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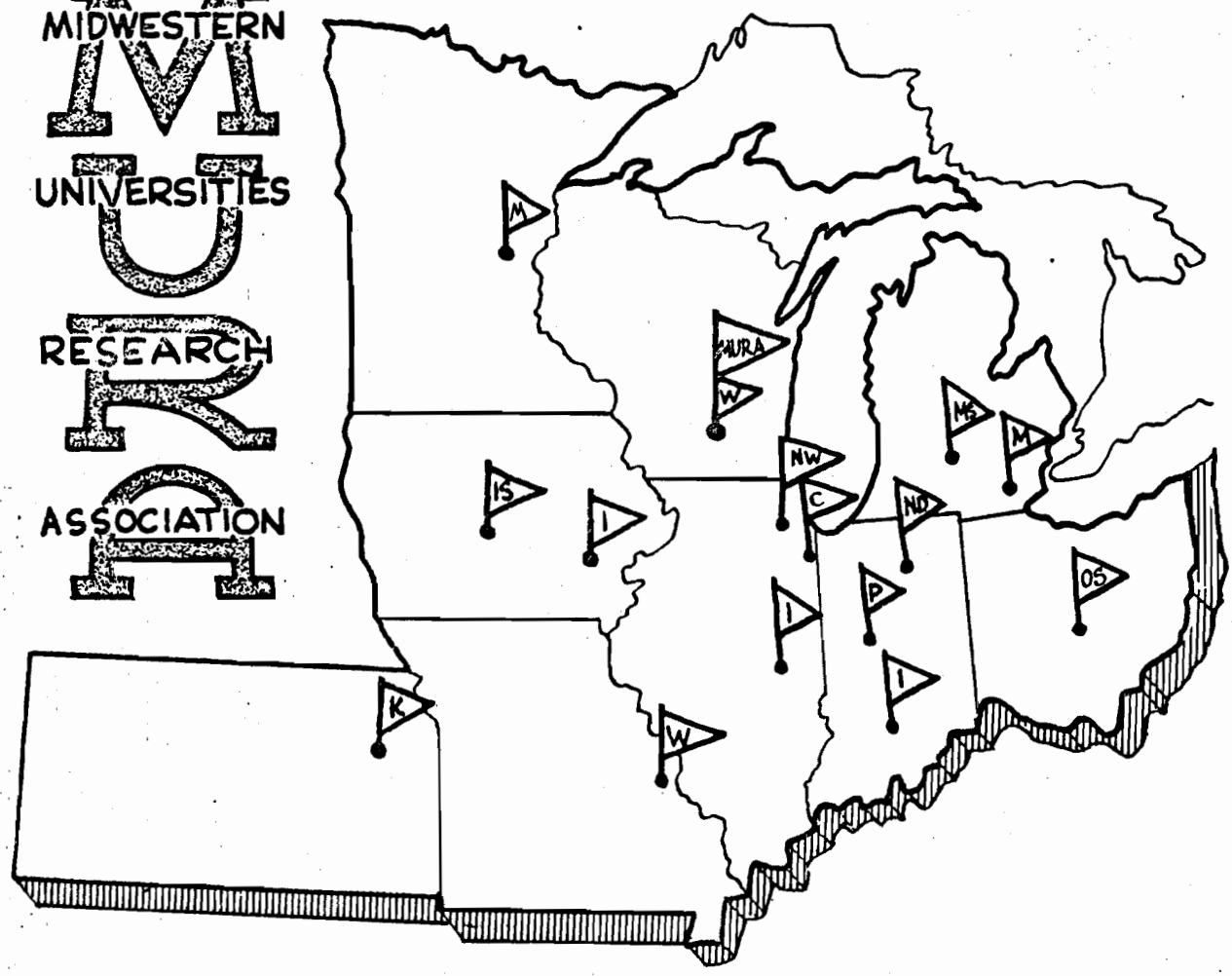

CONCERNING THE $\nu/N \longrightarrow 1/3$ RESONANCE, IV

THE LIMITING-AMPLITUDE SOLUTION OF THE EQUATION

$$\frac{d^2u}{d\phi^2} + (a + b\cos 2\phi)\,u + \frac{B_1}{2}(\sin 2\phi)\,u^2 = 0$$

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L. Jackson Laslett**

June 3, 1959

ABSTRACT

The equation shown in the title is reduced, by the transformations

$$v = \sqrt{\frac{N}{2\nu\beta}}\; u \quad \text{and} \quad t = \frac{N}{2\nu}\int_0^{\phi} \frac{d\phi}{\beta}\,, \quad \text{to the form}$$

