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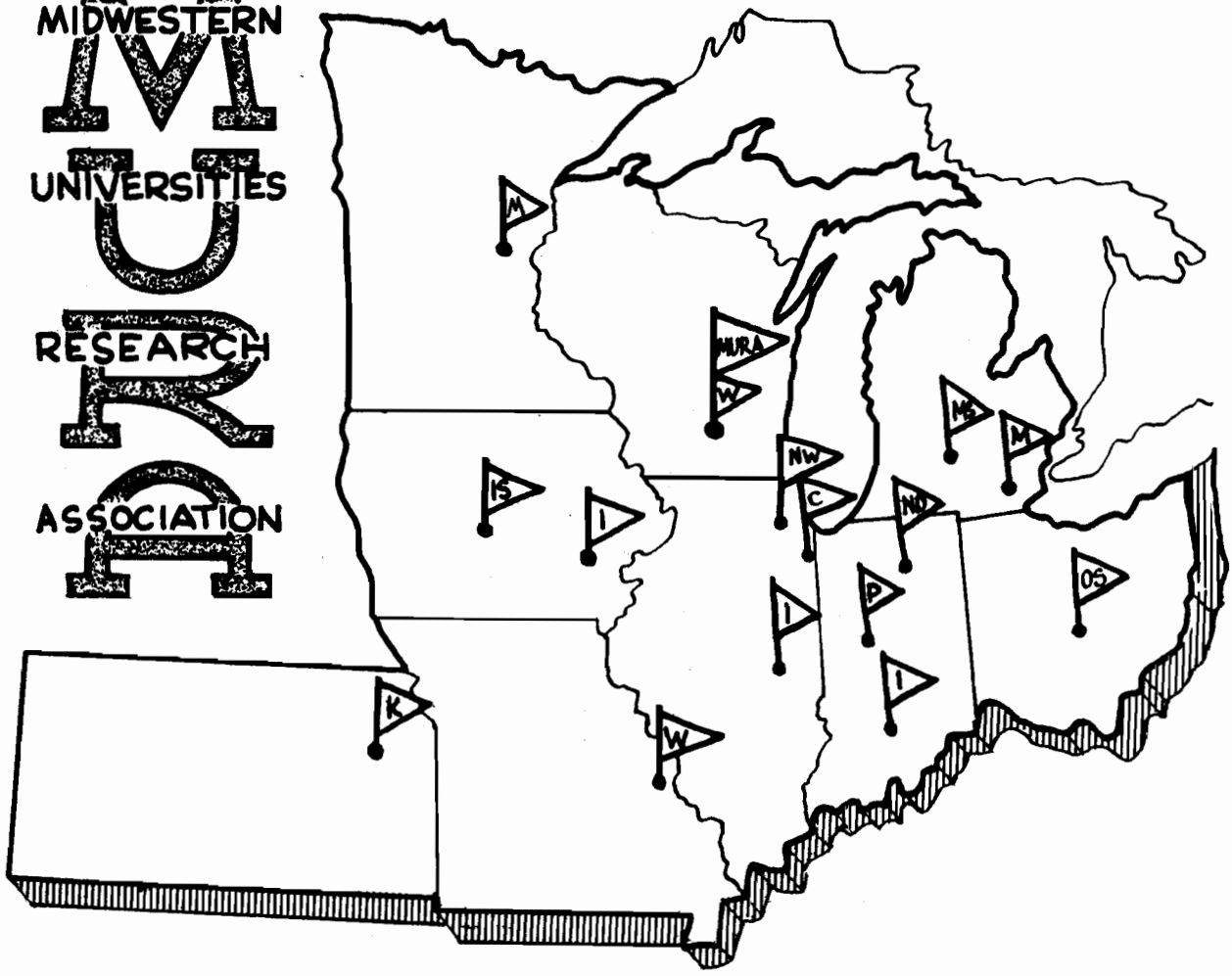

CONCERNING THE $\nu/N \rightarrow 1/3$ RESONANCE, III

Use Of The Moser Method To Estimate The Rotation Number, As A Function Of Amplitude, For The Equation

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{1}{2}(\sin 2t)\, v^2 = 0$$

REPORT

L. Jackson Laslett

NUMBER 461

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*
2203 University Avenue, Madison, Wisconsin

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ABSTRACT

The Moser method of analysis, as applied through terms of order $(\nu/N - 1/3)^2$ in an earlier report, is here employed to determine the variation of rotation number (or "tune") with amplitude for solutions of the non-linear differential equation given in the title. The result is given in terms of a complete elliptic integral of the first kind, with a modulus determined by the roots of a quartic equation. The rotation number is thus calculable in terms of an amplitude characterized by the value of the Moser t-independent Hamiltonian and this in turn may be related to some desired salient dimension of the phase curve of interest. This result, although by no means as convenient for hand calculation as the handy formulas sometimes employed for this purpose, is found to give results in very good agreement with numerical computations for a problem in which the small-amplitude frequency corresponds to $\nu/N = 0.3$. As is typical, the rotation number in this example departs initially from its small-amplitude value (0.3) by an amount proportional to the square of the oscillation amplitude and only near the stability limit undergoes a rapid variation to attain the value 1/3. The area enclosed by the phase curves, most specifically by the separatrix, is also briefly examined.

