MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

HIGH CURRENT EFFECTS IN FFAG ACCELERATORS

F. T. Cole, R. O. Haxby, L. W. Jones,** F. E. Mills,

C. E. Nielsen,+ A. M. Sessler,+ K. R. Symon and K. M. Terwilliger**

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ABSTRACT

The limitations on large currents and current densities in FFAG accelerators are reviewed. These limitations come from two sources, Liouville's theorem and space charge effects. Theoretical performance figures are given for typical 15 Bev radial and spiral sector accelerators.

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**University of Michigan

+The Ohio State University

I. INTRODUCTION

The high intensity possibilities inherent in the DC fields of FFAG accelerators have brought new interest to problems related to large particle densities. Some thought has been given in MURA to these problems. While it is certainly not true that they can be regarded as solved, some progress has been made in understanding the limits which such particle densities place on achievable output. These results are scattered through MURA documents having widely varying degrees of availability. It is the purpose of this report to collect these considerations for reference.

Let us note first that intensity goals can differ markedly among various applications. For example, in colliding beam use the desirable goal is mainly current density rather than total current, since the interaction rate relative to background depends on density. If extraction of a primary beam is desired, current density in phase space is desirable. For some single beam experiments with internal targets, maximum total current is the goal.

The restrictions which large particle densities put on orbit dynamics fall into two classes, effects due to Liouville's theorem and effects due to the Coulomb interactions between particles, which we shall call roughly "space charge" effects. We shall discuss them separately, though there are cases, as shall see, where they interact. We shall also limit our discussions to FFAG Accelerators.

II. LIOUVILLEAN RESTRICTIONS

Ideally, the mechanical system which is taken to represent particle motion in an accelerator is Hamiltonian and so Liouville's theorem applies;

the phase space volume occupied by a group of particles is a constant of the motion, as is the phase space density of particles. Then the density in phase is (ideally) completely determined by the original source of particles.

It is clearly impossible to exceed the Liouvillean limit unless some damping mechanism is present. Such a mechanism is the synchrotron radiation, which has important effects in electron accelerators, but is quite negligible for proton accelerators below 100 Bev. There are, however, mechanisms which reduce the phase density. Gas scattering increases the phase volume occupied by a given number of particles, thereby decreasing phase density. Even without dissipative forces, the useable density can be decreased by non-linear instabilities which "filament" the conserved volume so that some of it is useless and by transient phenomena, such as noise in the rf accelerating system. Thus in practice, the original source of particles determines an upper limit through Liouville's theorem.

To good approximation, the phase space volumes occupied by betatron and synchrotron oscillations are independent. This is true even with accelerating voltages which depend on radius, as has been shown by Robinson.[1] Non-linear forces can introduce coupling between the radial and vertical modes of betatron oscillations, but these effects are small except when close to coupling resonances. We can discuss the phase volume as roughly three separate phase areas.

Betatron oscillation phase space can be described by the radial and vertical amplitudes a_x and a_y (in centimeters, say) and the momenta p_x and p_y. If we call p the total momentum, p_x/p is the sine of the angle

which the projection of an orbit on the median plane makes with the equilibrium orbit (which lies in the median plane) and p_y/p is the sine of the angle which the projection of an orbit on the vertical plane tangent to the equilibrium orbit makes with the median plane.

The momenta are given in terms of the appropriate velocities by

$$\begin{aligned} p_x &= m\,\gamma\,\dot{r} \\ p_y &= m\,\gamma\,\dot{z} \\ \gamma &= (1-\beta^2)^{-1/2} \end{aligned} \tag{1}$$

