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MULTIPLE FREQUENCY ACCELERATORS

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ABSTRACT

A scheme for accelerating particles in an accelerator having a constant magnetic guide field is described which consists of a number of radio-frequency voltages each of a different and constant frequency. It is shown that the net accelerating voltage for an infinite number of radio-frequency voltages of constant frequency is equivalent to the accelerating voltage of a frequency modulated voltage which is modulated in amplitude and phase by a periodic function. This modulating function is Fourier analyzed and the coefficients of the first two terms are determined. The effects of the modulating function on the frequency modulated accelerating voltage is estimated. The fraction of time over which particles may be accepted into phase stable positions is estimated, and turns out to be remarkably high. It is pointed out that this principle of multiple frequency acceleration may be used to design very high intensity accelerators. Several examples of the application of this principle are given in the latter part of the report.

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MULTIPLE FREQUENCY ACCELERATORS

A type of accelerator is investigated which consists of a number of radio-frequency voltages, each of a different and constant frequency, in conjunction with a constant magnetic guide field.

For purposes of analysis we consider the accelerating system to be composed of an infinite number of radio-frequency voltages, with the separation between adjacent frequencies being ω_0. We assume that the range of frequencies of interest is sufficiently small such that only one harmonic order need be considered. The component of the accelerating voltage of interest may then be written:

$$V = V_N \sum_{n=-\infty}^{+\infty} e^{i[(\omega_m + n\omega_0)t + \phi_1 n + \phi_2 n^2 + \gamma]} e^{-i[h(\theta - \theta_1 n)]} \qquad (1)$$

V_N is the magnitude of the voltage of each frequency, which are assumed to be equal. ϕ_1 and ϕ_2 determine the time phase relation of the various frequencies. θ_1 represents the relation in azimuth of the radio-frequency voltages. h is the harmonic order.

$$V = V_N \sum_{n=-\infty}^{+\infty} e^{i[\omega_m t + n\omega_0 t + n(\phi_1 + h\theta_1 + 2\pi p) + \phi_2 n^2 + \gamma - h\theta]} \qquad (2)$$

where p is any integer. This may be written in the form

$$V = V_N e^{i\left[\omega_m t - \frac{\phi_1 + h\theta_1 + 2\pi p}{2\phi_2}\omega_0 t - \frac{\omega_0^2 t^2}{4\phi_2} - \frac{(\phi_1 + h\theta_1 + 2\pi p)^2}{4\phi_2} + \gamma - h\theta\right]}$$

$$\times \sum_{n=-\infty}^{+\infty} e^{i\left[\phi_2^{\frac{1}{2}} n + \frac{\omega_0 t}{2\phi_2^{\frac{1}{2}}} + \frac{\phi_1 + h\theta_1 + 2\pi p}{2\phi_2^{\frac{1}{2}}}\right]^2} \qquad (3)$$

This is equivalent to a frequency modulated accelerating voltage with azimuthal angular frequency

$$\omega = \frac{1}{h}\left[\omega_M - \frac{\phi_1 + h\theta_1 + 2\pi p}{2\phi_2}\omega_0 - \frac{\omega_0^2 t}{2\phi_2}\right] \tag{4}$$

with p any integer, modulated in amplitude and phase by the function

$$F = \sum_{n=-\infty}^{+\infty} e^{i\left[\phi_2^{\frac{1}{2}} n + \frac{\omega_0 t}{2\phi_2^{\frac{1}{2}}} + \frac{\phi_1 + h\theta_1 + 2\pi p}{2\phi_2^{\frac{1}{2}}}\right]^2}$$

It should be noted that the expression for F may be written in terms of the absolute value of ϕ_2 where the sign of the exponent is the same as the sign of ϕ_2,

$$F = \sum_{n=-\infty}^{+\infty} e^{\pm i\left[|\phi_2|^{\frac{1}{2}} n + \frac{\omega_0 t}{2|\phi_2|^{\frac{1}{2}}} + \frac{\phi_1 + h\theta_1 + 2\pi p}{2|\phi_2|^{\frac{1}{2}}}\right]^2} \tag{5}$$