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A. INTRODUCTION

In an earlier report,[1,*] hereinafter denoted as I, a differential equation of the form

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{b_1}{2} (\sin 2t) v^2 = 0 \tag{1}$$

was discussed, the dependent variable v being so scaled, for convenience, that $b_1 = 1$:

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{1}{2} (\sin 2t) v^2 = 0. \tag{2}$$

In that report[1] the Moser method[2] of solution was applied to eqn. (2), through terms of order $(\nu/N - 1/3)^2$, to obtain an approximate t-independent Hamiltonian

$$K_2 = -2\delta J_2 - (1/48) (N/\nu)^{3/2} J_2^{3/2} \cos 3\gamma_2 + (\alpha/2048) (N/\nu)^3 J_2^2 \quad , \tag{3a}^{**}$$

with

$$\alpha \equiv \frac{6\nu/N}{1 - (\nu/N)^2} - \frac{1}{1 + 3\nu/N} \tag{3b}^{***}$$

and

$$\delta \equiv 1/3 - \nu/N \ , \tag{3c}$$

the expression K_2 thus representing an approximate constant of the motion.

In I the results of the analysis were specifically applied to examine the character of the limiting amplitude solution of eqn. (2), resulting from the $\nu/N \to 1/3$ resonance--in the present report we apply the results of the same general analysis to examine the dependence of the "rotation number" on amplitude.

The Hamiltonian K_2 [eqn. (3a)] was obtained in Sec. D3 of I by a series of canonical transformations.

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*References are given in Section D.

**Eqn. (57) of I.

***Eqn. (25) of I.

Coordinate	Momentum
v	p
γ_o	J_o
γ_1	J_1
γ_2	J_2

in which $$\gamma_2 \equiv \gamma_1 - \frac{2}{3} t \cong \gamma_o - \frac{2}{3} t \qquad (4a)^*$$

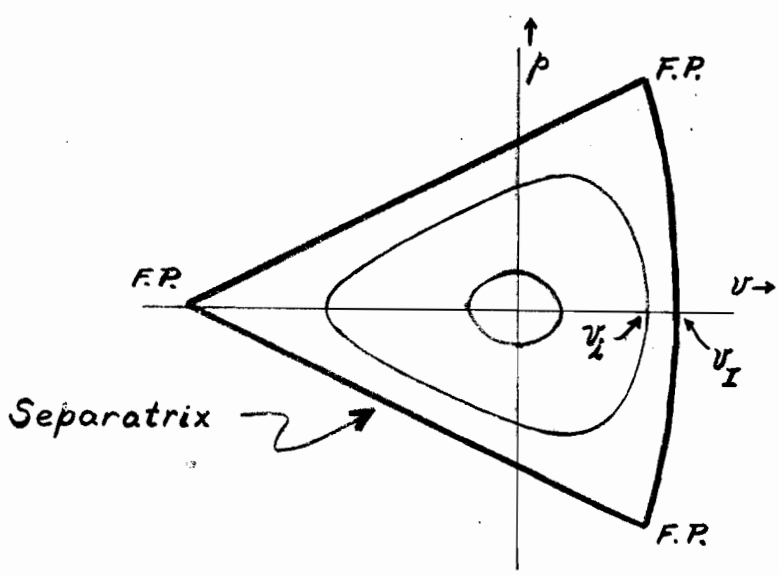
and $$J_2 \equiv J_1 \cong J_o \; , \qquad (4b)^{**}$$

with

$$v = (N/\nu)^{1/2} J_o^{1/2} \sin \gamma_o \qquad (5a)^{***}$$

and $$p = 2(\nu/N)^{1/2} J_o^{1/2} \cos \gamma_o \; . \qquad (5b)^{****}$$

Phase plots of solutions to eqn. (2), plotted in v, p-space at $t = 3\pi/4$, mod. π, show a transition in form from elliptical to roughly triangular curves (as illustrated) as the amplitude approaches the stability limit.



For $1/3 - \nu/N > 0$

*Eqns. (56b) and (52b) of I.

**Eqns. (56a) and (52a) of I.

***Eqns. (49c) of I.

****Eqn. (49d) of I.

Operationally, the amplitude may be characterized by the intercept v_i (see sketch), with v_I then serving to denote the value of this intercept for the separatrix. The corresponding values of J_2 or J_1 may be similarly designated. In the present report we shall examine analytically the dependence of the rotation number on $(J_2)_i$ and hence on v_i/v_I, specifically for a case in which the small-amplitude frequency is characterized by $\nu/N = 0.3$, and compare the results of this analysis with corresponding results obtained from computer solutions. A brief examination will also be made of the area enclosed by particular phase curves, in specific limiting cases.