As a particle is accelerated, its mass changes and ω, its frequency of revolution, changes. In the linear betatron oscillation approximation, the quantities

$$\begin{cases} J_x = m\,\gamma\,\omega\; a_x^2/\beta_x(\theta) \\ J_y = m\,\gamma\,\omega\; a_y^2/\beta_y(\theta) \end{cases} \tag{2}$$

are adiabatic invariants. β_x and β_y are functions with the period of the magnets which describe the A. G. oscillations. If we average over one magnet period $\left\langle \frac{1}{\beta_x} \right\rangle = \nu_x$ and $\left\langle \frac{1}{\beta_y} \right\rangle = \nu_y$, where ν_x and ν_y are the numbers of radial and vertical betatron oscillation waves per revolution, which must be constant during acceleration in order to avoid resonances. Then during acceleration a_x and a_y vary as

$$(m\,\gamma\,\omega)^{-1/2} \sim B^{-1/2} \tag{3}$$

where B is the magnitude of the average guide field. This is just the traditional adiabatic damping.[2] The momenta p_x and p_y vary as $B^{1/2}$ (since phase space is conserved), while the angles p_x/p and p_y/p vary as $(r^2B)^{-1/2}$ during acceleration. In a scaling FFAG accelerator non-linear amplitude

stability limits are proportional to radius and momentum stability limits to p, so that the region of phase space occupied by particles shrinks relative to the stability limits during acceleration.

In an FFAG accelerator, synchrotron oscillation theory can be written in Hamiltonian form.[3] The equations of motion are

$$\begin{aligned} \dot{W} &= e\,V \sin\varphi \\ \dot{\varphi} &= 2\pi\, h\,(f - f_o) \end{aligned} \tag{4}$$

where $h\,f_o$ is the oscillator frequency, f the particle frequency, h is the harmonic of the oscillator, φ the particle phase relative to the oscillator and

$$W = \int \frac{dE}{f} \tag{5}$$

is essentially the kinetic angular momentum about the z axis. Eqs. (4) follow from the Hamiltonian

$$H = e\,V \cos\varphi + 2\pi\, h \left[E(W) - W f_o \right] \tag{6}$$

Then, neglecting interaction between betatron and synchrotron oscillations, the area in $W - \varphi$ space occupied by a group of particles is a constant of the motion. If ΔW is small, it is related to the energy spread ΔE by

$$\Delta W = \frac{\Delta E}{f} \tag{7}$$

When an rf voltage is present, curves of constant H in $W - \varphi$ space are closed curves for certain values of H. H depends on time only through V and f_o. If these change adiabatically, particles remain on curves of constant H and therefore execute stable oscillations when on closed curves. The region of closed curves is called a "bucket." The area of a bucket

whose frequency is being modulated (a "moving" bucket) is

$$A = \frac{8}{f_o} \left[\frac{2\,E\,e\,V}{\pi\,h\,K} \right]^{1/2} \alpha_3\,(\Gamma), \tag{8}$$

where

$$\begin{cases} K = \dfrac{E}{f}\,\dfrac{df}{dE} \\ \Gamma = \sin \varphi_s \\ \alpha_3(\Gamma) \cong 1 - \Gamma\,, \end{cases} \tag{9}$$

φ_s being the stable phase angle. A bucket must of course be larger in phase area than the group of particles it is accelerating or it will lose some.

We now discuss the Liouvillean restrictions.

A. Restrictions at Injection

It is better at injection to fill betatron phase space, because betatron oscillations can aid in missing the injector. Trying to spread particles in synchrotron phase space would require inconveniently large voltages to attain enough bucket area to contain the energy spread, since bucket area varies as $\sqrt{V}$, from (8). Instead, one fills betatron phase space with the rf off, then adiabatically turns on the rf to capture all the particles.

The beam from an injector (assumed circular in cross sections), has a given half-width Δx, angular spread $\pm\,\Delta\theta$ and energy spread ΔE. The phase area occupied per betatron oscillation dimension is approximately $\pi\,p\,\Delta x\,\Delta\theta$, assuming that the area occupied in phase space is an ellipse with coordinate dimensions Δx and momentum dimension $\Delta p = p\,\Delta\theta$, and the total betatron phase volume occupied in the accelerator by one injected turn is

