use a simple counter in the main routine.

Backward integration is achieved by making h negative.

Timing: (4.12N + 4 [AUXILIARY SUBROUTINE time in milliseconds] + 0.59) milliseconds per integration step

```
      REM RKY4 MURA FIXED POINT RUNGE-KUTTA                          MU   RKY4
RKY4  CLA 1,4              PLANT SCALE FACTOR                      RKY4  0001
      STA RKY4+17          X                                       RKY4  0002
      SXD RKY4+46,1        SAVE INDEX REGISTERS                    RKY4  0003
      SXD TURKY4+1,2       X                                       RKY4  0004
      SXD TURKY4+2,4       X                                       RKY4  0005
      LXA RKY4+46,1        SET IR1 TO 4 (J=0)                      RKY4  0006
      LXA RKY4+47,2        SET IR2 TO N (I=0)                      RKY4  0007
      TSX WRKY4,4          ENTER AUXILLIARY SUBROUTINE             RKY4  0008
      CLA YRKY4,2          FORM K(IJ)-B(J)Q(IJ)                    RKY4  0009
      SUB ZRKY4,2          X                                       RKY4  0010
      TRA RKY4+38,1        X                                       RKY4  0011
      SUB ZRKY4,2          X                                       RKY4  0012
      STO TURKY4           TEMP. STORE                             RKY4  0013
      LDQ RKY4+42,1        FORM R(IJ)                              RKY4  0014
      MPR TURKY4           X                                       RKY4  0015
      ADD TURKY4           X                                       RKY4  0016
      STO TURKY4           X                                       RKY4  0017
      LRS **               TIMES 2EXP-M                            RKY4  0018
      ADD XRKY4,2          STEP Y(I)                               RKY4  0019
      STO XRKY4,2          X                                       RKY4  0020
      LDQ RKY4+46,1        FORM Q(IJ+1)                            RKY4  0021
      MPR YRKY4,2          X                                       RKY4  0022
      SUB YRKY4,2          X                                       RKY4  0023
      ADD ZRKY4,2          X                                       RKY4  0024
      ADD TURKY4           X                                       RKY4  0025
      ADD TURKY4           X                                       RKY4  0026
      ADD TURKY4           X                                       RKY4  0027
      STO ZRKY4,2          X                                       RKY4  0028
      TIX RKY4+8,2,1       I INTO I+1, N PASSES                    RKY4  0029
      TIX RKY4+6,1,1       J INTO J+1, 4 PASSES                    RKY4  0030
      LXD RKY4+46,1        RESTORE INDEX REGISTERS                 RKY4  0031
      LXD TURKY4+1,2       X                                       RKY4  0032
      LXD TURKY4+2,4       X                                       RKY4  0033
      TRA 2,4              EXIT                                    RKY4  0034
      TRA RKY4+11          B(J) SWITCH, B(0)                       RKY4  0035
      TRA RKY4+12          B(J) SWITCH, B(1)                       RKY4  0036
      TRA RKY4+12          B(J) SWITCH, B(2)                       RKY4  0037
      TRA RKY4+11          B(J) SWITCH, B(3)                       RKY4  0038
      OCT 600000000000     -1/2, A(0)                              RKY4  0039
      OCT 665011714640     -1/2EXP1/2, A(1)                        RKY4  0040
      OCT 265011714640     1/2EXP1/2, A(2)                         RKY4  0041
      OCT 725252252253     -5/6, A(3)                              RKY4  0042
      OCT 200000000000     1/2, C(0)                               RKY4  0043
      OCT 265011714640     1/2EXP1/2, C(1)                         RKY4  0044
      OCT 665011714640     -1/2EXP1/2, C(2)                        RKY4  0045
      OCT 200000000000     1/2, C(3)                               RKY4  0046
      HTR 4                CONSTANT                                RKY4  0047
      HTR NURKY4           X                                       RKY4  0048
```