This function is periodic with an angular frequency

$$\omega_1 = \frac{\pi}{|\phi_2|}\omega_0 \tag{6}$$

The properties of this modulating function may be investigated by a Fourier analysis

$$F = \sum_{m=-\infty}^{+\infty} A_m e^{i m \frac{\pi}{\phi_2}\omega_0 t} \tag{7}$$

$$A_m = \frac{1}{T}\int_0^T F\, e^{-i m \frac{\pi}{\phi_2}\omega_0 t}\, dt \qquad \text{with } T \text{ large} \tag{8}$$

$$A_m = \frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{+i\left[\frac{m\pi(\phi_1 + h\theta_1 + 2\pi p)}{\phi_2} - \frac{m^2\pi^2}{\phi_2}\right]}$$

$$\times \int_0^T e^{\pm\left[\frac{\omega_0 t}{2|\phi_2|^{\frac{1}{2}}} + |\phi_2|^{\frac{1}{2}} n + \frac{\phi_1 + h\theta_1 + 2\pi p}{2|\phi_2|^{\frac{1}{2}}} - \frac{m\pi}{|\phi_2|^{\frac{1}{2}}}\right]^2} dt \tag{9}$$

If the exponent of the integrand passes through zero in the range of integration, the integral will be given by:

$$\frac{2|\phi_2|^{\frac{1}{2}}}{\omega_0}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(1 \pm i) \tag{10}$$

Outside of this range the integral will be very small. The number of values of n for which the exponent passes through zero is given by $\frac{\omega_0 T}{2|\phi_2|}$

Then

$$A_m = \frac{1}{|\phi_2|^{\frac{1}{2}}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(1 \pm i)\, e^{-i\left[\frac{m^2\pi^2}{\phi_2} - \frac{m\pi(\phi_1 + h\theta_1 + 2\pi p)}{\phi_2}\right]}$$

$$A_m = \left(\frac{\pi}{|\phi_2|}\right)^{\frac{1}{2}} e^{-i\left[\frac{m^2\pi^2}{\phi_2} - \frac{m\pi(\phi_1 + h\theta_1 + 2\pi p)}{\phi_2} \mp \frac{\pi}{4}\right]} \tag{11}$$

The amplitude of the fundamental component (A_0) of the frequency modulated accelerating voltage is then given by:

$$V_0 = \left(\frac{\pi}{|\phi_2|}\right)^{\frac{1}{2}} V_N \tag{12}$$

The amplitude of the other Fourier components of the modulating function is also of this value.

The system of constant radio-frequency voltages may also be shown to be exactly equivalent to an infinite number of frequency modulated voltages, with the spacing between frequencies being ω_1, and the amplitude of each voltage being $\left(\frac{\pi}{|\phi_2|}\right)^{\frac{1}{2}} V_N$

During certain values of t for every value of p the frequency modulated accelerating voltage (Eq. (3)) forms a region of stable synchrotron oscillations for a particle of total energy E_s and orbital frequency ω_s (angular velocity). The average rate of change of the orbital frequency of

a particle in this stable region is equal to the rate of change of the azimuthal angular frequency of the accelerating voltage (Eq. (4)). Hence the average rate of acceleration is determined by the rate of modulation of the azimuthal angular frequency.

$$\frac{d\omega}{dt} = -\frac{\omega_0^2}{2h\phi_2} = \frac{d\omega}{dE}\frac{\overline{dE}}{dt} = \frac{\omega_s K_s}{E_s}\frac{\overline{dE}}{dt} \tag{13}$$

where K_s is determined by the characteristics of the orbits in the magnet structure, and is defined to be

$$K_s = \frac{E}{\omega}\frac{d\omega}{dE} \quad \text{for} \quad E = E_s \quad \text{and} \quad \omega = \omega_s \tag{14}$$

The average rate of change of energy is given by

$$\frac{\overline{dE}}{dt} = -\frac{\omega_0^2 E_s}{2hK\phi_2\omega_s} \tag{15}$$

and is dependent on ϕ_2 , the coefficient of the quadratic term in the phase relation of the different frequencies.