$$v_i = (\pi p \Delta x \Delta \theta)^2 \tag{10}$$

The accelerator has given maximum radial and vertical betatron oscillation amplitudes, a_{xo} and a_{yo}, due either to the finite aperture or to a non-linear stability limit. The smaller of the two (aperture or stability limit) for each dimension determines the maximum amplitude.* Corresponding to the maximum amplitudes are maximum momenta. If the oscillations are quasi-linear, these are

$$\begin{aligned} p_{x_o} &\cong p\, \nu_x \; a_x / r_1 \sqrt{f_x} \\ p_{y_o} &\cong p\, \nu_y \, a_y / r_1 \sqrt{f_y} \end{aligned} \tag{11}$$

where r_1 is the injection radius and f_x and f_y are the beat factors due to alternating gradient focusing ($\sqrt{f_x} \sim \sqrt{f_y} \sim 2$ in most cases).

The total betatron phase space available is then

$$\begin{aligned} V_i &\cong \pi\, a_{x_o} p_{x_o} \pi\, a_{y_o} p_{y_o} \\ &\cong \frac{\pi^2 p^2 \nu_x \nu_y a_{x_o}^2 a_{y_o}^2}{r_1^2 \sqrt{f_x f_y}} \end{aligned} \tag{12}$$

Clearly, Liouville's theorem restricts the number of turns which can be injected into the betatron phase space. This number is

$$n_B \leqslant \frac{V_i}{v_i} \tag{13}$$

If the injector current during injection is I_1, the total circulating current

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*For radial motion, the non-linear stability limit is usually smaller, while for vertical motion the aperture is usually smaller. In most cases the effect of coupling resonances ensures that a_{x_o} and a_{y_o} are of the same order.

which can be injected is $n_B I_1$. This injected beam has an energy spread ΔE_1 and occupies area $2\pi \Delta E_1/f_1$ in synchrotron phase space.

One may also consider filling separately the radial and vertical betatron phase spaces. If, for example, the beam is injected onto the median plane, no attempt is made to fill vertical betatron phase space at all. How far one should attempt to go in filling betatron phase space is usually determined by the space charge limit, which is discussed in Section III below.

It is hoped that 50 Mev linear accelerators now being built will give a 5 ma output current with $\Delta x \cong \pm 0.5$ cm. $\Delta\theta \cong \pm 1.5 \cdot 10^{-3}$, $\Delta E \cong 50$ kev, giving a phase area $0.75 \cdot 10^{-3} \pi$ p per betatron mode. A typical 15 Bev radial sector FFAG has $a_{x_o} \cong a_{y_o} \cong 15$ cm, $\nu_x \cong 25$, $\nu_y \cong 4$, $r \cong 2.10^4$ cm, $f_x \cong f_y \cong 4$. Then $V_{i_x} \cong 0.14\, p\, \pi$ and $V_{i_y} \cong 0.02\, \pi\, p$, so that radial betatron phase space alone can take 190 turns and vertical 30. A typical 15 Bev spiral sector FFAG has $a_{x_o} \cong a_{y_o} \cong 7.5$ cm, $\nu_x \cong \nu_y \cong 8$, $r \cong 7.5 \cdot 10^3$ cm, $f_x \cong f_y \cong 4$. Then $V_{i_x} \cong V_{i_y} \cong 0.03\, \pi\, p$, so that radial and vertical betatron phase spaces can each accommodate about 40 turns. We shall see later that about 130 turns fill the radial sector FFAG to its space charge limit, while about 460 turns are required to fill the spiral sector FFAG to its space charge limit.

B. Restrictions on Accelerated Beam

The time-average accelerated current is clearly determined by the restriction on injected current. Injection takes place during a time interval n_B/f_1. If the repetition time of the acceleration cycle is T, the accelerated time-average current, assuming no losses during acceleration, is $n_B I_1/f_1 T$.

