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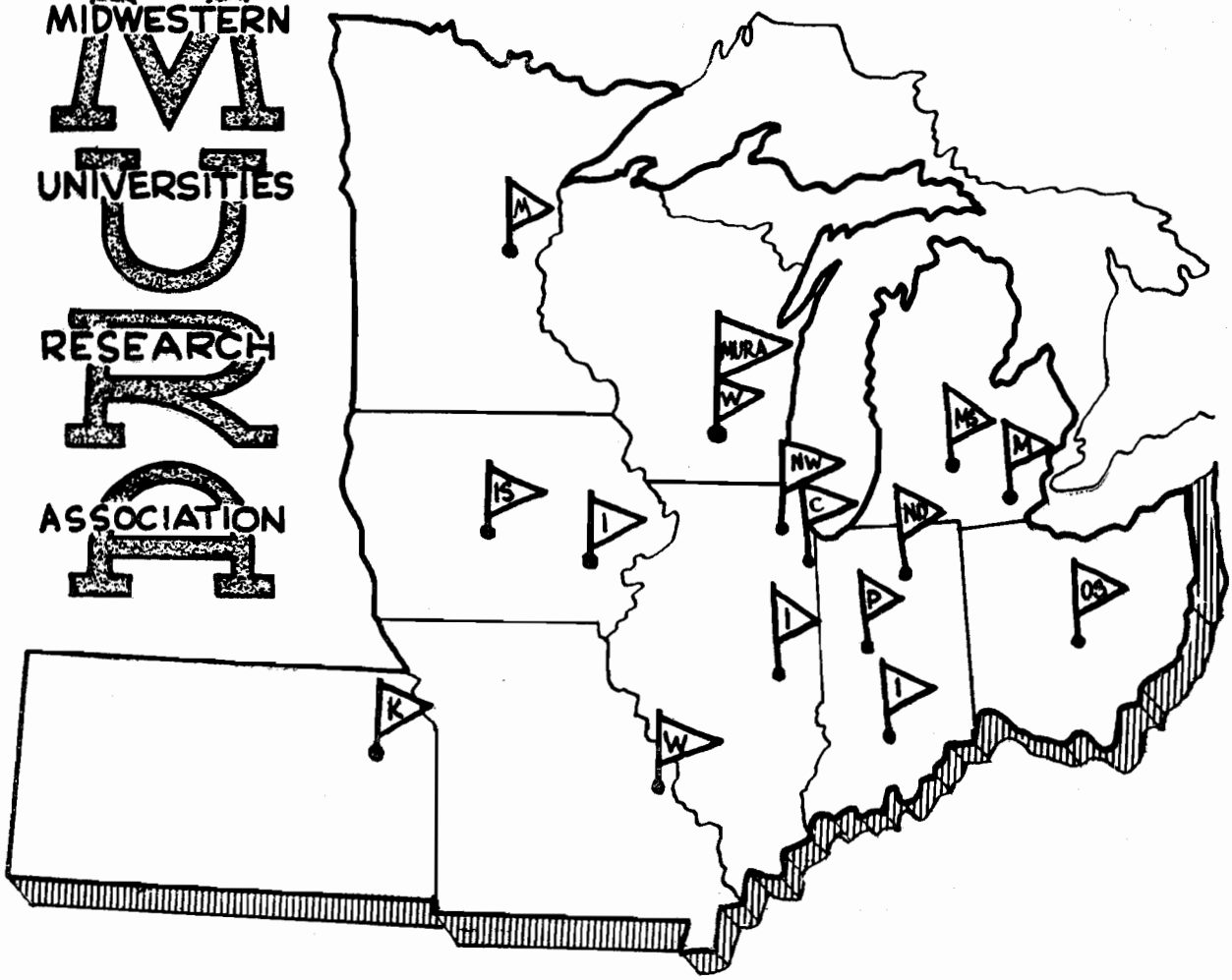

LONGITUDINAL SPACE CHARGE EFFECTS - PHASE BOUNDARY EQUATIONS & POTENTIAL KERNELS

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LONGITUDINAL SPACE CHARGE EFFECTS - PHASE BOUNDARY EQUATIONS & POTENTIAL KERNELS

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ABSTRACT

An assembly of particles subject only to external fields and long-range interactions may be represented as an incompressible fluid in phase space; it follows that the behavior in the special case described by uniform phase density out to sharp boundaries is completely determined by the boundary motion. For azimuthal particle motion this report gives a derivation of two coupled non-linear first order partial differential equations describing the motion of the phase boundaries when they are single valued functions of coordinate and time. It gives also a derivation of a potential function adequate for representation of the collective-field term in these equations even when charge density varies appreciably in a distance of the order of beam diameter; and it proposes computer programs for their solution. In the appendix certain implications of this potential kernel for the solutions in the small amplitude limit of MURA-441 are discussed; a numerical error in MURA-441 is noted.

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+Participant MURA Summer Study Group.