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{1}{2}\left[\sum_{m=1} b_m (\sin 2mt)\right] v^2 = 0\,.$$

Use is made of the results of an earlier report, in which the characteristics of the limiting-amplitude solution of this latter equation were obtained by a variational procedure and by application of the Moser method, to obtain corresponding information concerning the solution $u(\phi)$ of the first equation. The analytic work is carried through terms of order $(1/3 - \nu/N)^2$ and applied to an example in which

$$a = 0.1262875 \qquad b = 1.15$$
$$\nu/N = 0.2997 \qquad B_1 = 1.$$

Comparisons with the results of direct digital computation for this example indicate the results of the analytic theory are within a few (2 to 4) percent of computed values.

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Lagrangian

$$L_o\,(du/d\phi,\ u;\ \phi)$$
$$= (1/2)\,(du/d\phi)^2 - (1/2)\,k\,(\phi)\,u^2 - (B_1/6)\,(\sin 2\phi)\,u^3\,, \tag{4a}$$

in which

$$k\,(\phi) \equiv a + b\cos 2\phi\,. \tag{4b}$$

The transformation to follow then makes use of the constant $2\nu/N$, where $2\pi\nu/N$ ($\equiv \sigma$) represents the change in phase of the solutions of the linearized eqn. (3) when $\Delta\phi = \pi$, and also employs the function $\beta(\phi)$ commonly employed in the theory of A-G accelerators.[3, 4] We then introduce the variables[5]

$$v = \sqrt{\frac{N}{2\nu\beta}}\ u \tag{5a}$$

$$t = \frac{N}{2\nu}\int_0^{\phi} \frac{d\phi}{\beta}\,, \tag{5b}$$

the transformation of the independent variable being such that in an interval $\Delta\phi = \pi$ [i.e., in one period of the coefficients of eqn. (3)], $\Delta t = \frac{N\sigma}{2\nu} = \pi$ and the period in terms of t accordingly is the same as in terms of ϕ.

The Lagrangian in terms of the new variables is taken to be

$$L_1\,(dv/dt,\ v;\ t) = \frac{2\nu\beta}{N}\,L_o$$
$$= \frac{1}{2}\left(\frac{dv}{dt}\right)^2 + \frac{\nu}{N}\,\frac{d\beta}{dt}\,v\,\frac{dv}{dt}$$
$$+ \frac{1}{2}\left[\left(\frac{\nu}{N}\right)^2\left(\frac{d\beta}{d\phi}\right)^2 - \frac{k}{2}\left(\frac{2\nu\beta}{N}\right)^2\right] v^2$$
$$- \frac{B_1}{6}\left(\frac{2\nu\beta}{N}\right)^{5/2}(\sin 2\phi)\,v^3\,. \tag{6}$$

The Lagrangian (6) is then modified by subtraction of a perfect differential, to eliminate the term containing $v \frac{dv}{dt}$:

$$L_2 (dv/dt, v; t) = L_1 - \frac{d}{dt}\left[\frac{\nu}{2N} \frac{d\beta}{d\phi} v^2\right]$$

$$= \frac{1}{2}\left(\frac{dv}{dt}\right)^2 - \frac{1}{2}\left(\frac{2\nu}{N}\right)^2 \left[\frac{\beta}{2} \frac{d^2\beta}{d\phi^2} - \frac{1}{4}\left(\frac{d\beta}{d\phi}\right)^2 + k\beta^2\right] v^2$$

$$- \frac{B_1}{6}\left(\frac{2\nu\beta}{N}\right)^{5/2} (\sin 2\phi)\, v^3$$

$$= \frac{1}{2}\left(\frac{dv}{dt}\right)^2 - \frac{1}{2}\left(\frac{2\nu}{N}\right)^2 v^2 - \frac{B_1}{6}\left(\frac{2\nu\beta}{N}\right)^{5/2} (\sin 2\phi)\, v^3, \qquad (7)$$

the last reduction being accomplished by virtue of the relation[6]

$$\frac{\beta}{2} \frac{d^2\beta}{d\phi^2} - \frac{1}{4}\left(\frac{d\beta}{d\phi}\right)^2 + k\beta^2 = 1 . \qquad (8)$$

