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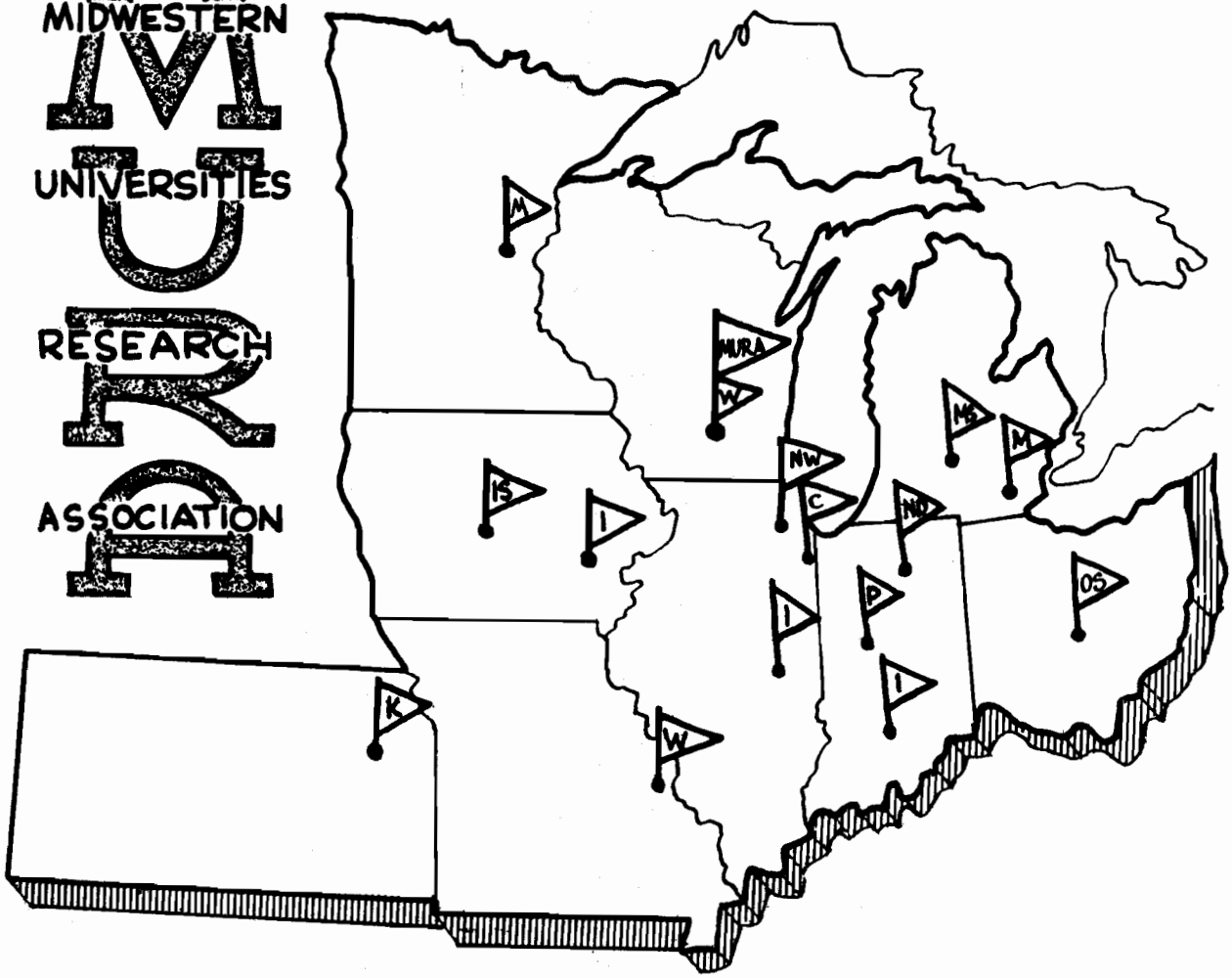

CONCERNING THE $\nu/N \longrightarrow 1/3$ RESONANCE, I

APPLICATION OF A VARIATIONAL PROCEDURE AND OF THE MOSER METHOD TO THE EQUATION

$$\frac{d^2v}{dt^2} + \left(\frac{2\,\nu}{N}\right)^2 v + \frac{1}{2}\,(\sin\,2\,t)\,v^2 = 0$$

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April 13, 1959

ABSTRACT

As an introduction to certain non-linear dynamical problems in which the 1/3-resonance plays a dominant role, the stability boundary for the equation

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{1}{2}(\sin 2t)\, v^2 = 0$$

has been studied analytically and by digital computation. Use of a relatively simple trial function in a variational procedure or with harmonic balance is shown to lead to simultaneous algebraic equations, for the coefficients in the trial function, whose solution affords a good estimate of the unstable fixed points. Application of the Moser method of solution is also carried through in detail, to include terms of the order $(\nu/N - 1/3)^2$, and the results compared with computer data for various values of ν/N.

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A. MOTIVATION

Simple applications of a variational method or of harmonic balance have been used previously[1*] to obtain "handy formulas" to indicate the stability limits for certain non-linear differential equations, with periodic coefficients, ocarring in the theory of cyclic accelerators. The applicability of the method described by Moser[2] has also been recognized and it may be noted that this latter method affords the opportunity of obtaining more detailed information concerning the solutions, since the previous methods are most simply applicable to the special problem of determining the unstable equilibrium solution, whose period is a multiple of that for the periodic coefficients in the differential equation.

Work currently in progress[3] concerning the possible practicality of injection into FFAG accelerators with a "field bump" deliberately introduced, with a period which is some integral multiple of the basic period of the unperturbed structure, has made it desirable to re-examine the analytic methods, in comparison with computational results, and to attempt to obtain analytic formulas of accuracy adequate to provide quantitatively useful orientation for detailed computational studies.

In the present report we develop analytic methods, which are compared with computational results, for solutions--particularly at the stability limit--to a simple type of differential equation for which the stability limit is determined by the one-third resonance($\nu/N \longrightarrow 1/3$). The application of these methods in the present case has been felt to be fruitful and later reports may make use of similar methods in more complicated situations.

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*References are given in Section E.

B. THE DIFFERENTIAL EQUATION EMPLOYED

In the theory of spirally-ridged FFAG accelerators the radial betatron motion, about the stable equilibrium orbit, may be convenienty represented by[4]

$$d^2u/d\theta^2 + \left[a + b \cos N\theta\right]u = b_1 (\theta)\, u^2/2 + \cdots, \tag{1}$$

where u denotes the departure from the stable equilibrium orbit, in units of the radius,

$$b \cong f/w\ , \text{ and} \tag{2a}$$

$$b_1 (\theta) \cong -(f/w^2) \sin N\theta\ . \tag{2b}$$

By introducing the scaled variables

$$t = (N/2)\,\theta\ , \tag{3a}$$

$$v = 4\,\frac{f}{w N^2}\,\frac{u}{w}\ , \tag{3b}$$

eqn. (1) assumes the form

$$d^2v/dt^2 + 4\left[\frac{a}{N^2} + \frac{f}{w N^2} \cos 2t\right] v + \frac{1}{2} (\sin 2t)\, v^2 = 0. \tag{4}$$

Although it is possible by a suitable transformation to remove the alternating-gradient feature of the linear term,[5] it is frequently convenient, in the interests of simplicity, to replace[6] the A-G coefficient by $(2\nu/N)^2$. The equation which results is, then,

$$d^2v/dt^2 + (2\nu/N)^2 v + (1/2) (\sin 2t)\, v^2 = 0. \tag{5}$$

It is this equation with which we shall work in the present report, being concerned in particular with the limiting-amplitude solution, governed by the one-third resonance ($\nu/N \rightarrow 1/3$). Results of a variational solution (or equivalently, of harmonic balance) and of application of the Moser procedure will be presented in Sections C and D, respectively, and compared with computational results.