1. INTRODUCTION

In a recent report[1] a computer program for the solution of the one-dimensional longitudinal space charge problem was proposed. This program was designed to solve the reduced Boltzmann equation with arbitrary distribution function $\psi(W, \varphi, t)$. Preliminary computer studies have indicated that it may be difficult to follow an arbitrary distribution for as long a time as would be interesting for some purposes because of limitations in the accuracy of determinations of derivatives imposed by the number of mesh points available in the computer. The present report therefore proposes as a supplementary program a procedure that should give more detailed information about a more limited class of distributions.

If we limit ourselves to distributions of constant density in phase space within arbitrary boundaries, the Boltzmann equation leads to two coupled first order partial differential equations describing the motion of these boundaries, subject to the restriction that boundaries do not become so convoluted that the boundary curves $W_b(\varphi, t)$ are multiple valued. We give a derivation of these boundary equations in Section II.

Next, since our previous space charge studies[2] have been limited in their applicability to problems of stability above transition energy by the use of a very simple potential approximation that fails when the charge density varies too rapidly in azimuth, we develop in Section III a more accurate approximation for use in the proposed program. We hope that this new expression for space charge potential will permit computer study of such problems as the capture of a circulating ring of particles above transition into rf buckets in the presence of space charge and the influence of space charge forces on the stability of buckets above transition.

In Section IV we describe the programs.

Finally, in the Appendix we show that the extension of the analysis of MURA-441 to short wave-lengths made possible with the improved potential function leads to the predictions that the effect of energy-spread in stabilizing a circulating beam is greater for very short wave-lengths, and that the growth rate of instability in a monoenergetic beam approaches a limiting value at short wave-lengths.

II. BOUNDARY EQUATIONS

The equations of boundary motion were first obtained[3] by consideration of the phase motion of the particles lying on the boundary. We give here a different derivation based upon the well known development of the Boltzmann equation into a series of equations in the successive moments.[4]

In order to make the present discussion as general as may be necessary to include all cases of interest, we suppose that the accelerating voltage and rate of energy gain as well as the space charge potential may be time varying. Thus we replace the nearly time-independent Hamiltonian of previous work (ref. 2, eq. 6) by the time-dependent,

$$H(W, \varphi, t) = \pi h \left(f \frac{df}{dE}\right)_s W^2 + e\, V(t) \cos\varphi + \dot{w}_s(t)\, \varphi + 2\pi e\, h\, U(\varphi, t) \qquad (1)$$

and the canonical equations become

$$\dot{W} = e\, V(t) \sin\varphi - \dot{w}_s(t) - 2\pi e\, h \frac{\partial U(\varphi, t)}{\partial \varphi} \equiv L(\varphi, t) \qquad (2)$$

$$\dot{\varphi} = 2\pi h \left(f \frac{df}{dE}\right)_s W \equiv M(t)\, W$$

(It may be noted in passing that this formulation in terms of the rotating coordinate system is advantageous for the present problem because coefficients are more easily evaluated than if one expressed the motion in terms of laboratory coordinates w, φ.)

The Boltzmann equation in terms of these values for $\dot{W}$ and $\dot{\varphi}$ is

$$\frac{\partial \psi}{\partial t} = -L(\varphi, t) \frac{\partial \psi}{\partial W} - M(t) \; W \frac{\partial \psi}{\partial \varphi} \tag{1}$$

We may first integrate over W from $-\infty$ to ∞. This gives

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \psi \, dW = -L(\varphi, t) \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial W} \, dW - M(t) \frac{\partial}{\partial \varphi} \int_{-\infty}^{\infty} W \psi \, dW$$

Since ψ must vanish at the limits, the first term on the right is zero.

If we define

$$\rho(\varphi, t) \equiv \int_{-\infty}^{\infty} \psi \, dW \tag{2}$$

$$j(\varphi, t) \equiv \int_{-\infty}^{\infty} W \psi \, dW$$

the equation becomes

$$\frac{\partial \rho}{\partial t} = -M \frac{\partial j}{\partial \varphi} \tag{3}$$

We next multiply by W and again integrate over W. The result is

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} W \psi \, dW = -L(\varphi, t) \left\{ \left[W \psi \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi \, dW \right.$$

$$- M(t) \frac{\partial}{\partial \varphi} \int_{-\infty}^{\infty} W^2 \psi \, dW$$

and if ψ falls off at infinity faster than W^{-1}, as it must for finite total number of phase points, $\left[W \psi \right]_{-\infty}^{\infty} = 0$. We have then the second equation

$$\frac{\partial j}{\partial t} = L\rho - M \frac{\partial}{\partial \varphi} \int_{-\infty}^{\infty} W^2 \psi \, dW \tag{4}$$

These two equations although true in general are not useful as a solution of the problem unless the second moment can be evaluated or neglected. We can therefore deal with the following two cases: If the distribution is nearly monoenergetic, the range of W may be small enough to allow neglecting the second moment in comparison with ρ. And if the distribution is constant within sharp boundaries, the

second moment is known from ρ and j. Only if density is constant do the boundaries determine the distribution.