As in a conventional frequency modulated accelerator, the average acceleration and the magnitude of the fundamental component of the frequency modulated voltage determine a synchronous phase angle.

$$\frac{2\pi}{\omega_s}\frac{\overline{dE}}{dt} = V_0 \sin\phi_s \tag{16}$$

where ϕ_s is the synchronous phase angle

$$V_0 \sin\phi_s = -\frac{\pi\omega_0^2 E_s}{hK\phi_2\omega_s^2} \tag{17}$$

The frequency of synchrotron oscillations in the fundamental component (A_0) of the accelerating voltage is given by:

$$\omega_p = \left[\frac{\omega_s^2 V_0 h \, |K \cos\phi_s|}{2\pi E_s} \right]^{\frac{1}{2}} \tag{18}$$

$$\omega_p = \omega_0 \left[\frac{|\cot\phi_s|}{2\,|\phi_2|} \right]^{\frac{1}{2}} \tag{19}$$

The ratio between the frequency of the lowest frequency component of the modulating function F and the frequency of synchrotron oscillations is given by:

$$\frac{\omega_1}{\omega_p} = \left[\frac{2\pi^2 |\tan\phi_s|}{|\phi_2|} \right]^{\frac{1}{2}} \tag{20}$$

Values of ϕ_s and ϕ_2 will be chosen to make this ratio in the neighborhood of 3 or 4. Typical values would be $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{8}$. Then the change in phase position produced by the modulating function will be small, and the energy and phase variations produced by the lowest frequency component in F may be calculated neglecting the change in the fundamental component due to the change in phase position. The energy variation produced by the A_1 term in the modulating function is given by:

$$\delta E = \frac{1}{\omega_1} \cdot \frac{\omega_s}{2\pi} V_0 = \frac{V_0 \, |\phi_2| \, \omega_s}{2\pi^2 \omega_0} \tag{21}$$

The phase displacement in azimuth produced by this energy variation is given by

$$\delta\theta = \frac{1}{\omega_1} \cdot \frac{\omega_s |K_s|}{E_s} \delta E = \frac{V_0 \, \phi_2^2 \, \omega_s^2 \, |K|}{2\pi^3 E_s \, \omega_0^2} \tag{22}$$

Using Eq. (17)

$$h\,\delta\theta = \frac{|\phi_2|}{2\pi^2 |\sin\phi_s|} \qquad \delta E = \left[\frac{-\phi_2 E_s V_0}{4\pi^3 h K \sin\phi_s} \right]^{\frac{1}{2}} \tag{23}$$

$h\delta\theta$ is the displacement in radio-frequency phase from the phase stable position. By proper choice of ϕ_s and ϕ_2, this phase displacement associated with the first harmonic of F may be made quite small. The phase displacement caused by the higher harmonics of F is reduced by m^2 and may be neglected.

Since a frequency modulated voltage with frequency equal to a given frequency occurs with an angular periodicity of ω_0, particles may be accepted into phase stable positions for acceleration with this periodicity. The fraction of time over which particles may be accepted into phase stable positions may be estimated from the ratio of the maximum allowable energy variation in a phase stable synchrotron oscillation, to the acceleration of the phase stable particle in one period. The maximum allowable energy variation in a stable synchronous oscillation is given by:

$$\Delta E^2 = \frac{E_s V_0}{\pi h K}\left[2\cos\phi_s - (\pi - 2\phi_s)\sin\phi_s\right] \tag{24}$$

The magnitude of the change of energy in one period is:

$$E_g = \frac{2\pi}{\omega_0}\left|\frac{dE}{dt}\right| = \frac{\pi E_s \omega_0}{h\,|K\,\phi_2|\,\omega_s} \tag{25}$$

Using Eq. (17)

$$E_g = \left[-\frac{\pi E_s V_0 \sin\phi_s}{h K \phi_2}\right]^{\frac{1}{2}} \tag{26}$$

then

$$\frac{\Delta E}{E_g} = \left[-\frac{\phi_2\left[2\cos\phi_s - (\pi - 2\phi_s)\sin\phi_s\right]}{\pi^2 \sin\phi_s}\right]^{\frac{1}{2}} \tag{27}$$

Also the ratio of the fluctuation in energy produced by the lowest frequency

component of F, to the energy spacing between groups of particles may be calculated.

$$\frac{\delta E}{E_s} = \frac{|\phi_2|}{2\pi^2 |\sin \phi_s|} \tag{28}$$

For numerical values $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{2}$

$$\frac{\Delta E}{E_s} = .26$$

For these values the phase and energy displacement produced by the lowest frequency component of the modulating function is:

$$h\,\delta\theta = \frac{\delta E}{E_s} = .113$$

For $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{4}$

$$\frac{\Delta E}{E_s} = .183$$

$$h\,\delta\theta = \frac{\delta E}{E_s} = +.056$$

The frequency spacing of the accelerating voltages may be determined by using Eq. (17) with $V_0 = \left(\frac{\pi}{|\phi_2|}\right)^{\frac{1}{2}} V_N$