Illustrative machine parameters might be

$$f = 1/4, \quad N = 33, \quad k \cong 79, \quad 1/w \cong 1252,$$

for which

$$k/N^2 = 0.0729, \quad \frac{f}{wN^2} = 0.2875,$$

and, from the approximate equations of motion, the frequencies of the small-amplitude (A-G) radial and axial oscillations are respectively such that

$$2\nu_x/N \equiv \sigma_x/\pi = 0.5994, \quad 2\nu_y/N \equiv \sigma_y/\pi = 0.1983;$$

in some of the work to follow the case $2\nu_x/N = 0.6$ will be specifically considered.

C. THE VARIATIONAL METHOD

1. Analytic Development

The unstable equilibrium orbit, or the associated "fixed points" characterizing the limiting-amplitude solution of eqn. (5),

$$d^2v/dt^2 + (2\nu/N)^2 v + (1/2)(\sin 2t) v^2 = 0, \tag{5}$$

may be sought by insertion of a trial function of suitable form into the variational statement

$$\delta\left[\langle (dv/dt)^2\rangle - (2\nu/N)^2 \langle v^2\rangle - (1/3)\langle v^3 \sin 2t\rangle\right] = 0. \tag{6}$$

We shall employ here the relatively simple three-term trial function

$$v \cong A_1 \sin 2t/3 + A_2 \sin 2t + A_3 \sin 10t/3, \tag{7}$$

in which the form of the last two terms may be suggested by insertion of the first term into the differential equation (5) and considerations of harmonic balance.

By insertion of the trial function (7) into the variational statement (6), or by harmonic balance in the differential equation (5), the following three simultaneous non-linear algebraic equations are obtained:

$$\left[\frac{4}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_1 + \frac{1}{8} A_1^2 - \frac{1}{2} A_1 A_2 + \frac{1}{4} A_1 A_3 - \frac{1}{4} A_2 A_3 = 0 \tag{8a}$$

$$\left[4 - \left(\frac{2\nu}{N}\right)^2\right] A_2 - \frac{1}{4} A_1^2 - \frac{3}{8} A_2^2 - \frac{1}{4} A_1 A_3 - \frac{1}{4} A_3^2 = 0 \tag{8b}$$

$$\left[\frac{100}{9} - \left(\frac{2\nu}{N}\right)^2\right] A_3 + \frac{1}{8} A_1^2 - \frac{1}{4} A_1 A_2 - \frac{1}{2} A_2 A_3 = 0 . \tag{8c}$$

A systematic solution of eqns. (8a-c) in ascending powers of $\epsilon \equiv (4/9) - (2\nu/N)^2$ may be obtained, but for operation an appreciable distance away from the $\nu/N \rightarrow 1/3$ resonance--i.e., when ϵ is not very small--it may be considered more satisfactory to solve these equations numerically.

For the case $2\nu/N = 0.6$, a direct numerical solution of eqns. (8a-c) leads to the values

$$A_1 = -0.5751_{517}$$

$$A_2 = +0.0229_{394}$$

$$A_3 = -0.0041_{574} ,$$

so that the approximate solution

$$v \cong -0.5751_{517} \sin 2t/3 + 0.0229_{394} \sin 2t - 0.0041_{574} \sin 10t/3, \tag{9a}$$

$$dv/dt \cong -0.3834_{345} \cos 2t/3 + 0.0458_{788} \cos 2t - 0.0138_{58} \cos 10t/3 \tag{9b}$$

is obtained.

An algebraic solution of eqns. (8a-c) in ascending powers of $\epsilon \equiv 4/9 - (2\nu/N)^2$ leads to the series

$$A_1 = -8\epsilon \Big[1 - 2\left(\frac{4}{P} + \frac{1}{Q}\right)\epsilon + 4\left(\frac{32}{P^2} + \frac{11}{PQ} + \frac{2}{Q^2}\right)\epsilon^2 - 4\left(\frac{652}{P^3} + \frac{292}{P^2Q} + \frac{79}{PQ^2} + \frac{10}{Q^3}\right)\epsilon^3 + 4\left(\frac{14912}{P^4} + \frac{8294}{P^3Q} + \frac{2808}{P^2Q^2} + \frac{577}{PQ^3} + \frac{56}{Q^4}\right)\epsilon^4 + \cdots \Big] \tag{10a}$$

$$A_2 = \frac{16\epsilon^2}{P} \Big[1 - \left(\frac{16}{P} + \frac{3}{Q}\right)\epsilon + \left(\frac{326}{P^2} + \frac{100}{PQ} + \frac{15}{Q^2}\right)\epsilon^2 - 4\left(\frac{1864}{P^3} + \frac{785}{P^2Q} + \frac{188}{PQ^2} + \frac{21}{Q^3}\right)\epsilon^3 + \cdots \Big] \tag{10b}$$

$$A_3 = -\frac{8\epsilon^2}{Q} \Big[1 - 4\left(\frac{3}{P} + \frac{1}{Q}\right)\epsilon + 4\left(\frac{56}{P^2} + \frac{27}{PQ} + \frac{5}{Q^2}\right)\epsilon^2 - 4\left(\frac{1234}{P^3} + \frac{744}{P^2Q} + \frac{219}{PQ^2} + \frac{28}{Q^3}\right)\epsilon^3 + \cdots \tag{10c}$$

where

$$\epsilon \equiv \frac{4}{9} - \left(\frac{2\nu}{N}\right)^2$$

$$P \equiv 4 - \left(\frac{2\nu}{N}\right)^2$$

$$Q \equiv \frac{100}{9} - \left(\frac{2\nu}{N}\right)^2 .$$

In the case considered previously, in which $2\nu/N = 0.6$, so that $\epsilon = 19/225 = 0.08444\cdots$, the series (10a-c) appear to converge rather slowly. Evaluation of the terms listed would suggest

$$A_1 = -0.6755\,555\left[1 - 0.2013\,012 + 0.0774\,000 - 0.0379\,715 + 0.0209\,305 + \cdots\right]$$

$$\cong -0.6755\,555 \times 0.8590\,578 = -0.5803\,413,$$

$$A_2 = +0.0313\,445\left[1 - 0.3947\,478 + 0.1945\,985 - 0.1074\,831 + \cdots\right]$$

$$\cong +0.0313\,445 \times 0.6923\,676 = +0.0217\,019,$$

$$A_3 = -0.0053\,061\left[1 - 0.3098\,062 + 0.1414\,688 - 0.0755\,171 + \cdots\right]$$

$$\cong -0.0053\,061 \times 0.7561\,455 = -0.0040\,122.$$

As was just mentioned, it is seen that the convergence of the expressions for A_1, A_2, and A_3 is quite slow in this example, each term being roughly minus 50 or 55 percent of the term before it, and only about two-figure accuracy is obtained* for the solution of the algebraic equations in this case without extension of the series to include terms beyond those shown here. The convergence, of course, would be markedly better if one were, say, one-third as far from the resonant frequency as was the case in the example considered here.

It may be noted in passing that retention of only the leading term in A_1 leads to

$$\text{Ampl. of } v \cong 8\,|\epsilon| = 8\,\left|(4/9) - (2\nu/N)^2\right|, \tag{11a}$$

or

$$\text{Ampl. of } u \cong (2\,w^2\,N^2/f)\,\left|(4/9) - (2\nu/N)^2\right| = (8\,w^2/f)\,\left|(N/3)^2 - \nu^2\right|, \tag{11b}$$

in agreement with the "handy formula" previously cited.[1]

*Cf. the results of the numerical solution which led to eqns. (9a, b).