Let the boundaries of the distribution be $W_1(\varphi, t)$ and $W_2(\varphi, t)$ within which $\psi = \sigma$ a constant, and outside of which $\psi = 0$. Then

$$\rho = \int_{-\infty}^{\infty} \psi \, dW = \sigma (W_2 - W_1)$$

$$j = \int_{-\infty}^{\infty} W \psi \, dW = \sigma (W_2^2/2 - W_1^2/2) \tag{7}$$

$$m = \int_{-\infty}^{\infty} W^2 \psi \, dW = \sigma (W_2^3/3 - W_1^3/3)$$

It is convenient to define new variables

$$\Delta = W_2 - W_1 \qquad \overline{W} = \frac{W_2 + W_1}{2} \tag{8}$$

in terms of which

$$\rho = \sigma \Delta$$

$$j = \sigma \overline{W} \Delta$$

$$m = \sigma \overline{W}^2 \Delta + \sigma \Delta^3/12$$

and the two equations for the two functions $\Delta(\varphi, t)$ and $\overline{W}(\varphi, t)$ become

$$\frac{\partial \Delta}{\partial t} = -M(t) \frac{\partial \overline{W} \Delta}{\partial \varphi} \tag{9}$$

$$\frac{\partial \Delta \overline{W}}{\partial t} = L \Delta - M \frac{\partial}{\partial \varphi} \left(\overline{W}^2 \Delta + \frac{\Delta^3}{12} \right)$$

The second can be simplified by carrying out the differentiation and then subtracting the first leaving (after removal of the common factor Δ)

$$\frac{\partial \overline{W}}{\partial t} = L(\varphi, t) - M(t) \frac{\partial}{\partial \varphi} \left(\frac{\overline{W}^2}{2} + \frac{\Delta^2}{8} \right) \tag{10}$$

It may seem at first glance surprising that density σ seems to vanish from these equations; but it will reappear when $L(\varphi, t)$ is evaluated. The

coefficient M(t) depends only upon the properties of the accelerator and the particles being accelerated and it need not concern us further. The coefficient $L(\varphi, t)$, however, contains both the external applied field and the electrostatic self-field.

In order to evaluate the self-field part of $L(\varphi, t)$ we must relate the phase space distribution to the coordinate space charge density and we must evaluate the field resulting from this charge density. In the following section the potential contribution of an element of charge q at a distance z is computed in the form of the kernel $K(|z' - z|)$ from which the total potential at a point z is

$$U(z) = \int_{-\infty}^{\infty} K(|z' - z|)\, \lambda(z')\, dz' \tag{11}$$

with $\lambda(z')$ defined as charge per unit length and K having dimensions of reciprocal length.

We make the assumption that the curvature of the charge distribution in its orbit is slight enough in a distance of the order of shield diameter, beyond which K falls rapidly to zero, to allow replacement of dz by R dθ.

From the definition of φ (ref. 2) $d\theta = \frac{1}{h} d\varphi$, and we therefore put $dz = R/h\, d\varphi$ and write $R/h\,(\varphi' - \varphi)$ for $z' = z$ in the expression for K.

The total charge in a uniform ring would be

$$Q = 2\pi R\lambda$$

and the corresponding phase area would be

$$S = 2\pi h (W_2 - W_1) = 2\pi h\Delta$$

whence total charge in terms of phase point density σ is

$$Q = e\sigma S = 2\pi e\sigma h\Delta$$

and

$$\lambda = \frac{e\sigma h\Delta}{R.}$$

We therefore put into the integral

$$\lambda(z')\,dz' = e\sigma\Delta(\varphi', t)\,d\varphi' \tag{12}$$

(Compare eq. 13, ref. 2). The coefficient $L(\varphi, t)$ is consequently

$$L(\varphi, t) = e\,V(t)\sin\varphi - \dot{w}_s(t) - 2\pi e^2\sigma h \frac{\partial}{\partial\varphi}\int_{-\pi h}^{\pi h} K(\varphi' - \varphi)\Delta(\varphi', t)\,d\varphi' \tag{13}$$

This expression together with the appropriate $M(t)$ and the value of K determined below puts the boundary equations into a form appropriate for solution as outlined in Section IV. For relativistic particles we multiply the last term by γ^{-2}, γ being defined by

$$\gamma = m/m_o$$

to correct for the change in self-field experienced by the particles.