The differential equation which follows from the Lagrangian (7) is seen to be

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{B_1}{2}\left(\frac{2\nu\beta}{N}\right)^{5/2} (\sin 2\phi)\, v^2 = 0 , \qquad (9)$$

and the associated Hamiltonian is

$$H = \frac{1}{2} p^2 + \frac{1}{2}\left(\frac{2\nu}{N}\right)^2 v^2 + \frac{B_1}{6}\left(\frac{2\nu\beta}{N}\right)^{5/2} (\sin 2\phi)\, v^3 \qquad (10)$$

with $p = dv/dt$. Accordingly, if one makes the expansion

$$B_1\left(\frac{2\nu\beta}{N}\right)^{5/2} (\sin 2\phi) = \sum_{m=1} b_m (\sin 2mt) , \qquad (11)$$

these results, (9) and (10), are in the form treated in II. We proceed, then, to an application of this analysis to a specific example in which, for comparison, computational solutions are available for the original differential equation. Specifically, we shall take

$$a = 0.126\,2875\,, \tag{12a}$$

$$b = 1.15\,, \quad \text{and} \tag{12b}$$

consider (as in I) the dependent variable to be so scaled that

$$B_1 = 1. \tag{12c}$$

The value of ν/N which is implied by this particular selection of values for a and b may be estimated analytically,[7] obtained from available tables,[4] or determined by a direct computation--in the present example we find

$$\nu/N = 0.2997\,, \tag{12d}$$

or substantially 0.3.

C. THE EXPANSION INVOLVING β

The function $\beta(\phi)$ may be estimated analytically,[7] obtained from tabulated[4] values of $\beta \cdot (\sin\sigma)$, or found by direct computation. In the present instance, with the governing parameters given by (12a, b), $\beta(\phi)$ itself may be represented by the expansion[8] (see Fig. 1)

$$\frac{2\nu}{N}\beta = 1.3956\,\Big[1 + 0.741\,13\cos 2\phi + 0.083\,56\cos 4\phi + 0.004\,54\cos 6\phi + \cdots\Big]\,. \tag{13}$$

It may be of interest to note in passing that the analytic results of reference 7 suggest that in the present case the quantity $\frac{2\nu}{N}\beta$ ranges between the maximum and minimum values (at $\phi = 0$ and at $\phi = \pi/2$, respectively) 2.539 and 0.474, while the values obtained by a direct computation are substantially 2.552 and 0.472.

In the present work we require the expansion of $B_1\left(\frac{2\nu\beta}{N}\right)^{5/2}\sin 2\phi$, as a trigonometric series in the variable t, with t related to ϕ by eqn. (5b).

The coefficients[8] of this expansion, (11), are as listed in Table I.

TABLE I

COEFFICIENTS, b_m, OF $\sin 2\,m\,t$ IN THE EXPANSION (11)

$a = 0.126\,2875$ $\qquad$ $b = 1.15$ $\qquad$ $B_1 = 1$

m	b_m
1	1.0645
2	1.3531
3	1.2396
4	0.9878
5	0.7278
6	0.5100
7	0.3450
8	0.2274
9	0.1470
$\geqslant 10$	< 0.01

These tabulated values may be employed, in application of the results given in II, to an examination of the expected limiting-amplitude solution to eqn. (3). The scale distortion in passing from the variable ϕ to the variable t is instrumental in effecting a pronounced peak in a plot of the (odd) function $\left(\frac{2\nu}{N}\beta\right)^{5/2} \sin 2\phi$ vs. t (Fig. 2), with a consequent enhancement of the higher-order Fourier coefficients b_m; the effect of the higher-order coefficients on the salient features of the phase plots, however, would not be expected to be great.

D. COMPARISON WITH COMPUTATIONAL RESULTS

For comparison with available computer results, we apply the procedure outlined above to the specific case for which* $a = 0.126\,2875$

$$b = 1.15$$

$$(\nu/N = 0.2997)$$

$$B_1 = 1,$$

particularly with respect to the location of the unstable fixed points which characterize the unstable equilibrium orbit at $t = 0$. In terms of the notation of II, then, we have

$$1/3 - \nu/N = 0.0336\,3333\ldots = 1.009/30\,, \tag{14a}$$

$$\alpha = 3.975\,962\,, \qquad \text{and} \tag{14b}$$

$$\eta_1 = 0.8201\,1582\,, \tag{14c}$$

making use of the values of b_m ($m \leq 9$) listed in Table I.