2. Computational Results

(a) The coefficients of the trial function: For comparison with the solution (9a, b) which was found in sub-section 1 by a variational method, the unstable equilibrium solution (period $\Delta t = \pi/3$) of eqn. (5) was found computationally for $2\nu/N = 0.6$ by means of the DUCK-ANSWER program[7] and subjected to Fourier analysis by aid of the FORANAL program.[8] The result of this computational work is given below:

$$v = -0.575116 \sin 2t/3 + 0.022944 \sin 2t - 0.004159 \sin 10t/3 + 0.000182 \sin 14t/3 - 0.000019 \sin 6t + \cdots, \tag{12a}$$

$$dv/dt = -0.383411 \cos 2t/3 + 0.045888 \cos 2t - 0.013865 \cos 10t/3 + 0.000851 \cos 14t/3 - 0.000116 \cos 6t + \cdots. \tag{12b}$$

It is seen that the coefficients found for the first three terms of v and dv/dt check quite closely the results obtained by hand calculation in sub-section 1 [eqns. (9a, b)] and that the remaining coefficients are relatively small.

(b) Coordinates of fixed points: The predicted coordinates of the unstable fixed points for $t = 0$ (mod. π), or alternatively for $t = \frac{3\pi}{4}$ (mod. π), may be obtained by substitution of these values into the expressions of eqns. (9a, b). The results in the first case, then, refer to solutions of

$$d^2v/dt^2 + (2\nu/N)^2 v + (1/2)(\sin 2t)\, v^2 = 0 \quad \text{at } t = 0, \text{ mod. } \pi,$$

and in the second case to

$$d^2v/d\tau^2 + (2\nu/N)^2 v - (1/2)(\cos 2\tau)\, v^2 = 0 \quad \text{at } \tau = t - 3\pi/4 = 0, \text{ mod. } \pi,$$

which are examples for which computer information has been obtained. The results are summarized in Table I. The agreement between the results of eqns. (9a, b) and the computational values is seen to be quite good in these examples.

TABLE I

COORDINATES OF UNSTABLE FIXED POINTS, AS CALCULATED FROM EQNS. (9a, b) AND AS OBTAINED FROM COMPUTER RESULTS

t	From Eqns. (9a, b)		From Computer*	
	v	dv/dt	v	dv/dt
0,	0	- 0.3514	0	- 0.3506
mod. π	$\pm$ 0.4945	0.2445	$\pm$ 0.4943	0.2440
$3\pi/4$,	- 0.6022	0	- 0.6024	0
mod. π	0.2667	$\mp$ 0.3201	0.2668	$\mp$ 0.3207

D. THE MOSER PROCEDURE

1. Outline of Method

The differential equation (5), with which we are concerned in the present report, may be derived from the Hamiltonian

$$H = (1/2)\, p^2 + (1/2)\, (2\,\nu/N)^2\, v^2 + (1/6)\, (\sin 2\,t)\, v^3 , \qquad (13)$$

with $p \equiv dv/dt$. It is the purpose of the work to transform the variables (v, p) in such a way that the time-dependence is removed from the cubic term in H; the resultant Hamiltonian through terms of this order (and including the time-independent part of the terms of next higher order) may

*In much of the computational work the variables actually employed were v/1.15 and (dv/dt)/1.15, representing respectively u/w and (2/N)(du/dθ)/w when $\frac{4\,f}{w\,N^2} = 1.15$. To avoid complexity, however, the results are presented here in terms of the variable v which is employed in the analysis of the present report.

then be taken as an approximate constant of the motion, from which invariant phase curves can be constructed and values of fixed points determined.

The work first will be outlined in terms of complex variables ($\zeta_0, \bar{\zeta}_0$; $\zeta_1, \bar{\zeta}_1$) of the sort introduced by Moser,[2] and secondly will be carried out in a way which may be somewhat simpler for the present purposes, using quantities akin to angle-action variables. The use of these two methods may be of some inherent interest and serves to check the algebraic work.

2. Use of ζ , $\bar{\zeta}$ Variables

(a) The forward transformations: Commencing with the Hamiltonian (13), which is expressed in terms of v, p, and t, a first transformation is made to variables ζ_0 , $\bar{\zeta}_0$ which are complex conjugate quantities (with v and p real) but which are to be regarded as independent for the purposes of Hamiltonian theory, with ζ_0 playing the role of a coordinate and $\bar{\zeta}_0$ representing the canonically-conjugate momentum. ζ_0 and $\bar{\zeta}_0$ are defined in terms of v and p as follows:

$$\zeta_0 = (\nu/N)^{1/2} \left[v + \frac{i}{2}\frac{N}{\nu} p \right] \tag{14a}$$

$$\bar{\zeta}_0 = (\nu/N)^{1/2} \left[v - \frac{i}{2}\frac{N}{\nu} p \right], \tag{14b}$$

and, correspondingly,

$$v = (1/2)\,(N/\nu)^{1/2} \left[\zeta_0 + \bar{\zeta}_0 \right] \tag{14c}$$

$$p = -\,i\,(\nu/N)^{1/2} \left[\zeta_0 - \bar{\zeta}_0 \right]. \tag{14d}$$

It is noted that the functional determinant is

$$\frac{\partial(\zeta_0, \bar{\zeta}_0)}{\partial(v,\ p)} = -\,i\,, \tag{15}$$

but that one can write

$$p\,dv = +\,i\,(\zeta_0\,d\,\bar{\zeta}_0) + \text{perfect differential;} \tag{16}$$

hence, although the transformation from v, p to ζ_0, $\bar{\zeta}_0$ is strictly not canonical, the pair ζ_0, $\bar{\zeta}_0$ may be regarded as canonically-conjugate in association with the Hamiltonian

$$\Omega_0 = -\,i\,H \tag{17a}$$

$$= -\,i\,(2\nu/N)\zeta_0\bar{\zeta}_0 - (i/48)\,(N/\nu)^{3/2}\,(\sin 2\,t)\,(\zeta_0 + \bar{\zeta}_0)^3\,. \tag{17b}$$

A canonical transformation from ζ_0, $\bar{\zeta}_0$ to ζ_1, $\bar{\zeta}_1$ is now performed by means of a generating function $F_1\,(\zeta_0, \bar{\zeta}_1)$. The generating function is so chosen as to remove from the Hamiltonian all time-dependence in the cubic term, save that associated with the resonance $\nu/N \rightarrow 1/3$, the coefficients of the transformation thus remaining finite as the resonance is approached; supplementary fourth-order terms are also included in the generating function in order that, to the order that the work is carried, the new variables ζ_1, $\bar{\zeta}_1$ conveniently will be complex conjugates.[9] The generating function selected is

$$\begin{aligned} F_1(\zeta_0,\bar{\zeta}_1) = {} & \zeta_0\bar{\zeta}_1 - (i/48)(N/\nu)^{3/2}\left[\Phi_0\zeta_0^3 + \Phi_1\zeta_0^2\bar{\zeta}_1 + \Phi_2\zeta_0\bar{\zeta}_1^2 + \Phi_3\bar{\zeta}_1^3\right] \\ & + (1/1152)(N/\nu)^3\left[\Psi_0\zeta_0^4 + \Psi_1\zeta_0^3\bar{\zeta}_1 + \Psi_2\zeta_0^2\bar{\zeta}_1^2 + \Psi_3\zeta_0\bar{\zeta}_1^3 + \Psi_4\bar{\zeta}_1^4\right], \end{aligned} \tag{18}$$

where Φ_0, $\cdots$ are taken as periodic solutions of the differential equations

$$i\,(d\Phi_0/dt) + 3\,(2\nu/N)\Phi_0 - (1/2)\,e^{-2\,it} = 0 \tag{19a}$$

$$i\,(d\Phi_1/dt) + (2\nu/N)\Phi_1 + (3/2)\,(e^{2\,it} - e^{-2\,it}) = 0 \tag{19b}$$