III. EVALUATION OF THE KERNEL

We may easily obtain the kernel $K(|z' - z|)$ in the non-relativistic approximation for a cylindrical beam in which charge density is a function of axial coordinate z only. What is required is the potential, on axis, of a uniformly charged disk of radius a, located inside a conducting cylinder of radius b, as indicated in Fig. 1. The solution to this problem may be easily obtained from the Green's function for a cylinder[5];

$$V = \frac{2q}{b^2}\sum_{r=1}^{\infty}\sum_{s=o}^{\infty}(2 - \delta_s^0)\,e^{-\mu_r|z|}\,\frac{J_s(\mu_r a)J_s(\mu_r \rho)}{\mu_r\left[J_{sH}(\mu_r b)\right]^2}\cos s(\varphi - \varphi_o) \tag{14}$$

where the notation is the same as in reference 5, but we have used e. s. u. The numbers μ_r are determined by the condition $J_o(\mu_r b) = 0$. Integrating over a disk and replacing q by $\frac{\lambda\delta z}{\pi a^2}$ we obtain:

$$\delta V(z) = \frac{4\lambda\delta z}{b^2\,a^2}\sum_r \frac{e^{-\mu_r|z|}}{\mu_r\left[J_1(\mu_r b)\right]^2}\int_0^a J_o(\mu_r x)\,x\,dx. \tag{15}$$

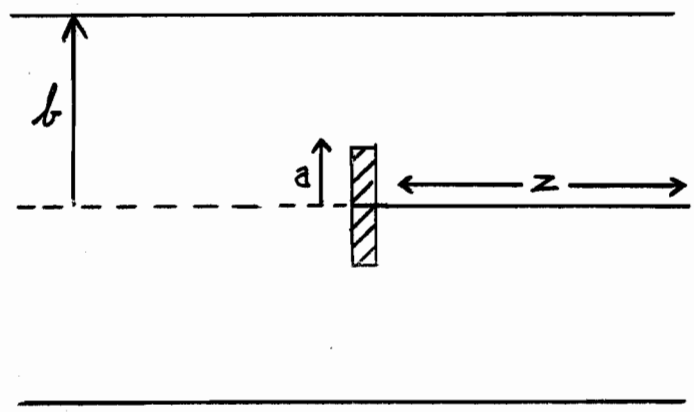

Fig. 1. Geometry for determination of the kernel.

The integral may be performed to yield:

$$\delta V(z) = \frac{4\lambda\delta z}{a} \sum_r \frac{e^{-\epsilon_r \frac{z}{b}} J_1\left(\epsilon_r \frac{a}{b}\right)}{\epsilon_r^2 J_1^2(\epsilon_r)} \tag{16}$$

in which the ϵ_r are the roots of the Bessel function J_o.

This series converges fairly rapidly for a/b not too small compared with unity, even when $z = 0$, and obviously even more rapidly for any given a/b as z/b becomes greater than zero. To illustrate we may give some numerical values for the case $a/b = 0.5$. The first three roots of J_o and the corresponding values of J_1 are

$$\epsilon_1 = 2.4048 \qquad J_1(\epsilon_1) = 0.5191$$

$$\epsilon_2 = 5.5201 \qquad J_1(\epsilon_2) = -0.3403$$

$$\epsilon_3 = 8.6537 \qquad J_1(\epsilon_3) = 0.2715$$

and accordingly the potential is in general

$$\delta V(z) = \frac{4\lambda\delta z}{a} \left\{ \frac{e^{-2.405 \frac{z}{b}} J_1\left(2.405 \frac{a}{b}\right)}{1.5583} + \frac{e^{-5.520 \frac{z}{b}} J_1\left(5.520 \frac{a}{b}\right)}{3.5287} + \frac{e^{-8.654 \frac{z}{b}} J_1\left(8.654 \frac{a}{b}\right)}{5.5200} + \ldots\ldots \right\} \tag{17}$$

For $a/b = 0.5$ this becomes

$$\delta V(z) = \frac{4\lambda\delta z}{a}\left\{ 0.3202\, e^{-2.405\frac{z}{b}} + 0.1198\, e^{-5.520\frac{z}{b}} - 0.0327\, e^{-8.654\frac{z}{b}} - \ldots\ldots \right\} \tag{18}$$

Evaluation of the series for $z/b = 0$ gives 0.3958 as the sum of the first ten terms in brackets; this may be compared with 0.4073 from the three terms written.

In this situation one naturally asks whether the first one or two terms of this series provide sufficiently good approximation to the potential. Evidently better approximation could be obtained with a given number of terms by treating coefficients and exponents as adjustable parameters. Two criteria for such adjustment follow from the fact that, as may readily be shown, the field from a charge distribution of constant density gradient is correctly given providing the integral over z of the potential kernel has the correct value (the approximation of ref. 2), which the field at a discontinuity in an otherwise uniform charge density is correctly given if the kernel is correct at $z = 0$.