The restrictions on current density in a stacked beam, which are of interest for colliding beams, are more complicated. There is first a Liouvillean restriction in synchrotron phase space. If a pulse of particles is injected at energy E_1 with an energy spread ΔE_1, this pulse occupies an area

$$A_1 = 2\pi \Delta W_1 = 2\pi \Delta E_1/f_1, \tag{14}$$

in the $W - \varphi$ plane, where f_1 is the frequency of revolution at injection. If it is desired to stack at energy E_2 within an energy spread ΔE_2, the phase area available is

$$A_2 = 2\pi \Delta W_2 = 2\pi \Delta E_2/f_2 \tag{15}$$

f_2 being the revolution frequency at energy E_2. Then at most

$$n_s = \frac{A_2}{A_1} = \frac{f_1 \Delta E_2}{f_2 \Delta E_1} \tag{16}$$

pulses can be stacked in the energy interval ΔE_2.

Liouville's theorem also demands that care be taken when a bucket approaches the stack. Since phase points move like an imcompressible fluid, if empty area is brought up to the stack, an equal area must be displaced downward in energy. When an area A moves through the stack, the previously stacked particles must displace in the opposite direction by an average amount given by Eq. (7):

$$\langle \Delta E \rangle_{av} = \frac{Af}{2\pi} \quad . \tag{17}$$

It is therefore important to arrange that the bucket area A be no larger than is needed to contain the particles being accelerated. As the stacking process continues, the energy width of the stack increases for two reasons. First, the average displacement given by Eq. (17) implies that after n buckets have been delivered to the stack, its width will be at least $n \langle \Delta E \rangle_{av}$. Secondly the particles phase displaced by a moving bucket have a statistical distribution in energy about the mean value given by Eq. (17). This energy straggling results in an additional increase in stack width proportional to $n^{1/2}$.

From the constancy of synchrotron phase area, the current I_2 at energy E_2 resulting from a current I_1 at energy E_1 is

$$I_2 = I_1 \frac{\Delta E_2}{\Delta E_1} \tag{18}$$

where ΔE_2 is the final energy spread and ΔE_1 the initial energy spread, which is determined by the injector. The final energy spread ΔE_2 occupies a certain spread in radii Δr_2. In a scaling FFAG,

$$\Delta E_2 = (k + 1) \beta_2^2 \gamma_2 \frac{\Delta r_2}{r_2} mc^2 \tag{19}$$

The particles were injected with a maximum radial betatron oscillasion amplitude a_1. This amplitude has been adiabatically damped to

$$a_2 = a_1 \left(\frac{B_1}{B_2}\right)^{1/2} = a_1 \left(\frac{r_1}{r_2}\right)^{k/2} \tag{20}$$

since $B \sim r^k$. As groups of particles are stacked close to each other in energy and radius, their betatron oscillations move particles in and out radially around the equilibrium radius for a given energy, so that both betatron and synchrotron phase spaces contribute to the current density. The center of the stack will have higher current density because particles from the edges will spend some fraction of their time there on betatron oscillations. All particles stacked with an energy difference such that $\Delta r_2 < 2 a_2$ will add to this central current density. Then clearly the maximum current density occurs when $\Delta r_2 = 2 a_2$. A plot of current density vs. radius will then be a triangle. If more groups are added to the stack, so that $\Delta r_2 > 2 a_2$ they will add to the total current, but not to the current <u>density</u>. A plot of current density vs. radius for $\Delta r_2 > 2 a_2$ will then be a trapezoid with a

horizontal top. The maximum current density is then (neglecting factors of order unity which depend on the shape of the beam)

$$J_{max} = \frac{I_2}{\pi a_2^2} \cong I_1 \frac{(k+1)\beta_2^2 \gamma_2 mc^2}{\Delta E_1 a_2 r_2} \tag{21}$$

Terwilliger[4] has shown that the maximum current density can be increased by introducing a non-scaling field perturbation at the stacking energy. If done properly, a coherent betatron oscillation is induced which brings orbits of all energies within ΔE_2 through a narrow range of radii.

III. SPACE CHARGE RESTRICTIONS

We shall calculate space charge effects with a simple model. The main effect on betatron oscillations is to increase the wavelengths of oscillations because of the repulsive electric force, which pushes a particle away from the center of the beam.