$$i\,(d\Phi_2/dt) - (2\nu/N)\Phi_2 + (3/2)\,(e^{2\,it} - e^{-2\,it}) = 0 \tag{19c}$$

$$i\,(d\Phi_3/dt) - 3\,(2\nu/N)\Phi_3 + (1/2)\,e^{2\,it} = 0, \tag{19d}$$

namely

$$\Phi_0 = \frac{1}{4[1 + 3\nu/N]} e^{-2it} \tag{20a}$$

$$\Phi_1 = \frac{3}{4[1 - \nu/N]} e^{2it} + \frac{3}{4[1 + \nu/N]} e^{-2it} \tag{20b}$$

$$\Phi_2 = \frac{3}{4[1 + \nu/N]} e^{2it} + \frac{3}{4[1 - \nu/N]} e^{-2it} = \Phi_1^* \tag{20c}$$

$$\Phi_3 = \frac{1}{4[1 + 3\nu/N]} e^{2it} = \Phi_0^* , \tag{20d}$$

and where

$$\Psi_0 = -(3/4)\Phi_0\Phi_1 \tag{21a}$$

$$\Psi_1 = -(3/2)\Phi_0\Phi_2 - (1/2)\Phi_1^2 \tag{21b}$$

$$\Psi_2 = -(9/4)\Phi_0\Phi_3 - (5/4)\Phi_1\Phi_2 \tag{21c}$$

$$\Psi_3 = -(3/2)\Phi_1\Phi_3 - (1/2)\Phi_2^2 = \Psi_1^* \tag{21d}$$

$$\Psi_4 = -(3/4)\Phi_2\Phi_3 = \Psi_0^* . \tag{21e}$$

The transformation equations then read

$$\bar{\xi}_0 = \partial F_1/\partial \xi_0 = \bar{\xi}_1 - (i/48)(N/\nu)^{3/2}[3\Phi_0\xi_0^2 + 2\Phi_1\xi_0\bar{\xi}_1 + \Phi_2\bar{\xi}_1^2]$$
$$+ (1/1152)(N/\nu)^3[4\Psi_0\xi_0^3 + 3\Psi_1\xi_0^2\bar{\xi}_1 + 2\Psi_2\xi_0\bar{\xi}_1^2 + \Psi_3\bar{\xi}_1^3] \tag{22a}$$

$$\xi_1 = \partial F_1/\partial \bar{\xi}_1 = \xi_0 - (i/48)(N/\nu)^{3/2}[\Phi_1\xi_0^2 + 2\Phi_2\xi_0\bar{\xi}_1 + 3\Phi_3\bar{\xi}_1^2]$$
$$+ (1/1152)(N/\nu)^3[\Psi_1\xi_0^3 + 2\Psi_2\xi_0^2\bar{\xi}_1 + 3\Psi_3\xi_0\bar{\xi}_1^2 + 4\Psi_4\bar{\xi}_1^3] \tag{22b}$$

and

$$\Omega_1 = \Omega_0 + \partial F_1/\partial t$$
$$= \Omega_0 - (i/48)(N/\nu)^{3/2}[\Phi_0'\xi_0^3 + \Phi_1'\xi_0^2\bar{\xi}_1 + \Phi_2'\xi_0\bar{\xi}_1^2 + \Phi_3'\bar{\xi}_1^3$$
$$+ (1/1152)(N/\nu)^3[\Psi_0'\xi_0^4 + \Psi_1'\xi_0^3\bar{\xi}_1 + \Psi_2'\xi_0^2\bar{\xi}_1^2 + \Psi_3'\xi_0\bar{\xi}_1^3 + \Psi_4'\bar{\xi}_1^4], \tag{23}$$

with primes denoting differentiation with respect to t. It is now necessary, of course, to solve eqns. (22a, b) algebraically with sufficient accuracy that eqn. (23) for Ω_1 may be expressed explicitly in terms of the new variables ζ_1, $\bar{\zeta}_1$.

The algebraic steps leading to the expression of Ω_1 in terms of ζ_1, $\bar{\zeta}_1$ are detailed in Appendix A, with the result

$$\Omega_1 = -i\,(2\nu/N)\,\zeta_1\bar{\zeta}_1 - (1/96)(N/\nu)^{3/2}\left[e^{2it}\zeta_1^3 - e^{-2it}\bar{\zeta}_1^3\right] - \alpha\,\frac{i}{2048}\,(N/\nu)^3\,\zeta_1^2\,\bar{\zeta}_1^2 \quad , \tag{24}$$

where

$$\alpha \equiv 6\,\frac{\nu/N}{1-(\nu/N)^2} - \frac{1}{1+3\nu/N} \tag{25}$$

and where we have only retained in the quartic term that part which involves $\zeta_1^2\,\bar{\zeta}_1^2$ and which is independent of t.[10]

It is now convenient to introduce variables γ and J, to play the roles of coordinate and momentum, defined as

$$\gamma \equiv \frac{1}{2i}\ln(\bar{\zeta}_1/\zeta_1) \tag{26a}$$

$$J \equiv \zeta_1\,\bar{\zeta}_1 \quad , \tag{26b}$$

so that, correspondingly,

$$\zeta_1 = J^{1/2}\,e^{-i\gamma} \tag{27a}$$

$$\bar{\zeta}_1 = J^{1/2}\,e^{i\gamma} \quad . \tag{27b}$$

In this case the functional determinant is

$$\frac{\partial(\gamma, J)}{\partial(\zeta_1, \bar{\zeta}_1)} = i \tag{28}$$

and

$$\begin{aligned}\bar{\zeta}_1\,d\zeta_1 &= -i\,J\,d\gamma + \tfrac{1}{2}\,dJ \\ &= -i\,J\,d\gamma + \text{perfect differential,}\end{aligned} \tag{29}$$

so the new variables γ , J may be referred to the Hamiltonian

$$H_1 = i\Omega_1 . \tag{30}$$

It is noted that the functional determinant of the over-all transformation from v, p to γ , J is

$$\frac{\partial(\gamma, J)}{\partial(v, p)} = \frac{\partial(\gamma, J)}{\partial(\zeta_1, \bar{\zeta}_1)} \frac{\partial(\zeta_1, \bar{\zeta}_1)}{\partial(\zeta_0, \bar{\zeta}_0)} \frac{\partial(\zeta_0, \bar{\zeta}_0)}{\partial(v, p)} = (i)(1)(-i) = 1, \tag{31}$$

so that the pair γ , J may be regarded as canonically related to the original pair v, p.

From the expression (24) for Ω_1 and the relation (30) which connects H_1 with Ω_1 , we immediately find

$$H_1 = (2\nu/N)\, J - (1/48)\,(N/\nu)^{3/2}\, J^{3/2} \sin(3\gamma - 2t) + (\alpha/2048)\,(N/\nu)^3\, J^2 . \tag{32}$$

A final canonical transformation to variables $\bar{\gamma}$, $\bar{J}$, defined by the generating function

$$F_2(\gamma, \bar{J}) = \bar{J}(\gamma - \tfrac{2}{3} t) , \tag{33}$$

leads to

$$J = \partial F_2/\partial\gamma = \bar{J} \tag{34a}$$

$$\bar{\gamma} = \partial F_2/\partial\bar{J} = \gamma - \tfrac{2}{3} t \tag{34b}$$

and

$$\begin{aligned} H_2 &= H_1 + \partial F/\partial t \\ &= H_1 - \tfrac{2}{3}\bar{J} \\ &= -\left(\tfrac{2}{3} - \tfrac{2\nu}{N}\right)\bar{J} - \frac{1}{48}\left(\frac{N}{\nu}\right)^{3/2} (\bar{J})^{3/2} \sin 3\bar{\gamma} \\ &\quad + \frac{\alpha}{2048}\left(\frac{N}{\nu}\right)^3 (\bar{J})^2 . \end{aligned} \tag{35}$$

By a sequence of transformations between the pairs of variables

Coordinate	Momentum
ν	p
$\mathcal{Y}_0$	$\bar{\mathcal{Y}}_0$
$\mathcal{Y}_1$	$\bar{\mathcal{Y}}_1$
γ	J
$\bar{\gamma}$	$\bar{J}$

we are thus led to a Hamiltonian H_2 [eqn. (35)] from the first two terms of which the independent variable t has been entirely removed and in the last term of which we have retained the t-independent part.* The retention of the last term in this form is believed to be desirable, since it can exert a significant influence on the J-dependence (or amplitude-dependence) of the oscillation frequency.[10] To the degree of approximation considered here, then, we take H_2 in the form expressed by eqn. (35) to be a constant of the motion. In this spirit invariant phase curves of the problem are determined.