Adjustment of two parameters to meet the two conditions just stated is so simple and effective that for present purposes we need not explore methods of improving the fit by imposing additional criteria for additional parameter adjustment. The quality of the approximation obtained in various ways is indicated in the following table showing (1) potential for various z computed from ten terms of the series (2) the first term of the series (3) the first two terms of the series (4) a single exponential term with coefficient and exponent adjusted to match the peak value and the integral (5) the first term of the series plus a second exponential with coefficient and exponent adjusted. The factor $\frac{2\lambda\delta z}{a}$, which is the potential at the center of an isolated disc, is taken out so that the numbers given represent potential

at various z/b inside a shield of radius $b = 2a$ relative to potential at the center of an unshielded disc.

TABLE I

z/b =	0	1/4	1/2	3/4	1
1. Ten terms	0.7915	0.4006	0.2065	0.1092	0.0587
2. First term	0.6403	0.3510	0.1924	0.1054	0.0578
3. Two terms	0.8799	0.4112	0.2075	0.1093	0.0587
4. One adj. exp.	0.7915	0.4077	0.2100	0.1082	0.0557
5. First term plus adj. exp.	0.7915	0.3974	0.2066	0.1098	0.0591

The first three lines in the table are computed by means of eq. 16, line four represents the single term

$$0.7915 \; e^{-2.653 \, z/b} \tag{18a}$$

and line five represents the two terms

$$0.6403 \; e^{-2.405 \, z/b} + 0.1512 \; e^{-4.721 \, z/b} \; . \tag{18b}$$

The quality of the representation provided by even a single adjusted term would appear to be more than adequate in this case.

Total potential at a point z being given by the integral of $\delta V(|z' - z|)$ over all z', we write the kernel[6]

$$K(|z' - z|) = \frac{\delta V(|z' - z|)}{\lambda(z') \, \delta z'} \tag{19}$$

For computer use one will probably enter the kernel as a table of numbers giving $K(|z' - z|)$ at the required $z' - z$.

The two term approximation could take the form

$$K(|z' - z|) = \frac{2}{a}\left\{\alpha_1 e^{-\beta_1(|z' - z|)/b} + \alpha_2 e^{-\beta_2(|z' - z|)/b}\right\}$$

or in terms of phase angle

$$K(|\varphi' - \varphi'|) = \frac{2}{a}\left\{\alpha_1 e^{-\beta_1 R(|\varphi' - \varphi|)/hb} + \alpha_2 e^{-\beta_2 R(|\varphi' - \varphi|)/hb}\right\} \tag{20}$$

IV. DIGITAL COMPUTER PROGRAMS

In Section II the equations (eqs. 9, 10, 13, 2) describing the motion of a band of particles in phase space were shown to be

$$\frac{\partial \Delta}{\partial t} = - M(t) \frac{\partial \Delta \overline{W}}{\partial \varphi}$$

$$\frac{\partial \overline{W}}{\partial t} = L(\varphi, t) - M(t) \frac{\partial}{\partial \varphi}\left(\frac{\overline{W}^2}{2} + \frac{\Delta^2}{8}\right), \tag{21}$$

where

$$M(t) = 2\pi h \left(f \frac{df}{dE}\right)_s$$

$$L(\varphi, t) = eV(t)\sin\varphi - \dot{w}_s(t) - \frac{2\pi e^2 \sigma h}{\gamma^2} \frac{\partial}{\partial \varphi}\int_{-\pi h}^{\pi h} K(\varphi' - \varphi)\Delta(\varphi', t)\, d\varphi', \tag{22}$$

and $K(\varphi' - \varphi)$ is given in Section III (eqs. 16, 19) by:

$$K(\varphi' - \varphi) = \frac{4}{a}\sum_r e^{-\epsilon_r R \frac{|\varphi' - \varphi|}{hb}} \frac{J_1(\epsilon_r a/b)}{\epsilon_r^2 J_1^2(\epsilon_r)} \tag{23}$$

It is suggested that solutions to these equations may be obtained by digital computation. We describe two possible computer programs in the remainder of this section, taking for convenience the case h = 1.

Program A

The functions Δ and $\overline{W}$ must be evaluated for a set of values φ_i at each time t_n. Given $\Delta(\varphi_i, t_n)$ and $\overline{W}(\varphi_i, t_n)$ at some time t_n and for all i the computer must evaluate $\Delta(\varphi_i, t_{n+1})$ and $\overline{W}(\varphi_i, t_{n+1})$ for all i. In the course of this the computer must evaluate $L(\varphi, t)$. Thus roughly 1/5 of the storage available in the fast memory (beyond that required for instructions) may be used for $\Delta(\varphi_i, t_n)$ for fixed n and all i; 1/5 for $\overline{W}(\varphi_i, t_n)$ for fixed n and all i; 1/5 for $K(\varphi_j - \varphi_i)$; 1/5 for $\sin \varphi_i$ (probably most conveniently stored rather than evaluated); and 1/5 for $\sum_j K(\varphi_j - \varphi_i)\Delta(\varphi_j, t_n)$. The integration in time must be performed using a stable method, such as the method developed by R. Christian for use with MURA-442.