1. Location of Unstable Fixed Points

For the fixed point on the symmetry axis (at $t = 0$) we calculate**

$$v = 0 \tag{15a}$$

$$p = -\frac{128}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2 \eta_1 \left\{1 - \left[\frac{2}{1 - \nu^2/N^2} - \frac{1}{1 + 3\,\nu/N} - 2\sum_{m=2} m\,\frac{b_m}{b_1}\left(\frac{1}{m^2 - 9\nu^2/N^2} - \frac{1}{m^2 - \nu^2/N^2}\right)\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right\} \tag{15b}$$

to obtain

$$v = 0 \tag{15a'}$$

$$p = -0.2874\,, \tag{15b'}$$

*Eqns. (12 a - d).

**Eqns. (31a, b) of II.

again making use of the values of b_m ($m \leqslant 9$) in Table I. Similarly, for the fixed points situated to the right and left of the symmetry axis (for $t = 0$) we calculate*

$$v = \mp \frac{32\sqrt{3}}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1 \left\{1 - \left[\frac{2}{1 - \nu^2/N^2} - \frac{1}{1 + 3\nu/N} - 2\sum_{m=2} m\,\frac{b_m}{b_1}\left(\frac{1}{m^2 - 9\nu^2/N^2} - \frac{1}{m^2 - \nu^2/N^2}\right)\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right\} \tag{16a}$$

$$p = + \frac{64}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1 \left\{1 + \left[\frac{10}{1 - \nu^2/N^2} + \frac{1}{1 + 3\nu/N} + 2\sum m\,\frac{b_m}{b_1}\left(\frac{5}{m^2 - 9\nu^2/N^2} + \frac{1}{m^2 - \nu^2/N^2}\right)\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right\}, \tag{16b}$$

to obtain

$$v = \mp 0.4153 \tag{16a'}$$

$$p = + 0.2759\,. \tag{16b'}$$

To transform the quantities v, p, found above, to the quantities u, $P \equiv du/d\phi$, which pertain to eqn. (3) and which essentially constitute the working variables in the computational work, we note from eqns. (5a, b) that

$$u = \sqrt{\frac{2\nu\beta}{N}}\; v = 1.5975\; v \tag{17a}$$

and

---

*Eqns. (32a, b) of II.

$$P \equiv \frac{du}{d\phi} = \frac{d}{d\phi}\left(\sqrt{\frac{2\nu\beta}{N}}\ v\right)$$

$$= \frac{\nu}{N}\sqrt{\frac{N}{2\nu\beta}}\ \frac{d\beta}{d\phi}\ v + \sqrt{\frac{N}{2\nu\beta}}\ \frac{dv}{dt}$$

$$= \frac{p}{1.5975} \tag{17b}$$

when (as here, at $t = 0$, $\phi = 0$) $\frac{2\nu\beta}{N} = 2.552$ and $d\beta/d\phi = 0$ (Fig. 1).

The resulting predicted fixed-point coordinates and the corresponding values obtained from digital computation are presented in Table II. The latter values were obtained with the MURA IBM 704 computer, by use of the DUCK-ANSWER[9] program. A phase plot, obtained from the computational results for $\phi = 0$ (mod. π), is given in Fig. 3.

TABLE II

COORDINATES OF UNSTABLE FIXED POINTS, AT $\phi = 0$,
As Obtained from the Analysis of this Report and from Computer Results

$a = 0.126\,2875$ $\quad b = 1.15$ $\quad \nu/N = 0.2997$ $\quad B_1 = 1$

FIXED POINT	From Analysis		From Computer	
	u	$P \equiv du/d\phi$	u	$P \equiv du/d\phi$
On Symm. Axis	0	-0.1799	0	-0.1866
R and L of Symm. Axis	$\mp$ 0.6634	+0.1727	$\mp$0.6866	+0.1765

It is noted from Table II that the values found by use of our formulas are some two to four per cent less in magnitude than those given by the computer--a situation similar to that shown in Table VI of I for an example with $\nu/N = 0.3$.

2. Representation of the Unstable Equilibrium Orbit

Our application of the results of II to eqn. (9) gives us, of course, a trigonometric (sine) series for v(t), from which, for example, eqn. (16a) would follow. In the present example the pertinent coefficients for such a development of v(t), and the similar (cosine) coefficients calculated separately for p(t) by the expressions in II with which our present eqns. (15b) and (16b) are consistent, are listed in Table III (by use of Table I, considering $m \leq 9$).