At first we shall neglect A.G. effects, as well as the effects of charges of opposite sign collected by the beam and of images in the conducting walls. We take a cylindrical beam at radius r with minor radius a. If ρ is the charge density in the beam, the electric field at radius x in the beam can be found by applying Gauss' law

$$\int \nabla \cdot \underline{E} \, d\tau = \int E_n \, d\sigma \tag{22}$$

to the Maxwell equation

$$\nabla \cdot \underline{E} = 4\pi\rho \tag{23}$$

Since we are interested in the transverse force, we take the integrals of (22) to be over a unit length of the beam. Then

$$E_n = 2\pi\rho x \tag{24}$$

To this repulsive electrostatic force must be added the attraction of parallel currents, which is just $-\beta^2$ times the electric force. For relativistic particles space charge effects are very small, approaching zero as v approaches c. Note that for two beams in opposite directions, as in a colliding beams accelerator, the electric and magnetic forces add, so that there are large beam-beam forces. Then we must add to each transverse equation of motion a force

$$\frac{2\pi\rho e}{\gamma^2}$$

times the displacement from the central orbit. If we denote by x the relevant betatron oscillation variable, the linearized smooth approximation equation of motion is

$$m\gamma \frac{d^2x}{dt^2} + m\gamma\omega^2\nu_0^2 x = \frac{2\pi\rho e}{\gamma^2} x \tag{25}$$

and if we change to $\theta = \omega t$ as independent variable, (25) becomes

$$\frac{d^2x}{d\theta^2} + \left(\nu_0^2 - \frac{2\pi e\rho}{m\omega^2\gamma^3}\right) x = 0 \tag{26}$$

The frequency has been lowered to

$$\nu^2 = \nu_0^2 - \frac{2\pi e\rho}{m\omega^2\gamma^3}$$

Charge can be added to the beam until the frequency has changed enough to reach a resonance. The total charge density is then

$$\rho = \frac{m\omega^2\gamma^3}{2\pi e} 2\nu\Delta\nu \tag{27}$$

where $\Delta\nu$ is the allowable frequency change. The total charge in the beam is

$$Q = \rho \cdot 2\pi r \cdot \pi a^2$$

and the total current is

$$I = \frac{Q\omega}{2\pi} = I_o \nu \Delta\nu \left(\frac{a}{r}\right)^2 (\beta\gamma)^3 \tag{28}$$

where $I_o = \frac{mc^3}{e} = 3.129 \cdot 10^7$ amps for protons and we have used $\omega = \frac{\beta c}{r}$. Note that we have done this calculation for one betatron oscillation dimension. From (28), the dimension with the smaller ν dominates. The space charge limited current depends strongly on energy and in practically all cases of interest the space charge limit at injection dominates. The effect of the AG. ripple has been calculated by Jones.[5] In cases of practical interest it is small.

For the 15 Bev radial sector FFAG discussed above, the space charge limit at injection (50 Mev) is 0.64 amps (with $\Delta\nu = \frac{1}{4}$). For the spiral sector FFAG, the space charge limit is 2.3 amps. The smaller size and consequent smaller wavelengths in the spiral sector are the reason for the difference. The radial machine can be filled to its space charge limit using only radial betatron phase space, while the spiral machine requires both radial and vertical filling. If they can be filled at injection to their space charge limit, a reasonable repetition rate of 10 pulses accelerated per second give for both radial and spiral sector accelerators a time-average output current of over 60 microamperes.

When the injected beam is bunched by an rf voltage, the local current increases. Then the space charge limit (28) is decreased by a factor F, which is the fraction of the azimuth in which the beam is bunched. F can be calculated in terms of bucket areas. It is approximately

$$F \cong 0.72 \sqrt{\frac{A_o}{A_1}} \tag{29}$$

where A_o is the area in W - φ space occupied by the beam and A_1 is the area given by (8) with $\alpha_3 = 1$. In practice $F = \frac{1}{4}$ is reasonably attainable. This will reduce the time-average output current attainable to about 15 microamperes for either radial or spiral accelerators.