(b) The separatrix: The assumed constancy of H_2 means that for any particular value of t and points homologous thereto (t taken modulo π), the quantity

$$-\left(\frac{2}{3} - \frac{2\nu}{N}\right) J - \frac{1}{48}\left(\frac{N}{\nu}\right)^{3/2} J^{3/2} \sin(3\gamma - 2t) + \frac{\alpha}{2048}\left(\frac{N}{\nu}\right)^3 J^2 = H_2 \quad (36)$$

is constant. If we introduce for convenience the scaled quantities

*A little reflection will show that only that part of the quartic terms in Ω_1 which have been retained in (24) make a t-independent contribution to the final H_2.

$$\xi^2 = \frac{1}{(48)^2\,(2/3 - 2\nu/N)^2\,(\nu/N)^3}\,J \tag{37a}$$

$$\mathcal{K} = -\frac{1}{(48)^2\,(2/3 - 2\nu/N)^3\,(\nu/N)^3}\,H_2 \tag{37b}$$

and

$$\lambda = (2/3 - 2\nu/N)\alpha$$

$$= \left(\frac{2}{3} - \frac{2\nu}{N}\right)\left[6\,\frac{\nu/N}{1-(\nu/N)^2} - \frac{1}{1+3\nu/N}\right],^{*} \tag{37c}$$

eqn. (36) assumes the more concise form

$$\xi^2 + \xi^3 \sin(3\gamma - 2t) - (9\lambda/8)\xi^4 = \mathcal{K}. \tag{38}$$

The fixed points for the motion, in particular, are characterized by expression (38) being stationary with respect to γ and ξ. For the unstable fixed points associated with the separatrix between stable and unstable regions, we take γ as having values for which**

$$\sin(3\gamma - 2t) = -1 \tag{39a}$$

$$[\gamma = -\pi/6 + 2t/3, \text{ mod. } 2\pi/3] \tag{39b}$$

and ξ to be the root ξ_1, near 2/3, of the quadratic equation

$$(9\lambda/2)\xi^2 + 3\xi - 2 = 0: \tag{40}$$

$$\xi_1 = \frac{\sqrt{1+4\lambda} - 1}{3\lambda} \tag{41a}$$

$$\doteq \frac{2}{3}\left[1 - \lambda + 2\lambda^2 - 5\lambda^3 + 14\lambda^4 - \cdots\right]. \tag{41b}$$

---

*For $\nu/N = 0.3$, $\alpha = 1.451706$, $\lambda = 0.0967804$, and $(9/8)\lambda = 0.108878$.

**With this choice of sign for $\sin(3\gamma - 2t)$, the value of ξ which we select is positive.

In the work to follow it will be convenient to employ a quantity, normally near unity,* which we denote as η_1 :

$$\eta_1 = \frac{3}{2}\xi_1 \tag{42a}$$

$$= \frac{\sqrt{1+4\lambda} - 1}{2\lambda} \tag{42b}$$

$$\doteq 1 - \lambda + 2\lambda^2 - 5\lambda^3 + 14\lambda^4 - \cdots , \tag{42c}$$

λ being defined by eqn. (37c). The associated value of κ is**

$$\kappa_1 = \xi_1^2 - \xi_1^3 - (9\lambda/8)\xi_1^4 \tag{43a}$$

$$= \xi_1^2 (2 - \xi_1)/4 \tag{43b}$$

$$= \frac{4}{27} \frac{\eta_1^2 (3 - \eta_1)}{2} . \tag{43c}$$

Associated with this value of κ there is a value of ξ , which we denote by ξ_2 and which is normally roughly $\xi_1/2$, which corresponds to setting $\sin(3\gamma - 2t) = +1$ in eqn. (38); if we write $\eta_2 = \frac{3}{2}\xi_2$, in analogy to eqn. (42a), η_2 will be roughly 1/2.***

In summary, then

$$J^{1/2} = 64\,(1/3 - \nu/N)\,(\nu/N)^{3/2}\,\eta \tag{44}$$

[cf. eqn. (37a)]; points on a particular phase curve specified by its value of κ , are then obtained by use of values of γ and J which are mutually consistent with κ through eqn. (36) or (38), evaluation of the corresponding values of ξ_1 , $\bar{\xi}_1$, and finally proceeding back through the transformations to obtain the associated values of v and p. Without continuation

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*For $\nu/N = 0.3$, $\xi_1 = 0.61225$ and $\eta_1 = 0.918374$.

**For $\nu/N = 0.3$, $\kappa_1 = 0.130049$.

***For $\nu/N = 0.3$, the value ξ_2 corresponding to $\kappa_1 = 0.130049$ is $\xi_2 = 0.31570$ and $\eta_2 = 0.47355$.

of the analysis beyond the transformations described here, it is pointless to express the results to terms beyond those which are second order in the quantity $(1/3 - \nu/N)$.

We give below, in Table II, such values of ζ_1, $\bar{\zeta}_1$, for the two types of locations considered in the examples of Section C 2 b, namely $t = 0$ (mod. π) and $t = 3\pi/4$ (mod. π).

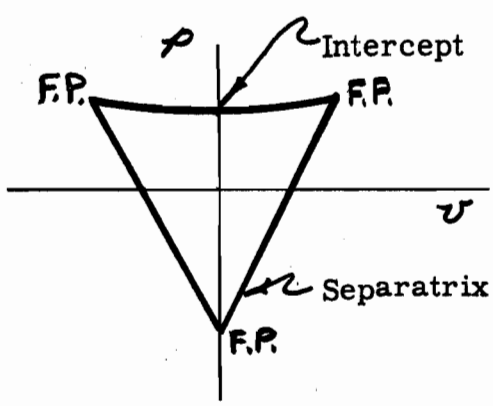
<u>(c) The reverse transformation to the original variables:</u> For evaluation of ζ_0, $\bar{\zeta}_0$, and hence of v, p, we now make use of the transformation equations previously exhibited. Since, by eqns. (14a, b), the quantities required for evaluating v and p are explicitly $\zeta_0 + \bar{\zeta}_0$ and $\zeta_0 - \bar{\zeta}_0$, respectively, we make use of eqn. (A4) of Appendix A,

$$\zeta_0 + \bar{\zeta}_0 \doteq \zeta_1 + \bar{\zeta}_1 + (i/48)(N/\nu)^{3/2}\left[(-3\Phi_0 + \Phi_1)\zeta_1^2 + (-2\Phi_1 + 2\Phi_2)\zeta_1\bar{\zeta}_1 + (-\Phi_2 + 3\Phi_3)\bar{\zeta}_1^2\right], \tag{45}$$

and the corresponding expression

$$\zeta_0 - \bar{\zeta}_0 \doteq \zeta_1 - \bar{\zeta}_1 + (i/48)(N/\nu)^{3/2}\left[(3\Phi_0 + \Phi_1)\zeta_1^2 + (2\Phi_1 + 2\Phi_2)\zeta_1\bar{\zeta}_1 + (\Phi_2 + 3\Phi_3)\bar{\zeta}_1^2\right] \tag{46}$$

obtained by subtraction of eqns. (A1) and (A2). It is a matter then of straight-forward algebra to evaluate $\Phi_0, \ldots, \Phi_3$ for the value of t which is of interest, to evaluate $\zeta_0 \pm \bar{\zeta}_0$ from the previously written ζ_1, $\bar{\zeta}_1$ (e.g., those listed in Table II), and thus determine v, p. The results, for the cases to which Table II pertains, are given in Table III.