TABLE III

COEFFICIENTS FOR A TRIGONOMETRIC EXPANSION OF v(t) AND p(t)
$m \leq 9$

Argument	m	Sine Coefficient in v	Cosine Coefficient in p
2 t/3	1	-0.477 435	-0.309 642
2 t	1	+0.018 056	+0.036 113
8 t/3	2	-0.005 580	-0.015 631
10 t/3	1	-0.003 329	-0.010 649
4 t	2	+0.005 343	+0.021 370
14 t/3	3	-0.001 687	-0.008 098
16 t/3	2	-0.001 567	-0.008 145
6 t	3	+0.002 148	+0.012 887
20 t/3	4	-0.000 665	-0.004 520
22 t/3	3	-0.000 744	-0.005 354
8 t	4	+0.000 959	+0.007 668
26 t/3	5	-0.000 291	-0.002 565
28 t/3	4	-0.000 362	-0.003 330
10 t	5	+0.000 451	+0.004 511
32 t/3	6	-0.000 135	-0.001 462
34 t/3	5	-0.000 180	-0.002 012
12 t	6	+0.000 219	+0.002 631
38 t/3	7	-0.000 065	-0.000 834
40 t/3	6	-0.000 091	-0.001 196
14 t	7	+0.000 109	+0.001 525
44 t/3	8	-0.000 032	-0.000 475
46 t/3	7	-0.000 046	-0.000 702
16 t	8	+0.000 055	+0.000 879
50 t/3	9	-0.000 016	-0.000 271
52 t/3	8	-0.000 024	-0.000 409
18 t	9	+0.000 028	+0.000 505
58 t/3	9	-0.000 012	-0.000 237

The conversion of $v(t)$ to $u(\phi)$ would appear to be rather tedious, involving as it does both the factor $\sqrt{\beta}$ and the non-linear relation between the independent variables t and ϕ. It is of interest to note from Table III, however, that $v(t)$ itself evidently should be rather well represented by its first one or two coefficients*--say by

$$v(t) \cong -0.477_{435} \sin 2t/3 + 0.018_{056} \sin 2t \,. \tag{18}$$

If a table of values of $u = \sqrt{\frac{2\nu\beta}{N}}\, v$, vs. ϕ, is constructed by hand computation, one finds that eqn. (18) suggests $u(\phi)$ should have a representation[8] in which the leading terms are roughly

$$u(\phi) \cong -0.533_9 \sin\frac{2\phi}{3} + 0.177_2 \sin\frac{4\phi}{3} + 0.015_5 \sin 2\phi$$
$$-0.040_0 \sin\frac{8\phi}{3} + \cdots \quad ; \tag{19}$$

this result, eqn. (19), may be compared with the direct computer analysis[10] of the limiting-amplitude solution for eqn. (3), namely (with $B_1 = 1$):

$$u(\phi) = -0.55231 \sin\frac{2\phi}{3} + 0.18429 \sin\frac{4\phi}{3} + 0.02167 \sin 2\phi$$
$$-0.04919 \sin\frac{8\phi}{3} + 0.00575 \sin\frac{10\phi}{3} + 0.00283 \sin 4\phi$$
$$-0.00140 \sin\frac{14\phi}{3} + \cdots \,. \tag{20}$$

As with the data of Table II, it is seen that the major calculated coefficients in the representation (19) are some three or four per cent less than the corresponding directly-computed values shown in eqn. (20).

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*Cf. the result of the numerical solution of eqns. (8a - c) in Sect. C 1 of I, or the computer results given by eqn. (12a) of that report ($\nu = 0.3$).

E. REFERENCES AND NOTES

1. L. Jackson Laslett, MURA-452 (April 13, 1959), herein designated as I.
2. L. Jackson Laslett, MURA-459 (May 20, 1959), herein designated as II.
3. β is a characteristic of the linearized eqn. (3) such that, in the notation employed in the Belford tables,[4] $\beta = M_{12}/\sin\sigma = 1/\Phi'$, where Φ is the "phase function."
4. G. Belford, L. Jackson Laslett, and J. N. Snyder, "Table Pertaining to Solutions of a Hill Equation," MURA Notes (April 3, 1956).
5. Cf. E. D. Courant and H. S. Snyder, Annals of Physics 3, 1-48 (January, 1958); eqns. (4.4) and (4.5), p. 18.
6. This relation (8) is implied by results appearing in
   (i) L. Jackson Laslett, MURA Notes (28 October 1954);
   (ii) L. Jackson Laslett, MURA-206 (November 21, 1956)--Appendix A; or
   (iii) Reference 5, Section 3(a).
7. L. Jackson Laslett and A. M. Sessler, MURA-252 (April 10, 1957). In this report an approximate solution of the linear (A-G) equation is given in terms of circular functions, explicitly those of argument $\nu\theta$, $(N \pm \nu)\theta$, and $(2N \pm \nu)\theta$, with $N\theta$ representing 2ϕ.
8. The Fourier expansion of tabulated function values is aided by use of the FORANAL program-- J. N. Snyder, FORANAL (IBM Program 52), MURA-228 (1957).
9. J. N. Snyder, DUCK-ANSWER (IBM Program 75), MURA-237 (1957). In the actual computations the value of B_1 was 1.15; the computer results for u and $du/d\phi$, accordingly, were multiplied by the factor 1.15 to obtain the values cited in the present report.
10. John McNall, DUCKNALL (IBM Program 219), MURA-438 (1958).