The functions M (t), $\dot{w}_s$ (t) and V (t) will be given as analytic functions and the program should allow for a number of choices for each of these functions. One might use, for example,

$$M(t) = M_o \left[1 + \frac{t}{\tau_0} + \frac{t^2}{\tau_1^2}\right]$$

$$\dot{w}_s(t) = \dot{w}_s(0)\left[1 + t/\tau_2^2\right] \tag{24}$$

$$V(t) = V_o\left[\frac{\dot{t}}{t + \tau_3}\right]$$

The kernel $K(\varphi' - \varphi)$ can be stored as a table of numbers and has the property that it is negligible for $|\varphi' - \varphi| > 2b/R$. (This is a number in the 50 Mev model of the order of $\frac{1}{20}$, i.e. if i takes on 1000 values $K(\varphi' - \varphi)$ is non-vanishing for about 50 values. For other accelerators b/R will in general be smaller.) The $\sum_j K(\varphi_j - \varphi_i)\Delta(\varphi_j, t_n)$ should be evaluated first and then the derivative with respect to φ taken. The special case of $K(\varphi_j - \varphi_i) = 0$ for $j \neq i$

should be programmed specially. All the differentiations should be performed using a fairly elaborate differentiation formula.

Print out should be arranged so that $\Delta(\varphi, t_n)$ and $W(\varphi, t_n)$ can be Fourier analyzed, and the Fourier components printed out after a specifiable number of iterations.

Program B

It may be advantageous to solve the equations describing a beam with constant phase density by means of a complete set of ortho-normal functions. The most obvious functions for this problem are $\frac{1}{\sqrt{\pi}} \sin n\varphi$ and $\frac{1}{\sqrt{\pi}} \cos n\varphi$ and we here formulate the equations, using these functions. The form of M(t), $\dot{w}_s(t)$, and V(t) is optional and could be taken as in Part A.

Let

$$\Delta(\varphi, t) = \sum_n \left[a_n(t) \frac{1}{\sqrt{\pi}} \sin n\varphi + \frac{b_n(t)}{\sqrt{\pi}} \cos n\varphi \right]$$
$$\bar{W}(\varphi, t) = \sum_n \left[c_n(t) \frac{1}{\sqrt{\pi}} \sin n\varphi + \frac{d_n(t)}{\sqrt{\pi}} \cos n\varphi \right] \tag{25}$$

In this case the first of eq. (21) becomes

$$\frac{1}{\sqrt{\pi}} \sum_n \left[\dot{a}_n \sin n\varphi + \dot{b}_n \cos n\varphi \right] =$$

$$- \frac{M(t)}{\pi} \sum_{n', n''} \frac{\partial}{\partial \varphi} \Big\{ a_{n'} \sin n'\varphi \; c_{n''} \sin n''\varphi + a_{n'} d_{n''} \sin n'\varphi \cos n''\varphi$$

$$+ b_{n'} c_{n''} \cos n'\varphi \; \sin n''\varphi + b_{n'} d_{n''} \cos n'\varphi \; \cos n''\varphi \Big\}$$

which is equivalent to

$$\frac{1}{\sqrt{\pi}} \sum_n \left[\dot{a}_n \sin n\varphi + \dot{b}_n \cos n\varphi \right] = - \frac{M(t)}{\pi} \frac{\partial}{\partial \varphi} \sum_{n', n''}$$

$$\left\{ \frac{a_{n'}c_{n''}}{2} \left[\cos (n' - n'')\varphi - \cos (n' + n'')\varphi \right] + \right.$$

$$\frac{a_{n'}d_{n''}}{2} \left[\sin (n' + n'')\varphi + \sin (n' - n'')\varphi \right] +$$

$$\frac{b_{n'}c_{n''}}{2} \left[\sin (n' + n'')\varphi + \sin (n'' - n')\varphi \right] +$$

$$\left. \frac{b_{n'}d_{n''}}{2} \left[\cos (n' + n'')\varphi + \cos (n' - n'')\varphi \right] \right\} ,$$

or

$$\frac{1}{\sqrt{\pi}} \sum_{n} \left[\dot{a}_n \sin n\varphi + \dot{b}_n \cos n\varphi \right] = - \frac{M(t)}{\pi} \sum_{n', n''}$$

$$\left\{ \frac{a_{n'}c_{n''}}{2} \left[- (n' - n'') \sin (n' - n'')\varphi + (n' + n'') \sin (n' + n'')\varphi \right] \right.$$

$$+ \ldots\ldots \quad \ldots\ldots \Big\}$$

Multiplying by $\frac{1}{\sqrt{\pi}} \sin n\varphi$ and integrating, and then by $\frac{1}{\sqrt{\pi}} \cos n\varphi$ and integrating yields:

$$\dot{a}_n = - \frac{M(t)\, n}{2\sqrt{\pi}} \sum_{n'} \left\{ -a_{n'} (c_{n'-n} + c_{n'+n}) + a_{n'}(c_{n-n'} + c_{-n-n'}) + \right.$$