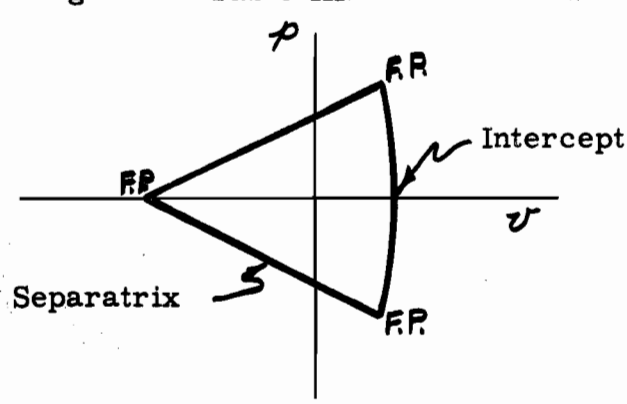


TABLE II

VALUES OF y_1 AND $\bar{y}_1$ CORRESPONDING TO THE SEPARATRIX OF EQUATION (5) FOR $t = 0$ (mod. π) AND FOR $t = 3\pi/4$ (mod. π)

The first lines apply to the unstable fixed points; the last line refers to the intercept of the separatrix with the symmetry axis of the v, p diagram.

ξ	γ	For $t = 0$, mod. π		For $t = 3\pi/4$, mod. π	
		y_1	$\bar{y}_1$	y_1	$\bar{y}_1$
ξ_1	$-\frac{\pi}{6}+\frac{2t}{3}$	$32(\sqrt{3}+i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$32(\sqrt{3}-i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$32(1-\sqrt{3}i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$32(1+\sqrt{3}i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$
	$-\frac{5\pi}{6}+\frac{2t}{3}$	$-32(\sqrt{3}-i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$-32(\sqrt{3}+i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$32(1+\sqrt{3}i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$32(1-\sqrt{3}i)\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$
	$\frac{\pi}{2}+\frac{2t}{3}$	$-64i\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$64i\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$-64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$	$-64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1$
ξ_2	$-\frac{\pi}{2}+\frac{2t}{3}$	$64i\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_2$	$-64i\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_2$	$64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_2$	$64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_2$

TABLE III

VALUES OF v AND p CORRESPONDING TO THE SEPARATRIX OF EQUATION (5) FOR $t = 0$ (mod. π) AND FOR $t = 3\pi/4$ (mod. π)

The first lines in each group give the coordinates of the unstable fixed points; the last line refers to the intercept of the separatrix on the axis of symmetry.

t	v	p
$t = 0$, mod. π	$\pm 32\sqrt{3}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1\left[1-\left(\frac{2}{1-\nu^2/N^2}-\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right]$	$64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left[1+\left(\frac{10}{1-\nu^2/N^2}+\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right]$
	0	$-128\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left[1-\left(\frac{2}{1-\nu^2/N^2}-\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right]$
	0	$128\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_2\left[1+\left(\frac{2}{1-\nu^2/N^2}-\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_2\right]$
$t=3\pi/4$, mod. π	$32\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1\left[1-\left(\frac{10\,\nu/N}{1-\nu^2/N^2}+\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right]$	$\mp 64\sqrt{3}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left[1+\frac{2\,\nu/N}{1-\nu^2/N^2}-\frac{1}{1+3\nu/N}\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right]$
	$-64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1\left[1+\left(\frac{2\,\nu/N}{1-\nu^2/N^2}-\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right]$	0
	$64\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_2\left[1-\left(\frac{2\,\nu/N}{1-\nu^2/N^2}-\frac{1}{1+3\nu/N}\right)\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_2\right]$	0

Since the foregoing results have not been carried consistently beyond terms of order $\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$, it may be considered sufficient to replace the coefficients of $\left(\frac{1}{3}-\frac{\nu}{N}\right)$ in the lastterm of the correction factors by the value which these coefficients assume as $\nu/N \to 1/3$. Thus the correction factor for the value of v given in the first line of Table III might be consistently written as $[1 - (7/4)(1/3 - \nu/N)]$. Indeed, since $\eta_1 \cong 1 - \lambda \cong 1 - (7/2)(1/3 - \nu/N)$, the factor η_1 outside the square bracket might be replaced by unity and a composite correction factor $[1 - (21/4)(1/3 - \nu/N)]$ employed in this case.* Although this contention cannot be gainsaid, we elect, however to leave our results in the form summarized in Table III, being guided, in part, by some computational results pertaining to the case $t = 0$, mod. π [Sect. D 4].

*Also $\eta_2 \cong (1/2)(1 - 5\lambda/8)$.

3. Use of Quantities Akin to Angle-Action Variables

(a) The forward transformations: We commence again with the Hamiltonian of eqn. (13),

$$H = (1/2)\, p^2 + (1/2)\, (2\,\nu/N)^2\, v^2 + (1/6)\, (\sin 2\,t)\, v^3 \,, \tag{13}$$

and make a series of canonical transformations from the conjugate pair v, p to γ_0 , J_0 ; γ_1 , J_1 ; and γ_2 , J_2 . The first transformation is defined by the generating function

$$G_0\,(v, \gamma_0) = (\nu/N)\, v^2 \operatorname{ctn} \gamma_0 \,, \tag{47}$$

so that

$$p = \partial G_0/\partial v = (2\,\nu/N)\, v \operatorname{ctn} \gamma_0 \tag{48a}$$

$$J_0 = -\,\partial G_0/\partial \gamma = (\nu/N)\, v^2 \csc^2 \gamma_0 \quad ; \tag{48b}$$

thus

$$\operatorname{ctn} \gamma_0 = \frac{N}{2\nu}\,\frac{p}{v} \tag{49a}$$

$$J_0 = \frac{1}{2}\left(\frac{N}{2\nu}\right) p^2 + \frac{1}{2}\left(\frac{2\nu}{N}\right) v^2 \,, \tag{49b}$$

$$v = (N/\nu)^{1/2}\, J_0^{\,1/2} \sin \gamma_0 \tag{49c}$$

$$p = 2\,(\nu/N)^{1/2}\, J_0^{\,1/2} \cos \gamma_0 \quad , \tag{49d}$$

and the new Hamiltonian is

$$\begin{aligned} K_0 &= H + \partial G_0/\partial t \\ &= H \\ &= 2\,(\nu/N)\, J_0 + (1/6)\,(N/\nu)^{3/2}\, J_0^{\,3/2} \sin^3 \gamma_0 \,\sin 2\,t \\ &= 2\,(\nu/N)\, J_0 + (1/48)\,(N/\nu)^{3/2}\, J_0^{\,3/2} \big[3 \cos(\gamma_0 - 2\,t) \\ &\qquad - 3 \cos(\gamma_0 + 2t) + \cos(3\gamma_0 + 2\,t) - \cos(3\,\gamma_0 - 2\,t)\big] \,, \end{aligned} \tag{50}$$