Fig. 1

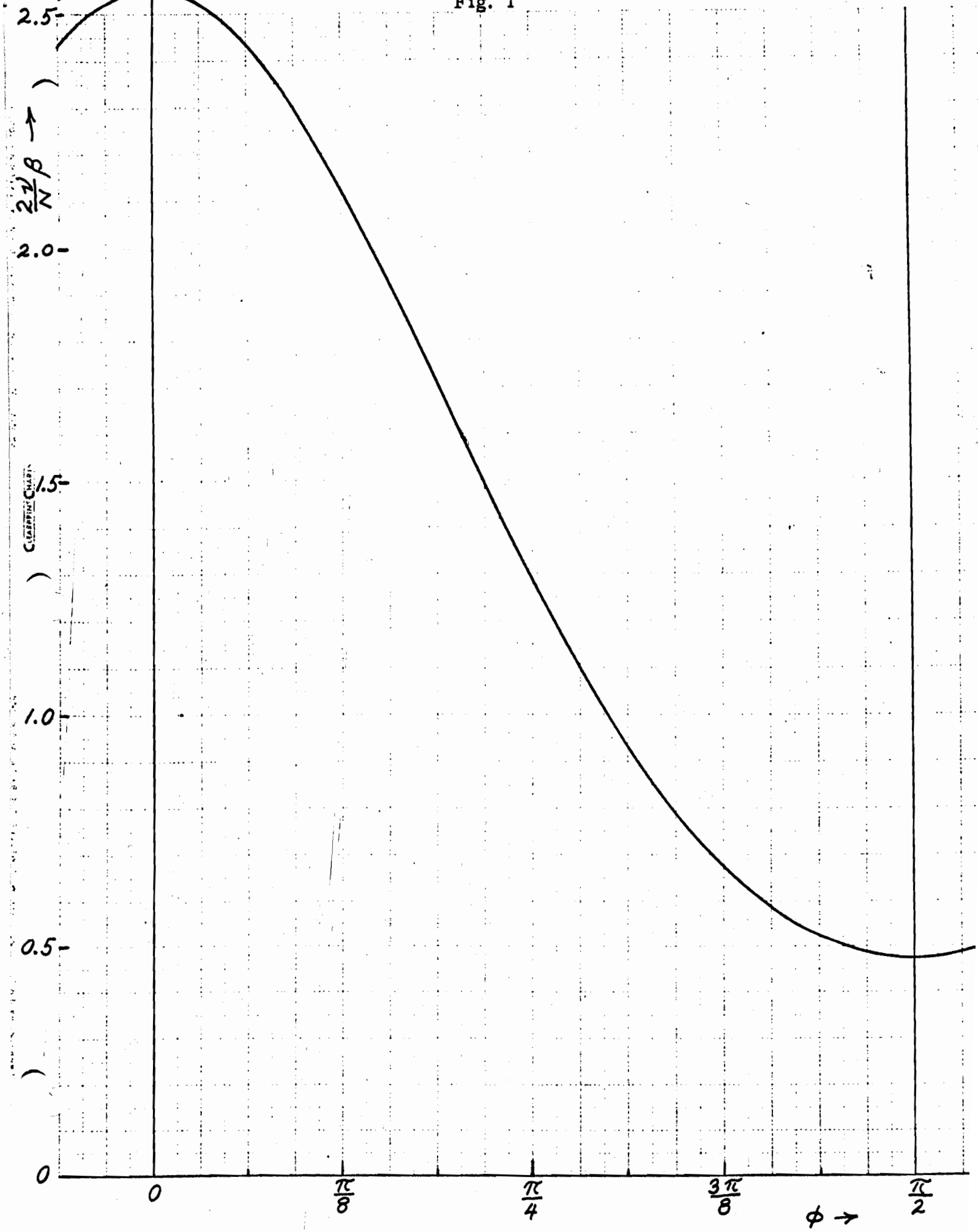

2.5
2.0
1.5
1.0
0.5
0
$\frac{2\nu}{N}\beta \rightarrow$
0
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
$\phi \rightarrow$