$$\left. - b_{n'} (d_{n-n'} + d_{-n-n'}) + -b_{n'} (d_{n'-n} + d_{n'+n}) \right\}$$

$$\dot{b}_n = - \frac{M(t)\, n}{2\sqrt{\pi}} \sum_{n'} \left\{ a_{n'} (d_{n-n'} - d_{-n-n'}) + \right.$$

$$+ a_{n'} (d_{n'-n} - d_{n'+n}) +$$

$$+ b_{n'} (c_{n-n'} - c_{-n-n'}) +$$

$$\left. + b_{n'} (c_{n+n'} - c_{-n+n'}) \right\} \tag{26}$$

The second of eq. (21) may be treated in a like manner. It is of course more complicated due to the integral in $L(\varphi, t)$. If we let

$$L(\varphi, t) = \sum_n \left(\frac{\alpha_n}{\sqrt{\pi}} \sin n\varphi + \frac{\beta_n}{\sqrt{\pi}} \cos n\varphi \right). \tag{27}$$

with the α_n and β_n to be evaluated below, then by a method similar to that used above we obtain:

$$\begin{aligned}
\dot{c}_n = \alpha_n - \frac{M(t)n}{2\sqrt{\pi}} \sum_{n'} \Big\{ & 1/2\, c_{n'} (c_{n-n'} - c_{-n-n'} - c_{n'-n} + c_{n'+n}) \\
& -1/2\, d_{n'} (d_{n-n'} - d_{-n-n'} + d_{n'-n} - d_{n'+n}) \\
& + 1/8\, a_{n'} (a_{n-n'} - a_{-n-n'} - a_{n'-n} + a_{n'+n}) \\
& - 1/8\, b_{n'} (b_{n-n'} - b_{-n-n'} + b_{n'-n} - b_{n'+n}) \Big\} \\
\dot{d}_n = \beta_n - \frac{M(t)n}{2\sqrt{\pi}} \sum_{n'} \Big\{ & (1/2\, c_{n'} (d_{n-n'} + d_{-n-n'} + d_{n'-n} + d_{n'+n}) \\
& + 1/2\, d_{n'} (c_{n-n'} + c_{-n-n'} + c_{n+n'} + c_{-n+n'}) \\
& + 1/8\, a_{n'} (b_{n-n'} + b_{-n-n'} + b_{n'-n} + b_{n'+n}) \\
& + 1/8\, b_{n'} (a_{n-n'} + a_{-n-n'} + a_{n+n'} + a_{-n+n'}) \Big\} .
\end{aligned} \tag{28}$$

To evaluate the α_n and β_n we use the expression for $L(\varphi, t)$ given in eq. (22) (with h = 1) together with $\Delta(\varphi, t)$ from eq. (25). We shall obtain expressions for α_n and β_n for two forms of $K(\varphi' - \varphi)$ which, in view of the generality of the second, seem to cover most cases of interest.

<u>Case i</u>. $K(\varphi' - \varphi) = K_o \delta(\varphi' - \varphi)$

$$L(\varphi, t) = e\, V(t) \sin\varphi - \dot{w}_s(t) - \frac{2\pi e^2 \sigma}{\gamma^2} K_o \frac{\partial \Delta(\varphi, t)}{\partial \varphi} \tag{29}$$

and consequently

$$\alpha_n = + \frac{2\pi e^2 \sigma}{\gamma^2} \; n \, K_o \, b_n \, (t) + \sqrt{\pi} \, e \, V \, (t) \; \delta \, (n-1)$$

$$\beta_n = - \frac{2\pi e^2 \sigma}{\gamma^2} \; n \, K_o \, a_n \, (t) - \sqrt{\pi} \, \dot{w}_s \, (t) \; \delta \, (n-0) \tag{30}$$

<u>Case ii.</u>

$K(\varphi' - \varphi)$ is given as a table of numbers. In this case we may Fourier analyze the kernel and then express α_n and β_n in terms of the Fourier coefficients of the kernel. If we let

$$K(\varphi' - \varphi) = \sum_m K_m \cos m \, (\varphi' - \varphi), \tag{31}$$

then

$$\frac{\partial}{\partial \varphi} \int_{-\pi}^{\pi} K(\varphi' - \varphi) \Delta (\varphi', t) \; d\varphi'$$

is given by

$$\sum_{m,n} \frac{K_m}{\sqrt{\pi}} \; m \int_{-\pi}^{\pi} \sin m \, (\varphi' - \varphi) \left[a_n \sin n\varphi' + b_n \cos n\varphi' \right] d\varphi',$$

or

$$\sum_{m,n} \frac{K_m m}{\sqrt{\pi}} \int_{-\pi}^{\pi} \left[\sin_m \varphi' \cos_m \varphi - \cos_m \varphi' \sin_m \varphi \right] x$$