If we substitute (28) for the I_1 of (21) and use the adiabatic damping law (20), the maximum theoretical current density can be written

$$J = I_o \frac{a_2 k \nu \Delta \nu F \gamma_2^2}{P r_2^3} \tag{30}$$

where P is the relative energy spread of the injector. We have made the approximations $\beta_2 \cong 1$, $\nu_1 \cong 1 + \frac{1}{2}\beta_1^2$, $k \gg 1$.

In both the 15 Bev accelerators discussed above, adiabatic damping reduces the amplitudes by a factor of approximately 7 during the acceleration from the 50 Mev injection energy to 15 Bev. Then we take $a_2 \cong 2$ for the radial sector and $a_2 \cong 1$ for the spiral sector. We use $\Delta\nu = \frac{1}{4}$ and $F = \frac{1}{4}$ for both. The radial sector has $k \cong 200$, while the spiral sector has $k \cong 75$. (30) then gives 100 amps/cm^2 for the radial sector accelerator and 710 amps/cm^2 for the spiral sector accelerator. These are, it should be noted, theoretical upper limits. Any non-adiabatic rf processes, for example, will mix empty phase space with the beam and reduce the current densities.

Charges of opposite sign (mostly electrons in a proton accelerator) will be attracted to the beam and will tend to neutralize the space charge. Because they do not rotate with the beam, these charges cancel only the

electrostatic force. They leave the magnetic force essentially unaltered and thus can even lower the space charge limit. If we take only the magnetic attraction into account, the space charge limited current (28) is replaced by

$$I = I_o \, \nu \Delta \nu \left(\frac{a}{r}\right)^2 \beta \gamma \tag{31}$$

Effects of space charge and positive ion attraction to the beam have been observed and the frequencies changes measured by Mills[6] on the MURA spiral sector electron model. The theory appears to be in semi-quantitative agreement with experiment.

The neutralized space charge limit given by (31) has its most serious effect at the stacking energy. For a 15 Bev accelerator, the limiting current given by (31) is over 200 times smaller than the limiting current (28) without neutralization. For the 15 Bev radial sector accelerator, (31) gives 5 amps with a = 2 (from adiabatic damping) while for the spiral sector, with $a_2 = 1$, (31) gives 9 amps. In order to attain useable current densities, it is necessary to clear the neutralizing particles with an electric field. The electric field due to a relativistic beam is (in volts/cm)

$$E = \frac{60\, I}{a} \quad , \tag{32}$$

where I is the current in amperes and a the radius in centimeters. The clearing field must then be greater than the value given by (32). The effects of this field on particle dynamics can be made very small by reversing it with a period equal to that of the magnets. Such clearing fields undoubtedly also have interesting pumping effects.

Images of the stacked beam in the walls can have important effects on the space charge limit.[7] A linear charge density λ halfway between conducting walls separated by a distance 2 t has images of alternating sign ($\pm \lambda$) at distances 2 t, 4 t, etc. above and below the beam. A current I halfway between walls of infinite permeability separated by a distance 2 G produces image currents I of constant sign at distances 2 G, 4 G, etc. above and below the beam. Both electrostatic and magnetostatic image forces are radially focusing and vertically defocusing. The resultant forces can be given in terms of the quantities

$$\alpha_- = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cong 0.803$$

$$\alpha_+ = \sum_{n=1}^{\infty} \frac{1}{n^2} \cong 1.447$$

The betatron oscillation frequency with images and charges collected by the beam (which we shall call ions) is

$$\nu^2 = \nu_0^2 - \frac{2\pi e^2}{m\gamma\omega^2} \left\{ n_e \left[\left(1 \mp \frac{a^2 \alpha_-}{2t^2}\right) - \beta^2 \left(1 \pm \frac{a^2 \alpha_+}{2G^2}\right) \right] - n_i \left[1 \mp \frac{a^2 \alpha_-}{2t^2} \right] \right\} \tag{33}$$

where a is the beam radius, n_e and n_i are beam and ion densities and the upper sign refers to radial oscillations and the lower to vertical.