B. THE ROTATION NUMBER

1. Analytic

To illustrate the procedure to be followed in obtaining a rotation number to characterize a particular solution, we may first note that, due to the non-linear character of the differential equation [eqn. (2)], J_2 is not a constant of the motion but is governed by the following differential equation:

$$dJ_2/dt = -\partial K_2/\partial \gamma_2$$

$$= -(1/16)(N/\nu)^{3/2} J_2^{\ 3/2} \sin 3\gamma_2, \qquad (6)$$

and $d\gamma_2/dt$ is similarly given by $\partial K_2/\partial J_2$. In the course of integration of dJ_2/dt, J_2 may go from an extreme value (say a minimum value) corresponding to its value $(J_2)_i \equiv a$ at the intercept v_i to a second extreme value (say its maximum value) b in an interval $\Delta t = T$. The corresponding changes of the variables of interest are then as listed below:

Δt	$J_2 = J_1$	γ_2	$\Delta\gamma_1$
0	a	0	0
T	b	$-\pi/3$	$2T/3 - \pi/3$

A frequency of revolution may then be taken as

$$\nu' = \Delta\gamma_1/T$$
$$= \frac{2}{3} - \frac{\pi}{3T}$$

or, since we consider $N = 2$ in eqns. (1) or (2), a "rotation number" introduced as

$$\frac{\nu'}{N} = \frac{1}{3} - \frac{\pi}{6T} \quad . \tag{7}$$

This quantity, ν'/N, will be seen to vary from the small-amplitude value, ν/N, to 1/3 as the amplitude increases to the value corresponding to the stability limit.

The differential equation (6) may be integrated by making use of the constancy of K_2 [given by eqn. (3a)] to eliminate γ_2:

$$dJ_2/dt = -\partial K_2/\partial\gamma_2$$

$$= -(1/16)(N/\nu)^{3/2} J_2^{3/2} \sin 3\gamma_2$$

$$= \frac{1}{16}\left(\frac{N}{\nu}\right)^{3/2}\sqrt{J_2^3 - 2304\left(\frac{\nu}{N}\right)^3\left[-K_2 - 2\delta\cdot J_2 + \frac{\alpha}{2048}\left(\frac{N}{\nu}\right)^3 J_2^2\right]^2} \tag{8a}$$

$$T = 16\left(\frac{\nu}{N}\right)^{3/2}\int_a^b \frac{dJ_2}{\sqrt{J_2^3 - 2304\left(\frac{\nu}{N}\right)^3\left[\frac{\alpha}{2048}\left(\frac{N}{\nu}\right)^3 J_2^2 - 2\delta\cdot J_2 - K_2\right]^2}}$$

$$= \frac{2^{11}}{3\alpha}\left(\frac{\nu}{N}\right)^3\int_a^b \frac{dJ_2}{\sqrt{\begin{aligned}&-J_2^4 + \left(\frac{2^7}{3\alpha}\right)^2\left(\frac{\nu}{N}\right)^3\left[1 + \frac{9}{2}\alpha\cdot\delta\right]J_2^3 + \left[\frac{2^{12}K_2}{\alpha}\left(\frac{\nu}{N}\right)^3\right.\\ &\left.- \left(\frac{2^{12}\delta}{\alpha}\right)^2\left(\frac{\nu}{N}\right)^6\right]J_2^2 - \left(\frac{2^{12}}{\alpha}\right)^2\left(\frac{\nu}{N}\right)^6\delta\cdot K_2J_2\\ &- \left(\frac{2^{11}}{\alpha}\right)^2\left(\frac{\nu}{N}\right)^6 K_2^2 \quad .\end{aligned}}} \tag{8b}$$

In the particular case that $\nu/N = 0.3$, $\delta = 1/3 - 0.3 = 1/30$ and $\alpha = 1.4517_{06}$;* eqn. (8b) above then assumes the form

$$T = 12.696\,785\,71 \int_a^b \frac{dJ_2}{\sqrt{\begin{array}{l} -J_2^4 + 28.401\,684\,09\, J_2^3 \\ + (76.180\,714\,27\, K_2 - 6.448\,334\,695) J_2^2 \\ - 193.450\,0409\, K_2 J_2 - 1450.875\,307\, K_2^2 \end{array}}} \tag{8c}$$

$$= 12.696\,785\,71 \int_a^b \frac{dJ_1}{\sqrt{(J_1 - a)(b - J_1)(c - J_1)(d - J_1)}} \tag{8c'}$$

$$= \frac{25.393\,571\,42}{\sqrt{(c - a)(d - b)}}\ \mathbf{K}(k)\,, \tag{8c''}$$

where a, b, c, d represent the roots of the equation obtained by setting the denominator of the integrand in eqn. (8c) equal to zero ($a < b < c < d$),

$$k = \sqrt{\frac{(b - a)(d - c)}{(c - a)(d - b)}}\,, \tag{8d}$$

and $\mathbf{K}(k)$ denotes the complete elliptic integral of the first kind (modulus k).[3] The values of T computed from eqn. (8c'') may then be substituted into eqn. (7) to obtain the estimated rotation number, ν'/N, for this case.