Then

$$\omega_0 = \omega_s \left[\frac{h V_N}{\pi^{\frac{1}{2}} E_s} |\phi_2|^{\frac{1}{2}} |K \sin \phi_s| \right]^{\frac{1}{2}} \tag{29}$$

The energy spacing is given by:

$$E_0 = \frac{E_s \omega_0}{h |K| \omega_s} = \left[\frac{E_s V_N}{h\, \pi^{\frac{1}{2}}} |\phi_2|^{\frac{1}{2}} \left| \frac{\sin \phi_s}{K} \right| \right]^{\frac{1}{2}} \tag{30}$$

For the values $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{2}$

$$E_0 = \left[\frac{E_s V_N}{2 h |K|} \right]^{\frac{1}{2}} \tag{31}$$

For $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{4}$

$$E_o = \left[\frac{E_s V_N}{2^{\frac{3}{2}} R |K|}\right]^{\frac{1}{2}} \tag{32}$$

From the phase displacement concept, the multiple frequency rf system would also be expected to decelerate particles which have initial values of energy and phase outside the acceptance region for acceleration. For ϕ_2 small, the rate of deceleration would be small compared to the rate of acceleration. For $\phi_2 = \pi$ particles would be accelerated and decelerated in an identical manner.

Since the repetition frequency ω_o will be generally very high compared with conventional frequency modulated accelerators, the principle of multiple frequency acceleration may be used to design very high intensity accelerators.

As a first application of multiple frequency acceleration, we consider extending the energy range for protons of a conventional cyclotron, by use of several rf frequencies. We consider an rf system with 6 frequencies, applied to separate electrodes, as shown below.

For 100 kv peak voltage on each electrode $V_N \approx$ 200 kv. We take $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{2}$ and $\frac{E_s}{K_s} =$ 630 Mev. This assumes that the radially decreasing magnetic field produces a frequency change of 50% of that produced by the relativistic increase in mass. Then

$$E_o = \left[\frac{E_s V_N}{2 R |K|}\right]^{\frac{1}{2}} = \left(\frac{6.3 \times 10^8 \times 2 \times 10^5}{2}\right)^{\frac{1}{2}} = 8.0 \text{mev} \tag{33}$$

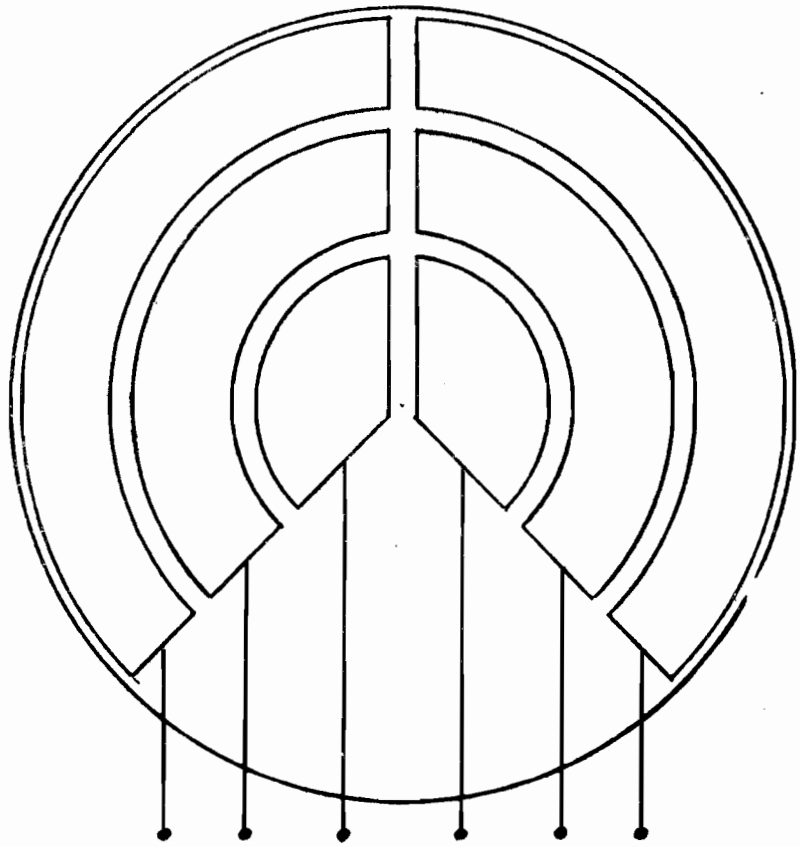

Fig. 1

Since six frequencies are used, the total energy would be about 48 Mev.