If the beam size is comparable to the apertures, the cancellation between electrostatic and magnetic forces can be lost and the space charge limit lowered. A great deal of the image effects can be cancelled by suitable

pole-face currents, whose magnitude, of course, depends on the beam intensity. Very little work has been done on such possibilities.

Note that all of the betatron oscillation space charge calculations outlined above are traditional in their basic approximation that the distribution of particles in phase space is stationary. In this way they are not different from the very earliest such calculations.[8] They do have other physical effects taken into account.

The synchrotron space charge limit has been calculated by Nielsen and Sessler[9] for a self-consistent stationary distribution. Here the longitudinal force must be calculated, but only those particles which are close in azimuth are effective, because of the shielding of the walls. For non-relativistic particles and a uniform distribution inside buckets, Nielsen and Sessler find for the total charge which can be contained in the h bunches of the hth harmonic

$$Q = \frac{V r}{2 \pi g h} \quad B(\Gamma) , \tag{34}$$

where

$$\begin{cases} B(\Gamma) \cong 2\pi (1 - \Gamma)^2 \\ g \cong 1 + 2 \ln \dfrac{4t}{\pi a} \end{cases} \tag{35}$$

g is a geometrical factor (t and a have the same meaning as in Eq. (33)) which is of order 3. For either the radial or spiral sector 15 Bev accelerators above, the longitudinal space charge limited current is about 3 amps at injection with V = 50 kilovolts.

Some investigation of dynamic effects of space charge has taken place. These concern the stability against small perturbations of the static distributions discussed above. So far results have been obtained only for the longitudinal (synchrotron) motion,[10,11]. A simple physical argument shows that in an accelerator operating above the transition energy (so that $\frac{\partial \omega}{\partial E} < 0$), a monoenergetic distribution is longitudinally unstable when particle-particle forces are considered. A particle approaching a bunch from behind is repelled by it, loses energy and therefore increases its frequency of revolution to catch up with the bunch, just as if it had negative mass. This is clearly an unstable situation which will cause the beam to collapse into very small bunches. A linearized Boltzmann equation treatment shows that there is a critical energy spread of the beam above which the beam is longitudinally stable. The critical energy spread is roughly

$$\Delta E \gtrsim \frac{1}{\pi\left(\frac{\partial \omega}{\partial E}\right)_s} \left[\frac{e^2 g N}{\gamma^2 r}\left(\omega \frac{\partial \omega}{\partial E}\right)_s\right]^{1/2}, \ldots \qquad (36)$$

where N is the total number of particles in the beam and s denotes evaluation for a typical particle in the stationary distribution.

When two beams interpenetrate, as in a colliding beams accelerator, "two-stream amplification" processes similar to those in travelling-wave amplifiers can occur. Such processes produce instability only for very intense beams where the number of particles is greater than

$$N_0 \sim \frac{\beta^2 \gamma^5}{g} \frac{r}{(e^2/mc^2)} . \qquad (37)$$

This number is very much larger than any beam so far contemplated in an accelerator.

As noted above, the electrostatic and magnetic space charge forces add for two oppositely directed beams, instead of cancelling as they do for

parallel beams. If two beams of current I intersect at N points around the accelerator, there is a detuning effect on the vertical betatron oscillations. The change of ν_y is

$$\Delta \nu_y = - \frac{4 I N r}{10 \pi \nu_y \cdot B \cdot \rho \, a \tan \phi} , \tag{37}$$

where ρ is the radius of curvature of the beam in the guide field and ϕ is the angle of intersection of the beams.

This effect can be large enough to move the betatron oscillation to a resonance, especially in a two-way radial sector accelerator, where $r/\rho \sim 6$. One can overcome this by introducing a vertical equilibrium orbit scalloping by radial magnetic fields in the median plane, so that the beams intersect only a few times per revolution. When the two beams do not intersect, they induce in each other an additional vertical equilibrium orbit displacement per pass of amount

$$\Delta z = \frac{8 \pi I r}{10 B \rho \nu_y \tan \phi} . \tag{38}$$

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