Fig. 2

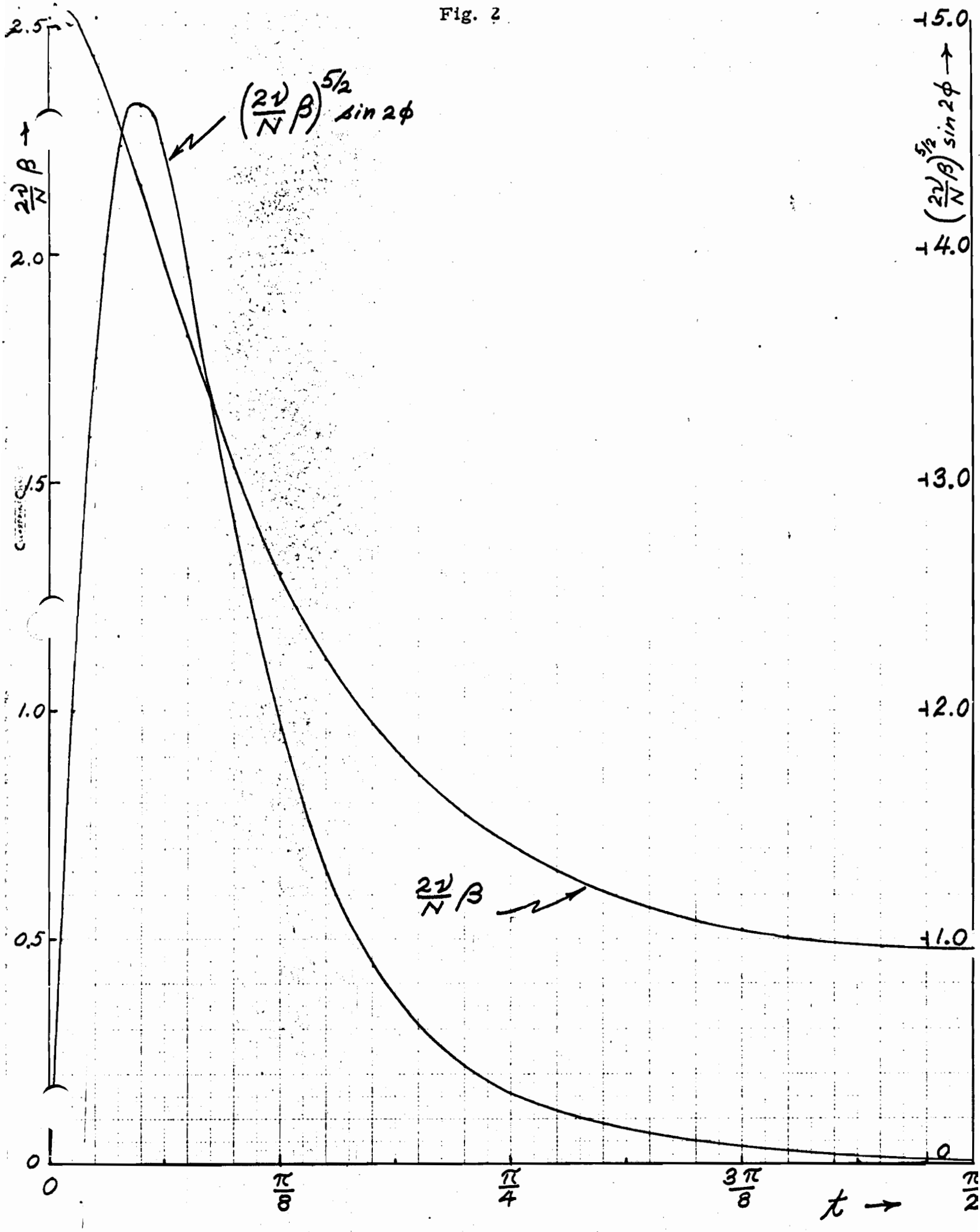
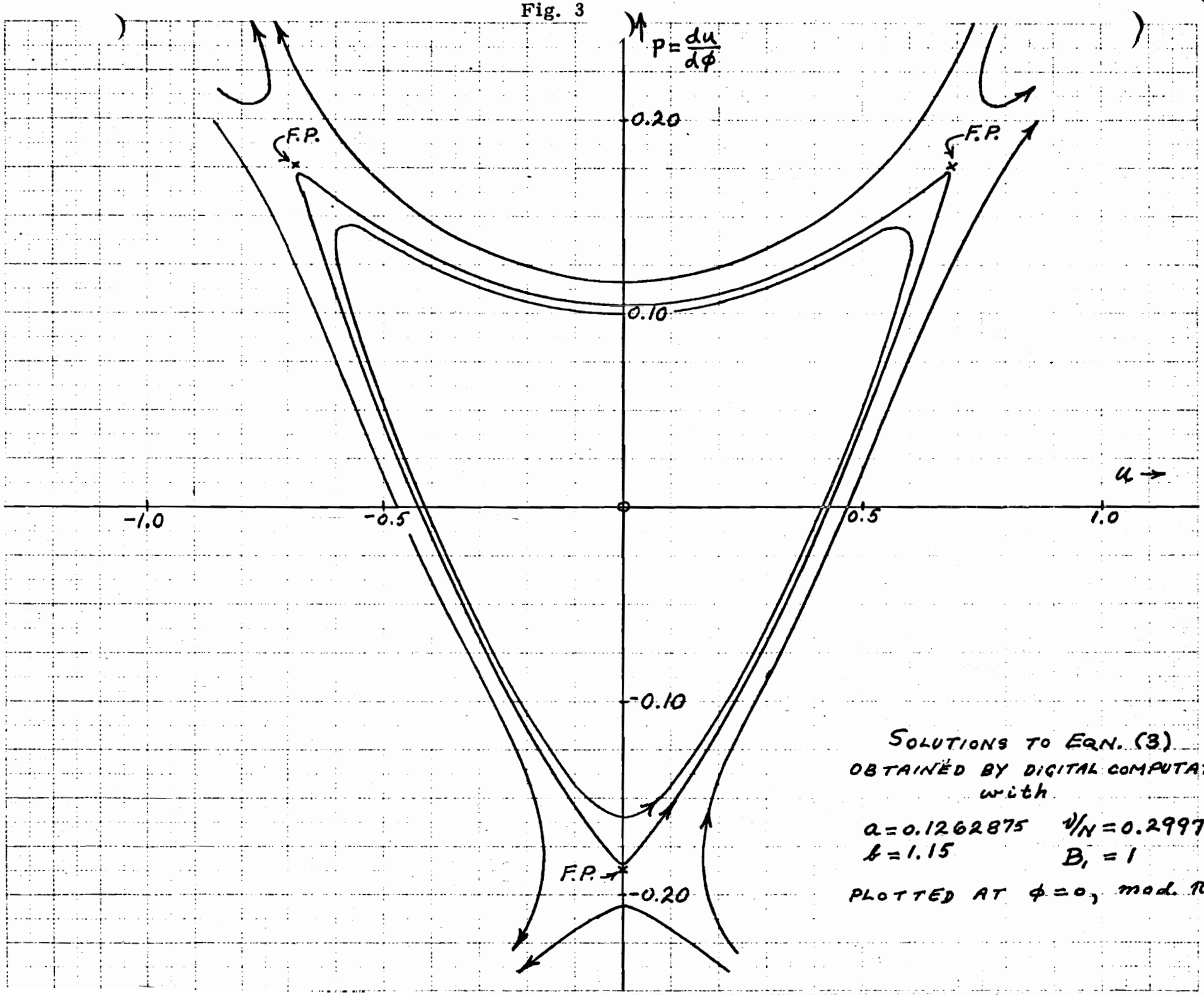

2.5
2.0
1.5
1.0
0.5
0
$\frac{2\nu}{N}\beta \rightarrow$
$\left(\frac{2\nu}{N}\beta\right)^{5/2} \sin 2\phi$
$\frac{2\nu}{N}\beta$
+5.0
+4.0
+3.0
+2.0
+1.0
0
$\left(\frac{2\nu}{N}\beta\right)^{5/2} \sin 2\phi \rightarrow$
0
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
$t \rightarrow$

Fig. 3


$p = \frac{du}{d\phi}$
0.20
0.10
-0.10
-0.20
F.P.
F.P.
F.P.
$u \rightarrow$
-1.0
-0.5
0.5
1.0
SOLUTIONS TO EQN. (3)
OBTAINED BY DIGITAL COMPUTATION
with
$a = 0.1262875$
$\nu/N = 0.2997$
$b = 1.15$
$B_1 = 1$
PLOTTED AT $\phi = 0$, mod. π