The frequency spacing is

$$\frac{8.0}{630} = 1.3\%$$

At a frequency of 20 m.c., the repetition rate would then be about 260 kc.

As a second application we consider the design of a proton accelerator using a FFAG strong-focusing magnet structure. The momentum compaction factor is assumed to be sufficiently small such that variations in radius may be neglected in determining changes in angular frequency. For non-relativistic particles with kinetic energy $T \ll m_0 c^2$, $\frac{E_s}{K} = 2T$.

We take $\phi_s = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{2}$ $h = 1$

then $$\omega_0 = \omega_s \left(\frac{V_N \, |K|}{2 E_s} \right)^{\frac{1}{2}} = \frac{\omega_s}{2} \left(\frac{V_N}{T} \right)^{\frac{1}{2}}$$

Since $$\omega_s \sim \text{Velocity} \sim T^{\frac{1}{2}}$$

then the frequency spacing is independent of energy for constant V_N, ϕ_s, ϕ_2 and the total number of frequencies required is given by:

$$N = \frac{\omega_{s\,MAX}}{\omega_0} = 2 \left(\frac{T_{MAX}}{V_N} \right)^{\frac{1}{2}} \tag{34}$$

We consider a 100 Mev proton accelerator with a radius of 150 inches, and a magnet structure with a compaction factor of .08. With injection at 400 kv, the total radial aperture is given by:

$$150 \left[\left(\frac{10^8}{4 \times 10^5} \right)^{\frac{1}{2} \times .08} - 1 \right] = 35 \text{ inches} \tag{35}$$

however, the back leg of the magnet for the return flux need only be about 12 inches. The rf system is taken to be of the drift tube type, with six sets of accelerating electrodes located around the circumference. The cross section of the vacuum chamber with accelerating electrodes would be somewhat as shown below.

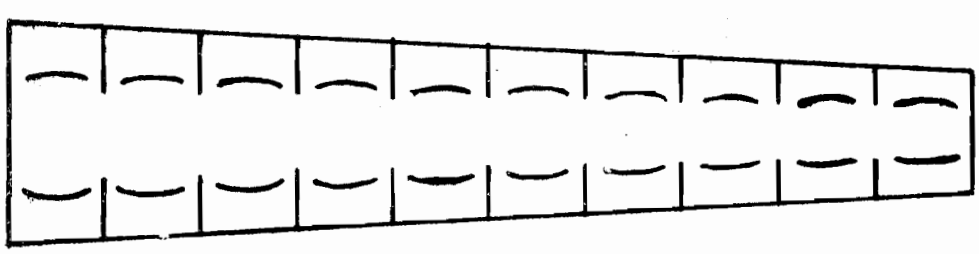

Fig. 2

As the voltages of the various frequencies are only applied in the region where they are effective, difficulties due to different harmonic orders of the same frequency are eliminated. We assume a peak voltage of 100 kv on each electrode, then as each electrode is about one-sixth of the circumference, V_N will be about 100 kv. Then

$$N \simeq 2\left(\frac{10^8}{10^5}\right)^{\frac{1}{2}} = 62 \text{ frequencies} \tag{36}$$

There will be about ten sets of electrodes located on a radius. The frequency range would be about 5.6 m.c. to 350 kc. A rough computation gives a total rf power requirement in the neighborhood of 2 mw for the system. Then 10% beam loading would be an average current of 2 m.a. and a beam power of 200 kw.

As an application of larger magnitude, we consider a 5 Bev FFAG proton accelerator with injection at 2 Mev. We take the average radius to be 2000 inches and a momentum compaction factor of 1/100. The total radial aperture is about

$$2000\left[\left(\frac{6\times10^9}{6.3\times10^7}\right)^{\frac{1}{100}} - 1\right] \simeq 95 \text{ inches} \tag{37}$$

The width of the magnet back leg would be about 20 inches. For relativistic energies

$$\frac{E_s}{k} = \frac{E_s^3}{(m_0c^2)^2}\left[1 - \left(\frac{m_0c^2}{E_s}\right)^2\right] \tag{38}$$