2. Comparison with Computational Results

In applying the results of the previous sub-section, the value $a = (J_2)_i = (J_1)_i$ may be related to a corresponding value of J_o by aid of eqn. (52) of I and thence directly to the intercept coordinate, v_i. The quantities $(v_i/v_I)^2$ and $(J_1)_i/(J_1)_I$ will, of course, be roughly proportional to one another. The root "a" will have the value

*From Eqn. (3b), or from p. 16 of I.

$(J_1)_i$; for small $(J_1)_i$ the roots a and b each approach zero, while for $(J_1)_i$ near the limiting value $(J_1)_I$ the roots b and c each approach 0.103 6384 and a = 0.027 5557.

The results for a series of selected values of $(J_1)_i$ are listed in Table I. For small values, the modulus k varies directly as $(J_1)_i^{3/4}$, being approximately equal to $4(J_1)_i^{3/4}$ - - see Fig. 1. Observed rotation numbers from a series of computer runs, made with the MURA I. B. M. - 704 computer by use of the DUCK-ANSWER program,[4] were obtained from examination of suitably numbered points on phase plots of the output data - - see Fig. 2. - - and are included in Table I. The results are expressed in terms of v_i/v_I, or $(v_i/v_I)^2$, using the value of v_I reported previously in I.[1]

The variation of rotation number with "amplitude" (or amplitude squared) is, finally, depicted in Fig. 3, in which the curve has been drawn to pass through the calculated values listed in Table I and the circles represent the results obtained from the machine computations. The agreement between the calculated curve and the computer results is seen to be close.[5]

Since the enclosed phase-space area is proportional to K_2, to a reasonable approximation, an effective average value of ν'/N may be taken as given by $\int (\nu'/N)\, dK_2 / \int dK_2$ - - i.e., by an average of ν'/N sampled in equal intervals of K_2. For the case considered, there thus results the effective value

$$\langle \nu'/N \rangle \cong 0.306 \quad . \tag{9}$$

TABLE I

CALCULATED AND OBSERVED VALUES OF ν'/N

v_i/v_I	$(v_i/v_I)^2$	$(J_1)_i^{1/2}$	$(J_1)_i$	-10^3K_2	a, b, c, d	$k^{(*)}$	K	T	ν'/N Calc.	ν'/N Obs.
0	0	0	0	0	0 0 .228 8851 28.172 7990	0	$\pi/2 =$ 1.5708	$10\pi/2 =$ 15.708	1/3-1/30 = 0.3	0.3
$\frac{.050}{.268} = .186\ 567$	0.3481	.030 918	.000 9559	0.06745	.000 9559 .001 0797 .227 0228 28.172 6257	.02331	1.5710	15.808	0.3002	0.3002
$\frac{.100}{.268} = .373\ 134$	.13923	.061 860	.003 8267	0.28474	.003 8267 .004 9413 .220 8204 28.172 0957	.07139	1.5728	16.155	0.3009	0.3010
.5427	.2945	.09	.0081	0.63071	.008 1 .012 0251 .210 3242 28.171 2348	.13883	1.5784	16.796	0.3 022	- - -
$\frac{.150}{.268} = .559\ 701$	.31327	.092 826	.008 6167	0.67391	.008 6167 .012 9909 .208 9491 28.171 1274	.14725	1.5794	16.886	0.3023	0.3023
$\frac{.200}{.268} = .746\ 269$	.55692	.12382	.015 3314	1.25661	.015 3314 .028 0399 .188 5344 28.169 7784	.27010	1.6007	18.411	0.3049	0.3051
.7534	.5676	.125	.015 625	1.28289	.015 6250 .028 9322 .187 5105 28.169 6164	.27746	1.6024	18.502	0.3050	- - -
$\frac{.250}{.268} = .932\ 836$	.87018	.15483	.023 972	2.05366	.023 972 .060 689 .149 319 28.167 704	.54037	1.7093	23.125	0.3107	0.3111

TABLE 1
(continued)

v_i/v_I	$(v_i/v_I)^2$	$(J_1)_i^{1/2}$	$(J_1)_i$	-10^3K_2	a, b, c, d	$k^{(*)}$	K	T	ν'/N Calc.	ν'/N Obs.
.9640	.9293	.16	.025 6	2.20878	.025 600 .071 529 .137 236 28.167 319	.64067	1.7849	25.593	0.3129	- - -
1	1	.166	.027 556	2.39707	.027 5557 .103 6384 .103 6384 28.166 8515	1	∞	∞	$\frac{1}{3}$	$\frac{1}{3}$

(*) For small $(J_1)_i$, k is proportional to $(J_1)_i^{3/4}$, being approximately $4(J_1)_i^{3/4}$.