In analogy to the procedure followed in Section D 2 in formulating the transformation from ζ_0 , $\bar{\zeta}_0$ to ζ_1 , $\bar{\zeta}_1$, we now introduce a second generating function

$$G_1(\gamma_1, J_1) = J_1 \cdot \gamma_0 + (1/96)(N/\nu)^{3/2} J_1^{3/2} \left[3\frac{\sin(\gamma_0 - 2t)}{1 - \nu/N} + 3\frac{\sin(\gamma_0 + 2t)}{1 + \nu/N} - \frac{\sin(3\gamma_0 + 2t)}{1 + 3\nu/N} \right], \quad (51)$$

so that

$$J_0 = \partial G_1/\partial \gamma_0$$

$$= J_1 + (1/32)(N/\nu)^{3/2} J_1^{3/2} \left[\frac{\cos(\gamma_0 - 2t)}{1 - \nu/N} + \frac{\cos(\gamma_0 + 2t)}{1 + \nu/N} - \frac{\cos(3\gamma_0 + 2t)}{1 + 3\nu/N} \right] \quad (52a)$$

$$\gamma_1 = \partial G_1/\partial J_1$$

$$= \gamma_0 + (1/64)(N/\nu)^{3/2} J_1^{1/2} \left[3\frac{\sin(\gamma_0 - 2t)}{1 - \nu/N} + 3\frac{\sin(\gamma_0 + 2t)}{1 + \nu/N} - \frac{\sin(3\gamma_0 + 2t)}{1 + 3\nu/N} \right], \quad (52b)$$

and

$$K_1 = K_0 + \partial G_1/\partial t$$

$$= K_0 + (1/48)(N/\nu)^{3/2} J_1^{3/2} \left[-3\frac{\cos(\gamma_0 - 2t)}{1 - \nu/N} + 3\frac{\cos(\gamma_0 + 2t)}{1 + \nu/N} - \frac{\cos(3\gamma_0 + 2t)}{1 + 3\nu/N} \right] \quad (53)$$

The new Hamiltonian, K_1, can be expressed in terms of the new variables γ_1 , J_1 without much difficulty [Appendix B], with the result

$$K_1 = 2(\nu/N) J_1 - (1/48)(N/\nu)^{3/2} J_1^{3/2} \cos(3\gamma_1 - 2t) + (1/2048)(N/\nu)^3 J_1^2 \left[\frac{6\nu/N}{1 - \nu^2/N^2} - \frac{1}{1 + 3\nu/N} \right], \quad (54)$$

in which we have retained[10] only terms independent of t and of γ_1 in the term involving J_1^2 .

It now only remains to introduce a third generating function,

$$G_2(\gamma_1, J_2) = J_2(\gamma_1 - \frac{2}{3}t), \tag{55}$$

which effects the transformation

$$J_1 = \partial G_2/\partial\gamma_1 = J_2 \tag{56a}$$

$$\gamma_2 = \partial G_2/\partial J_2 = \gamma_1 - \frac{2}{3}t \tag{56b}$$

with

$$\begin{aligned} K_2 &= K_1 + \partial G_2/\partial t \\ &= K_1 - \frac{2}{3}J_2 \\ &= -(2/3 - 2\nu/N)J_2 - (1/48)(N/\nu)^{3/2} J_2^{3/2}\cos 3\gamma_2 \\ &\qquad + (\alpha/2048)(N/\nu)^3 J_2^2 , \end{aligned} \tag{57}$$

where, as previously,

$$\alpha \equiv \frac{6\nu/N}{1-\nu^2/N^2} - \frac{1}{1+3\nu/N} \qquad [\text{cf. eqn. (25)}]$$

and t-dependent terms have been omitted[10] from the term involving J_2^2 .

This final Hamiltonian K_2, as expressed by eqn. (57) and which we shall take to be substantially a constant of the motion, is seen to be identical in form to the Hamiltonian H_2 of eqn. (35), as developed in Section D 2 save that the sine function is here fortuitously replaced by the cosine. It remains to perform with the present variables the reverse transformations required to carry particular values of γ_2 , J_2 back to the original quantities v, p--both the forward transformation and the reverse transformation which follows, however, appear to be somewhat simpler algebraically than the corresponding steps required with the ζ , $\bar{\zeta}$ variables.

(b) The separatrix: To initiate the reverse transformation in this case, we shall focus our attention as before [Section D 2 b] on the particular salient points of the separatrix:*

For the Fixed Points

$$\gamma_1 = \begin{cases} \pi/3 + 2t/3, \\ -\pi/3 + 2t/3, \\ \pi + 2t/3, \end{cases} \quad \text{with } J_1^{1/2} = 64\,(1/3 - \nu/N)(\nu/N)^{3/2}\,\eta_1\,; \tag{58}$$

For the Intercept of the Separatrix

$$\gamma_1 = 0 + \frac{2t}{3}, \qquad \text{with } J_1^{1/2} = 64\,(1/3 - \nu/N)\,(\nu/N)^{3/2}\,\eta_2\,. \tag{59}$$

(c) The reverse transformation to the original variables: For evaluation of the original variables v, p one notes from eqns. (49c, d) that the quantities explicitly required are $\sin\gamma_0$ and $\cos\gamma_0$, in addition to $J_0^{1/2}$. To the degree of accuracy with which we are concerned in the present work, it is sufficient for this purpose to refer to eqn. (52b) and write

$$\begin{aligned}\sin\gamma_0 &\doteq \sin\gamma_1 - (\cos\gamma_1)(\gamma_1 - \gamma_0) \\ &\doteq \sin\gamma_1 - \frac{\cos\gamma_1}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2}\left[3\frac{\sin(\gamma_1 - 2t)}{1 - \nu/N} + 3\frac{\sin(\gamma_1 + 2t)}{1 + \nu/N} - \frac{\sin(3\gamma_1 + 2t)}{1 + 3\nu/N}\right]\end{aligned} \tag{60a}$$

and

$$\begin{aligned}\cos\gamma_0 &\doteq \cos\gamma_1 + (\sin\gamma_1)(\gamma_1 - \gamma_0) \\ &\doteq \cos\gamma_1 + \frac{\sin\gamma_1}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2}\left[3\frac{\sin(\gamma_1 - 2t)}{1 - \nu/N} + 3\frac{\sin(\gamma_1 + 2t)}{1 + \nu/N} - \frac{\sin(3\gamma_1 + 2t)}{1 + 3\nu/N}\right]\end{aligned} \tag{60b}$$

---

*Because of the presence of $\cos 3\gamma_2$ in eqn. (57), in contrast to the presence of $\sin 3\bar{\gamma}$ in eqn. (35), the values of γ_2 which are of interest here may be related to the corresponding values of $\bar{\gamma}$ by $\gamma_2 = \bar{\gamma} + \pi/2$, or, similarly, $\gamma_1 = \gamma + \pi/2$. This distinction between γ_2 and $\bar{\gamma}$ of course could have been avoided by introduction of a phase shift in the generating function G_2.

while obtaining $J_o^{1/2}$ by [cf. eqn. (52a)]

$$J_o^{1/2} = J_1^{1/2}\left\{1 + \frac{1}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2}\left[\frac{\cos(\gamma_1 - 2t)}{1 - \nu/N} + \frac{\cos(\gamma_1 + 2t)}{1 + \nu/N} - \frac{\cos(3\gamma_1 + 2t)}{1 + 3\nu/N}\right]\right\}. \quad (60c)$$