In this case the frequency spacing would vary over a wide range if ϕ_s, ϕ_2, V_N are maintained constant. We assume that the frequency spacing is varied in multiples of the smallest spacing, which occurs at the maximum energy, in

order to maintain ϕ_s , ϕ_2 approximately constant. The energy spacing between frequencies is given by:

$$E_0 = \left[\frac{V_N E_s}{2h|K|}\right]^{\frac{1}{2}} \quad \text{with} \quad \phi_s = \frac{\pi}{4}, \quad \phi_2 = -\frac{\pi}{2} \tag{39}$$

The number of frequencies will be given by:

$$N = \left(\frac{2h}{V_N}\right)^{\frac{1}{2}} \int_0^{T_{max}} \frac{dT}{\left[\frac{E_s}{|K|}\right]^{\frac{1}{2}}} \tag{40}$$

$$N = \left(\frac{2h m_0 c^2}{V_N}\right)^{\frac{1}{2}} \int_0^{T_{max}/m_0c^2} \frac{dx}{[x(1+x)(2+x)]^{\frac{1}{2}}}$$

For $T_{max}/m_0c^2 = 5$ the integral is approximately 1.83

$$N = 1.83 \left(\frac{2h m_0 c^2}{V_N}\right)^{\frac{1}{2}} \tag{41}$$

We take the harmonic order $h = 5$, with 30 frequencies located around one circumference with 100 kv on each electrode. Then

$$V_N = 100 \text{ Kv}$$

N = <u>578 frequencies</u>

Then there would be about 20 different frequency electrodes located along a radius.

It may be desirable to also have the smallest frequency spacing at the very low energies following injection. This would enable particles to be injected with a high duty factor. After the phase space has been reduced by adiabatic damping, the frequency spacing would be increased. Then only those phase stable positions in the intermediate frequency range which are accepted into the high frequency range would contain particles.

In all of these applications of multiple frequency acceleration, injection would be very easy due to the relatively high fractional energy gain per turn following injection. Beam extraction at high energy would present the usual difficulties.

For the 5 Bev proton accelerator a rough computation gives a total rf power requirement for about 600 frequencies with 100 kv peak voltage for each as about 20 mw. At 20% beam loading this would be an average current of .8 m.a. The average rate of acceleration is about 100 kv per turn so a total of about 5×10^4 turns is required. If the beam is not extracted at maximum energy and the highest frequency corresponds to an energy within the aperture, then the particles will be decelerated by the rf system after reaching maximum energy. An estimate from the area in phase space which is accelerated, gives the rate of deceleration to be 0.2 of the rate of acceleration with $\phi_3 = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{2}$. Then the total circulating current would be

$$6 \times .8 \times 10^{-3} \times 5 \times 10^4 = \underline{240 \text{ amperes}} .$$

If the magnet structure were suitably designed, the rf system could be used to accelerate particles in both directions and the circulating beams used for colliding beam experiments.

Multiple frequencies could also be used to design a high intensity electron accelerator. We consider a 300 Mev electron accelerator with an average radius of 150 inches. The momentum compaction factor is taken to be 1/40 and the injection energy to be 3 Mev, so as to be above the phase transition.

The total radial aperture is

$$150\left[\,(100)^{\frac{1}{40}} - 1\,\right] \simeq 18 \text{ inches} \tag{42}$$

The width of the back leg of the magnet for the return flux would be about 4 inches.

For relativistic electrons

$$\frac{E_s}{K} = \frac{T}{\alpha}$$

with α the momentum compaction factor. We assume the frequency spacing is varied in multiples of the smallest spacing so as to maintain ϕ_s, ϕ_2 approximately constant. The total number of frequencies is then given by

$$N = \left(\frac{2h\alpha}{V_N}\right)^{\frac{1}{2}} \int_0^{T_{max}} \frac{dT}{(T)^{\frac{1}{2}}} = 2\left(\frac{2h\alpha T_{max}}{V_N}\right)^{\frac{1}{2}} \tag{43}$$

We choose $h = 1$ and $V_N = 100\ \text{Kv}$. Then

$$N = 2\left(\frac{2 \times \frac{1}{40} \times 3 \times 10^8}{10^5}\right)^{\frac{1}{2}} = 25 \text{ frequencies} \tag{44}$$

Since the fractional frequency variation over the energy range is small in this case, either drift tube type of electrodes, or re-entrant cavities could be used for acceleration. The total rf power required may be about 2 mw, and an average current of 1 m.a. at 300 Mev could possibly be obtained.