C. THE PHASE SPACE AREA

1. Analytic Introduction

It may be of some interest to inquire concerning the area, S, in phase space included within a curve of constant K_2, taking, as before, K_2 as given in eqn. (3a). We thus investigate

$$S \equiv \oint p\, dv$$
$$= 6 \int_0^{\pi/3} J_2\, d\gamma_2 , \tag{10}$$

with K_2 given in terms of J_2 $(=J_1)$ and γ_2 by [from eqn. (3a) with $\nu/N = 0.3$]

$$K_2 = -\frac{1}{15} J_2 - 0.126\,787\,629\, J_2^{3/2} \cos 3\gamma_2 + 0.026\,253\,363\,72\, J_2^2 . \tag{11}$$

Equation (11) may be used to eliminate γ_2 from eqn. (10), with the result (written in terms of J_1)

$$S = 38.090\,35713 \int_a^b \frac{-3K_2 - (1/15)\, J_1 - 0.026\,253\,363\,72\, J_1^2}{\sqrt{(J_1 - a)(b - J_1)(c - J_1)(d - J_1)}}\, dJ_1 , \tag{12}$$

where a, b, c, d have the same meaning as before [i.e., in connection with eqn. (8c')] .

If one were to undertake to evaluate the integral of eqn. (12) directly, it appears that the (complete) elliptic integral of the third kind would appear[6] and we shall not further pursue this matter with such generality here. The character of the integral, and hence the value of the area S may, however, be examined with some interest in the case (i) that K_2 is small and (ii) in the case that $K_2 = (K_2)_I$, corresponding to the separatrix which encloses the entire stable area of phase space.

(i) For K_2 small, the numerator of the integrand in eqn. (12) is approximately $-3K_2 - (1/15)\, J_2$ or $(2/15)\, J_1$, and is approximately constant, while $\sqrt{(c - J_2)(d - J_2)} \cong \sqrt{cd} = 2.539\,357$ [cf. Table I] . Accordingly, in this limit, we may write eqn. (12) as

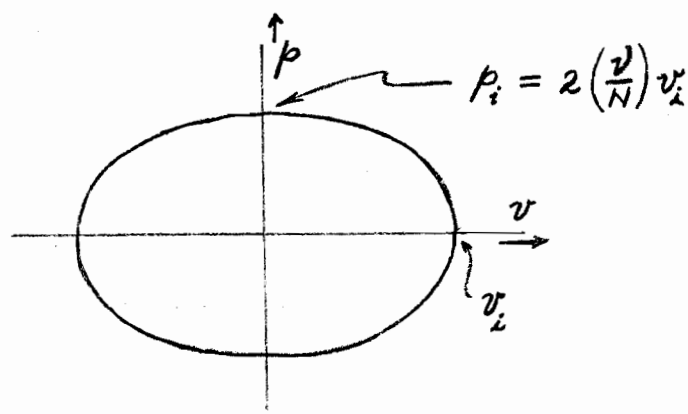
$$S \doteq 2\langle J_1\rangle \int_a^b \frac{dJ_1}{\sqrt{(J_1 - a)(b - J_1)}}$$

$$= 2\pi(J_1) \qquad \text{(by an elementary integration)}$$

$$= 2\pi J_o$$

$$= 2\pi(\nu/N)\, v_i^2 \qquad \text{[by eqn. (5a)]} \quad . \qquad (13)$$

This result is immediately seen to be correct, for the area enclosed within an elliptical phase curve of semi-axes v_i , $2(\nu/N)\, v_i$ [cf. eqn. (5b)] , and thus, to a degree, constitutes a check of eqn. (12).



(ii) When K_2 assumes the value $(K_2)_I$ characterizing the separatrix, b = c and the numerator of the integrand in eqn. (12) moreover may be factored to give us

$$S = \int_a^b \frac{(b - J_1)(J_1 + 2.642\,995)\, dJ_1}{\sqrt{(J_1 - a)(b - J_1)(c - J_1)(d - J_1)}} \quad , \text{ with } c = b$$

$$= \int_a^b \frac{J_1 + 2.642\,995}{\sqrt{(J_1 - a)(d - J_1)}}\, dJ_1$$

$$= \left[(5.28599 + a + d)\, \tan^{-1}\sqrt{\frac{J_1 - a}{d - J_1}} - \sqrt{(J_1 - a)(d - J_1)}\right]_a^b$$

$$= (5.28599 + a + d)\, \tan^{-1}\sqrt{\frac{b - a}{d - b}} - \sqrt{(b - a)(d - b)}$$

$$= 0.2805 \quad , \qquad (14)$$

by use of the values $a = 0.027\,5557$, $b = c = 0.103\,6384$, and $d = 28.166\,8515$ listed in Table I.

2. Computer Result for Area Within Separatrix

From computer results obtained in connection with the work reported previously in I, one finds (after scaling of those results so as to apply to the case $b_1 = 1$ under consideration here) that the area enclosed within the separatrix (estimated from the original plot in the v, p-plane) is approximately

$$S_{computer} \cong 0.296. \tag{15}$$

This area is some 5 or 6 percent greater than that suggested by the analytic result, eqn. (14), as might be expected in view of the observation that the computer values for salient coordinates and momenta on the separatrix were found correspondingly to be a few percent greater than the values derived from the Moser theory employed here [see, f. ex., Table IV or the first line of Table VI in I].