$$\left[a_n \sin_n \varphi' + b_n \cos n\varphi' \right] \; d\varphi'$$

Performing the integral yields

$$\sum_n \sqrt{\pi} \, n \, K_n \, (a_n \cos_n \varphi - b_n \sin_n \varphi).$$

Thus we obtain[7]

$$\alpha_n = + \frac{2\pi^2 e^2 \sigma}{\gamma^2} \; n \, K_n \, b_n \, (t) + \sqrt{\pi} \, e \, V \, (t) \; \delta \, (n-1)$$

$$\beta_n = - \frac{2\pi^2 e^2 \sigma}{\gamma^2} \; n \, K_n \, a_n \, (t) - \sqrt{\pi} \, \dot{w}_s \, (t) \; \delta \, (n-0) \tag{32}$$

We note that these expression differ from those in the special delta function case only in the replacement of the K_o by πK_n.

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APPENDIX[8]

The expansion of the potential kernel in a Fourier series as in Eq. (31) may be combined with the linear theory of MURA-441 to yield an interesting result. Examination of the analysis contained in Eqs. (19) to (31) of that report shows that the dispersion relation becomes[6]:

$$1 = \frac{4\pi^2 e^2}{\gamma^2} \left(f \frac{df}{dE}\right)_s \pi K_n \, n \int \frac{\frac{\partial \psi_0(\dot{\varphi})}{\partial \dot{\varphi}} \, d\dot{\varphi}}{n\dot{\varphi} - \omega} \, . \tag{A-1}$$

For a delta-function kernel the coefficients K_n are all equal to $\frac{g}{\pi R}$, and Eq. (A-1) reduces to Eq. (31) of MURA-441.

Following through the analysis of MURA-441 we find that the frequency associated with wave number n (eq. 36.) becomes for a rectangular distribution

$$\omega = n \left\{ \dot{\varphi}_A \pm \left[\frac{2\pi^2 e^2 N}{\gamma^2} \left[f \frac{df}{dE} \right]_s K_n + \pi^2 \left(\frac{df}{dE} \right)_s^2 (\Delta E)^2 \right]^{1/2} \right\} \tag{A-2}$$

Now the kernel $K(|\varphi' - \varphi|)$ has a discontinuity in its derivative at $\varphi = \varphi'$, but otherwise is continuous with continuous derivative. Consequently[9] K_n asymptotically goes as $1/n^2$ and ω becomes real for sufficiently large wave numbers n, only provided that $\Delta E \neq 0$. This physically reasonable result is very important from a computational point of view as it ensures that stability of the numerical procedures is attainable with a sufficiently large but not infinite number of Fourier components.

As an example, consider the first term of the kernel for a/b = 0.5 from Eq. (18)

$$K(\varphi) = \frac{8}{b} \quad (0.3202) \, e^{\frac{-2.405 \, R}{b} \varphi} \, . \tag{A-3}$$

This leads to Fourier coefficients

$$K_n = \frac{16(0.3202)\mu}{b\pi(\mu^2 + n^2)} \left[1 - (-1)^n e^{-\pi\mu} \right] \tag{A-4}$$

where

$$\mu = \frac{2.405 \, R}{b} \, . \tag{A-5}$$

The term $e^{-\pi\mu}$ expresses the fact that the kernel has a discontinuity in slope at $\varphi = \pm\pi$ as well as at $\varphi = 0$. This discontinuity is negligible however, as $e^{-\pi\mu} \ll 1$ for all accelerators. From Eq. (A-4) one can see that K_n is constant for small n (namely for $n < 2.4\ R/b$), and then decreases for large values of n. The critical value of n, such that all waves of greater wave number are stable, can be obtained in any case of interest from Eq. (A-2) using Eqs. (A-4) and (A-5).

If the energy spread is small enough so that wave number n exceeds μ without stabilization occurring, then the growth rate approaches the value

$$\omega \approx \left[\frac{2\pi^2 e^2 N}{\gamma^2} \left(f \ \frac{df}{dE} \right)_s \right]^{1/2} \cdot \left[\frac{16\ (0.3202)\mu}{b\pi} \right]^{1/2}$$

independent of n.

Although this conclusion is derived here only in the small amplitude approximation, it seems probable that non-linear large amplitude behavior is likewise characterized by a limiting wavelength below which disturbances grow at a nearly constant rate; and possibly they grow only to some maximum amplitude at which stabilization occurs.

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6. Eq. (25) of MURA-441 should contain $eg/\gamma^2 R^2$ and not $eg/2\pi\gamma^2 R^2$; subsequent equations in that report involving $\mathcal{E}$ are likewise in error by 2π.

7. Dr. R. Christian first pointed out to the authors the simplicity with which this expansion of the kernel could be carried through.

8. The possibility of the analysis of this appendix was pointed out by Dr. Peter Lax.

9. M. J. Lighthill, Fourier Analysis & Generalized Functions (Cambridge University Press, Cambridge 1958) p. 56.