Finally, it may be noted in closing that if the small-amplitude result,

$$\begin{aligned} S &\doteq 2\pi[15(-K_2)] \\ &= 30\pi(-K_2)\ , \qquad [\text{for } K_2 \text{ small}] \end{aligned} \tag{16}$$

of eqn. (13) had been applied in this form to the large value $K_2 = -0.002397$ which corresponds to the separatrix, one would have obtained the result

$$S = 0.2259 \quad ;$$

the value obtained in this way thus would have been some 20 percent lower than that calculated by eqn. (14).

D. REFERENCES AND NOTES

1. L. Jackson Laslett, MURA-452 (April 13, 1959), hereinafter denoted by I.

2. Jürgen Moser, Nach. Gött, Akad. (Math. - Phys. Kl.) Nr. 6, 87-120 (1955).

3. Cf. B. O. Peirce, "A Short Table of Integrals", Ed. 3 (Ginn and Company, Boston, Massachusetts), Formula 552, p. 70.

4. J. N. Snyder, DUCK-ANSWER (I. B. M. Program 75), MURA-237 (1957). In the actual use of this program for the work reported here, the coefficient b_1 in eqn. (1) was given the value 1.15; the computational values of v and p, accordingly, each required multiplication by the factor 1.15 to bring them into agreement with the quantities employed in the analytic work presented here.

5. Although the analytic approach outlined in the present report is of interest as an illustration of the applicability of Moser methods, and the results appear to be quantitatively quite accurate, the results obtained here [eqn. (8c"), etc.] cannot be regarded as particularly convenient for numerical evaluation. It therefore may be of interest to recall, as Dr. G. Parzen has kindly pointed out (private communication, 27 May 1959), that a "handy formula" has been proposed to describe the variation of "tune" in cases such as we consider here. One form of this formula is such that one would write for the present problem

$$(\nu'/N)^2 \doteq (1/3)^2 - \left[(1/3)^2 - (\nu/N)^2\right] \sqrt{1 - (A/A_I)^2} \quad , \qquad (17)$$

where A and A_I respectively denote the "amplitudes" of the actual oscillation and of the limiting stable motion. In the present instance we might, perhaps somewhat arbitrarily, identify A^2 as proportional to K_2 and write

$$(\nu'/N)^2 \cong (1/3)^2 - \left[(1/3)^2 - (0.3)^2\right] \sqrt{1 - K_2/(K_2)_I} \quad . \qquad (18)$$

We now may make a comparison, presented below, of (i) the results derived in the body of the text, (ii) the prediction of the handy formula noted here, and (iii) the rotation number derived from the computer results:

$K_2/(K_2)_I$	ν'/N		
	Formula of text	Handy formula	Computer
0	.3	.3	.3
.028 139	.3002	.3005	.3002
.118 786	.3009	.3021	.3010
.281 137	.3023	.3053	.3023
.524 226	.3049	.3107	.3051
.856 737	.3107	.3211	.3111
1	1/3	1/3	1/3

Although the handy formula certainly represents correctly the general trend of ν'/N, it appears to be somewhat inferior quantitatively, at least as applied here, to the more elaborate result given in the text.

6. See, f. ex., W. Gröbner and N. Hofreiter, "Integraltafel", Ed. 2 (Springer, Vienna, 1957) in regard to integrals such as they denote by $\int \frac{x^n}{y} dx$, $n \geqslant 1$ [as in Pt. I, Indef. Int., Sect. 244, pp. 81 ff].

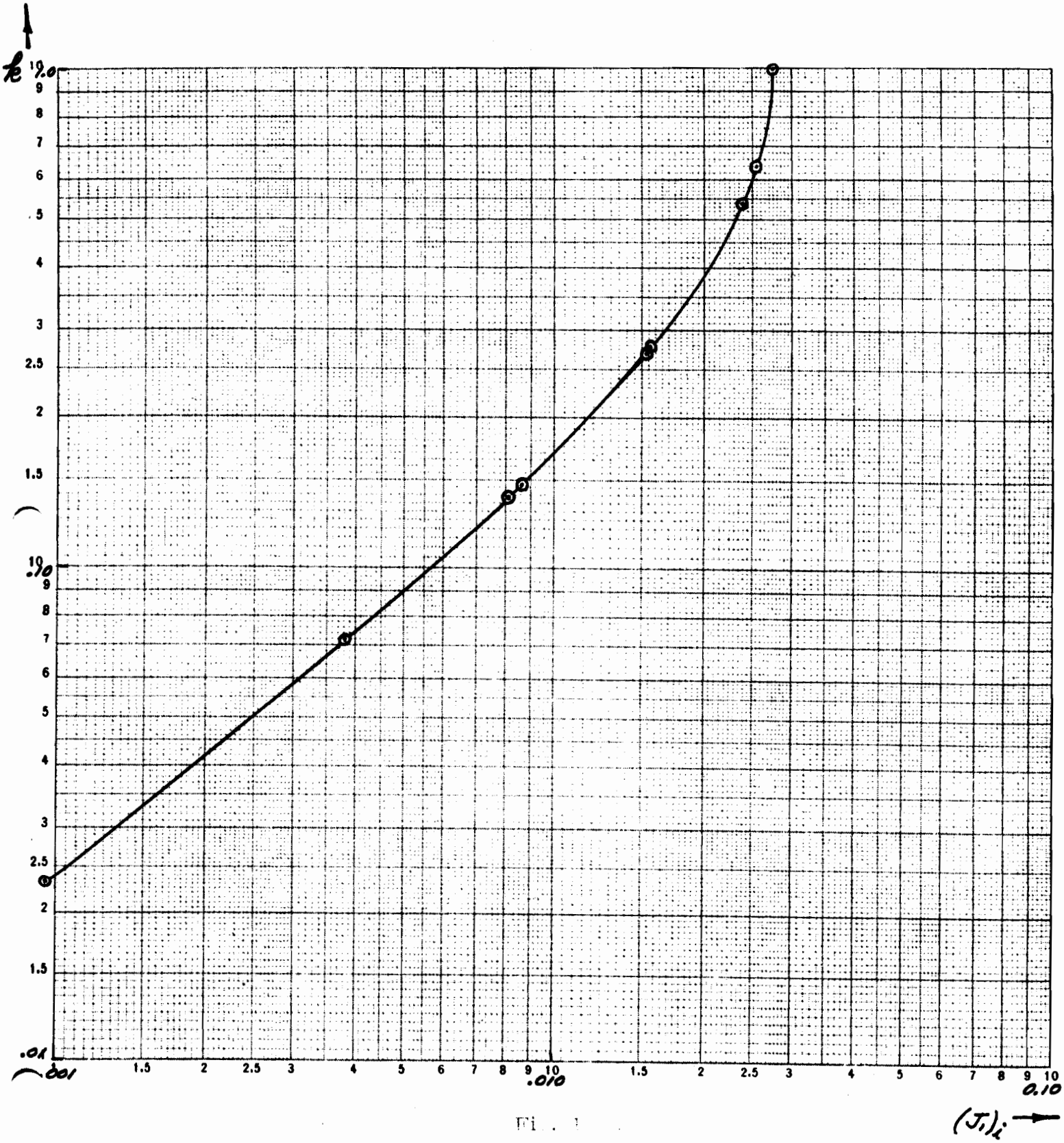

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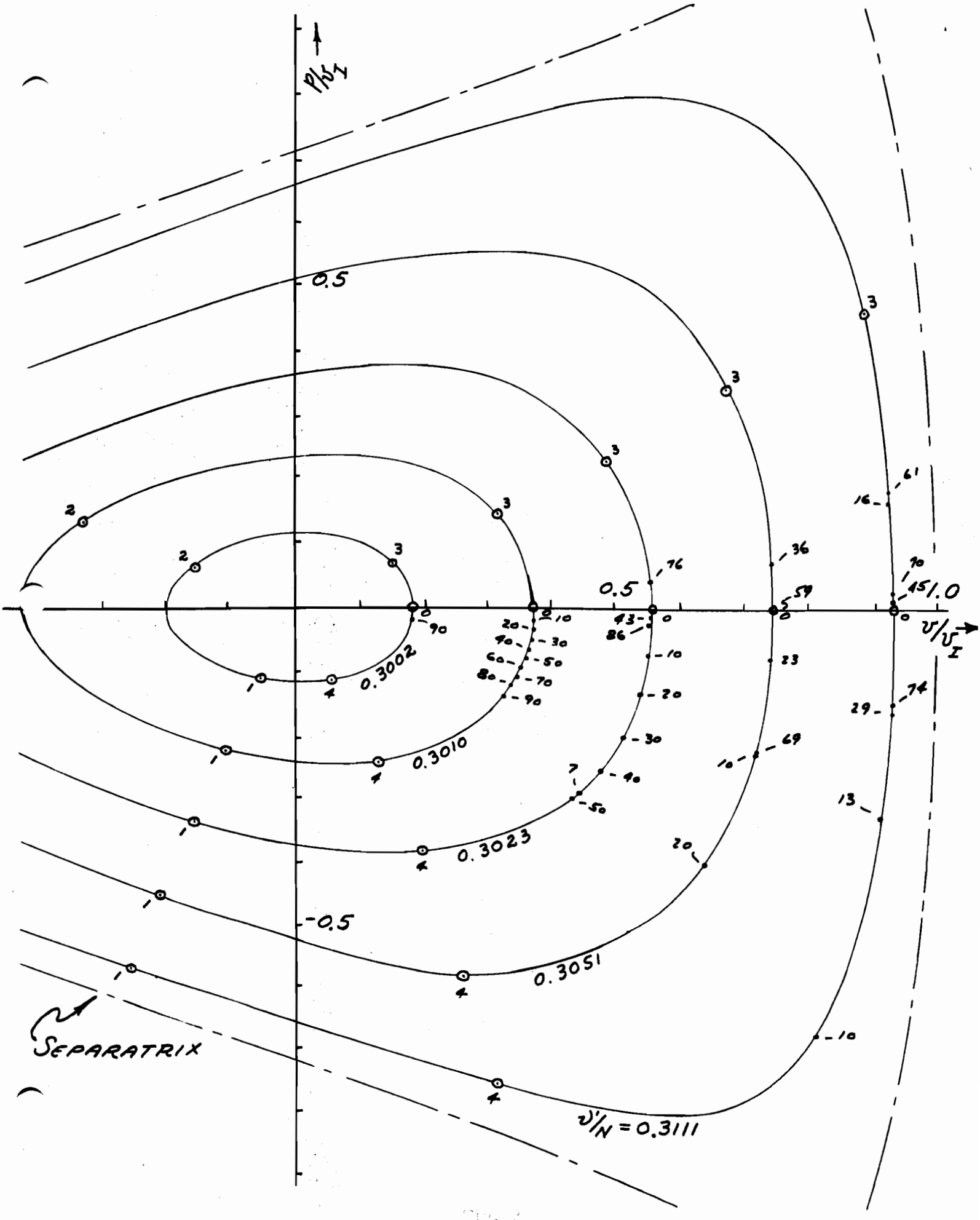
Fig. 1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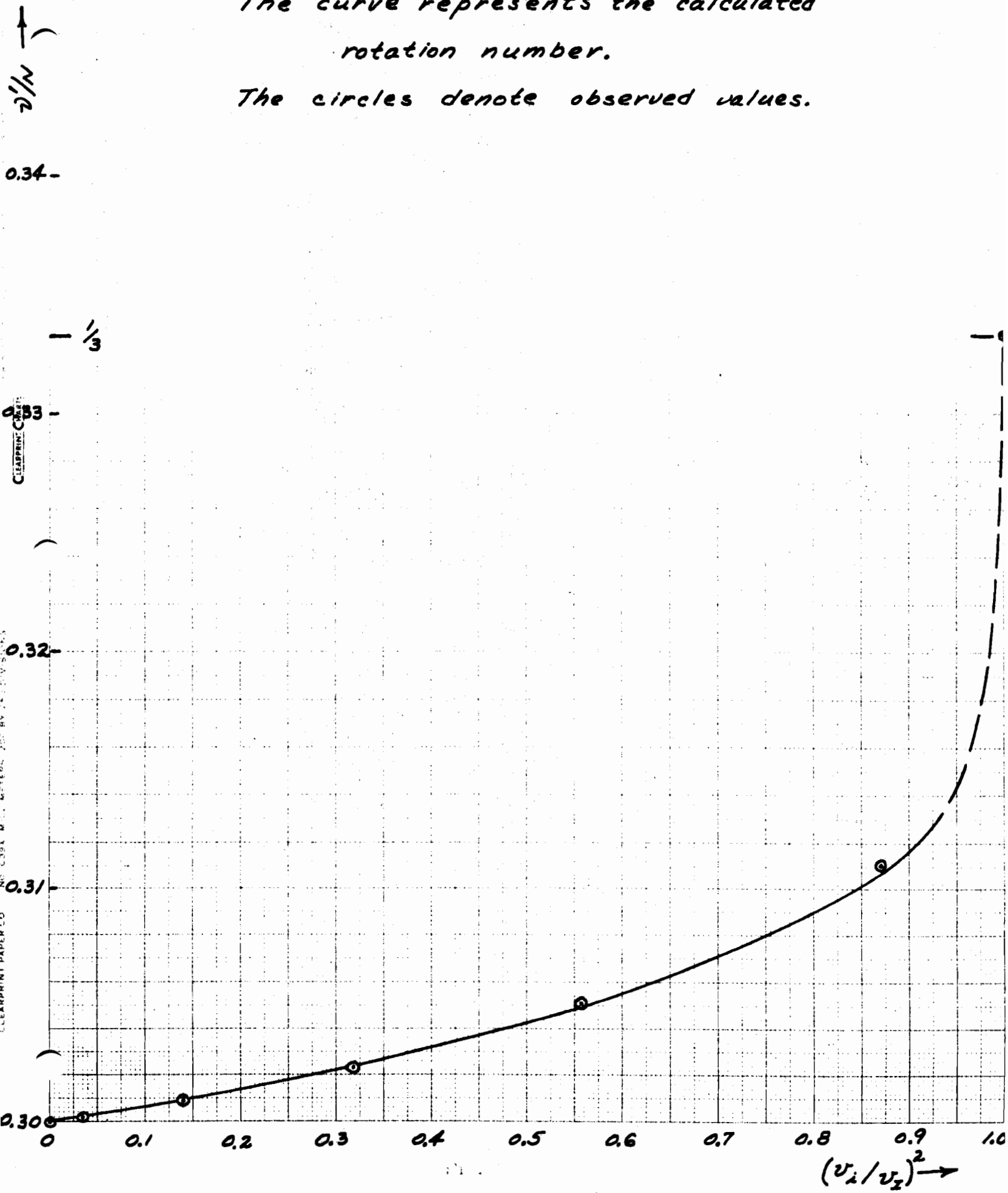
0.3023
0.3051
v'/N = 0.3111
SEPARATRIX

The curve represents the calculated rotation number.

The circles denote observed values.