Thus for $t = 0$ and $\gamma_1 = \pm \pi/3$, eqns. (60a-c) give

$$\sin\gamma_0 = \pm \frac{\sqrt{3}}{2}\left[1 - \frac{3}{1 - \nu^2/N^2}\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right],$$

$$\cos\gamma_0 = \frac{1}{2}\left[1 + \frac{9}{1 - \nu^2/N^2}\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right],$$

and

$$J_o^{1/2} = 64\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1\left[1 + \left(\frac{1}{1 - \nu^2/N^2} + \frac{1}{1 + 3\nu/N}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right],$$

so that, by eqns. (49c, d), the fixed point coordinates

$$v = \pm 32\sqrt{3}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1\left[1 - \left(\frac{2}{1 - \nu^2/N^2} - \frac{1}{1 + 3\nu/N}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right]$$

$$p = 64\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left[1 + \left(\frac{10}{1 - \nu^2/N^2} + \frac{1}{1 + 3\nu/N}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right]$$

are obtained. Similarly for the next case in the list (58), with $t = 0$ and $\gamma_1 = \pi$,

$$\sin\gamma_0 = 0$$

$$\cos\gamma_0 = -1$$

$$J_o^{1/2} = 64\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2}\eta_1\left[1 - \left(\frac{2}{1 - \nu^2/N^2} - \frac{1}{1 + 3\nu/N}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right],$$

so that

$$v = 0$$

$$p = -128\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left[1 - \left(\frac{2}{1 - \nu^2/N^2} - \frac{1}{1 + 3\nu/N}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right].$$

In this same way one finds complete agreement with all the results listed in Table III.

(d) The unstable equilibrium orbit: The procedure just followed can, of course, be employed in general to provide, as a function of t, the equation of the unstable equilibrium orbit, which is represented (mod. π) by fixed points as listed in sub-section (b).

For the unstable equilibrium orbits, in particular, the Hamiltonian equations which follow from K_2 [eqn. (57)] permit γ_2 and J_2 to be constant, with, let us say,

$$\gamma_2 = \pi \qquad \text{and } J_2^{1/2} = 64\,(1/3 - \nu/N)\,(\nu/N)^{3/2}\eta_1 \quad [\text{cf. (b)}].$$

Then

$$\gamma_1 = \gamma_2 + \frac{2t}{3} \tag{61a}$$

$$= \pi + \frac{2t}{3}\,,$$

$$J_1^{1/2} = J_2^{1/2}$$

$$= 64\,(1/3 - \nu/N)\,(\nu/N)^{3/2}\eta_1 \qquad [\text{cf. eqns. (58)}]\,. \tag{61b}$$

By making use of eqns. (60a-c), in conjunction with eqn. (49c), the equation for the unstable equilibrium orbit, v(t), is then found to be

$$v(t) = -64\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1\left\{\sin\frac{2t}{3} - \left[\frac{\sin 2t/3}{1-\nu/N} + 4\frac{(\nu/N)\sin 2t}{1-\nu^2/N^2} - \left(\frac{1}{1+\nu/N} - \frac{1}{1+3\nu/N}\right)\sin\frac{10t}{3}\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1\right\}, \tag{62}$$

through quantities of the order of $(1/3 - \nu/N)^2$. The expression (62) is seen to contain circular functions of argument 2 t/3, 2 t, and 10 t/3, as was the case for the trial function (7) employed in the variational treatment of Section C. By substitution of particular values of t, the specific values of v for the fixed points listed in Table III may be obtained.

It may be noted, however, that differentiation of (62), which results in

$$p \equiv dv/dt = -128\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{1}{3}\ \frac{\nu}{N}\right)\eta_1\left\{\cos\frac{2t}{3}-\left[\frac{\cos 2t/3}{1-\nu/N}+12\frac{(\nu/N)\cos 2t}{1-\nu^2/N^2}\right.\right.$$
$$\left.\left.-5\left(\frac{1}{1+\nu/N}-\frac{1}{1+3\nu/N}\right)\cos\frac{10t}{3}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right\}, \quad (63)$$

does not lead exactly to the specific forms listed in Table III, although the forms become coincident through $(1/3 - \nu/N)^2$ when $\left(\frac{1}{3}\ \frac{\nu}{N}\right)$ is expanded as $\left(\frac{\nu}{N}\right)^2\left[1 + 3\,(1/3 - \nu/N)\right]$. An expression for p may be obtained directly from eqn. (49d) of course, just as eqn. (62) was obtained from eqn. (49c), with the result

$$p = -128\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left\{\cos\frac{2t}{3}+\left[\frac{\cos 2t/3}{1-\nu/N}-4\frac{\cos 2t}{1-\nu^2/N^2}\right.\right.$$
$$\left.\left.+\left(\frac{1}{1+\nu/N}+\frac{1}{1+3\nu/N}\right)\cos\frac{10t}{3}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right\}, \quad (64)$$

from which the "momenta" for the fixed points listed in Table III follow for the special cases.

4. Computational Results

(a) The unstable equilibrium orbit: To establish a connection with Section C, in which the results of the variational method were presented, we note first that for $\nu/N = 0.3$ eqn. (62) leads to the unstable equilibrium orbit as given by

$$v\,(t) = -0.56206\sin 2t/3 + 0.02373\sin 2t - 0.00437\sin 10t/3\,, \quad (65)$$

while the alternative forms for p [eqn. (63), obtained by differentiation of eqn. (62), or eqn. (64), obtained directly from γ_2, J_2] are

$$p \equiv \frac{dv}{dt} = -0.37470\cos 2t/3 + 0.04745\cos 2t - 0.01457\cos 10t/3 \quad (66)$$

or

$$p = -0.36808 \cos 2t/3 + 0.04745 \cos 2t - 0.01399 \cos 10t/3 . \qquad (67)$$

These expressions may be compared with the Fourier analysis of computer results for this case, as given by eqns. (12a, b) of Section C 2. There is, of course, no fundamental basis for choosing between formulas (66) and (67) since, as noted previously, eqns. (63) and (64) are identical through terms in $(1/3 - \nu/N)^2$. It is in any event clear that the present results differ by a few percent from the computer results for $\nu/N = 0.3$.

(b) The fixed points: The results presented in Table III for the unstable fixed points at $t = 0$ (mod. π) and at $t = 3\pi/4$ (mod. π) have been subjected to computational checks for $\nu/N = 0.3$ and for $\nu/N = 0.3275$. Computational data pertaining to the fixed points at $t = 0$ (mod. π) have also been obtained for a series of values of ν/N, ranging from 0.30 to 0.36, in order to exhibit the dependence of the accuracy on the proximity to the $\nu/N \rightarrow 1/3$ resonance. We present these results below, to be followed in the suceeding sub-section by data for $\nu/N = 0.3$ which pertain to the "intercept" of the separatrix on the symmetry axis of the phase diagrams.

The coordinates of the fixed points, as calculated by the expressions listed in Table III, are compared with computer results for $\nu/N = 0.3$ in Table IV. The agreement with the computer results is seen to be poorer in Table IV than was obtained by the variational method summarized in Table I for $\nu/N = 0.3$.

TABLE IV

COORDINATES OF UNSTABLE FIXED POINTS,
AS CALCULATED FROM THE EXPRESSIONS OF TABLE III
AND AS OBTAINED FROM COMPUTER RESULTS

$\nu/N = 0.3$

t	From expressions of Table III		From Computer	
	v	p	v	p
0, mod. π	0 $\pm$ 0.48297	- 0.33461 0.2384$\dot{9}$	0 $\pm$ 0.4943	- 0.3506 0.2440
$3\pi/4$, mod. π	- 0.59015 0.25949	0 $\mp$ 0.30665	- 0.6024 0.2668	0 $\mp$ 0.3207

To illustrate results applying to operation nearer the $\nu/N \rightarrow 1/3$ resonance, the coordinates of the fixed points, as calculated by the expressions listed in Table III, are similarly compared in Table V with computer results for $\nu/N = 0.3275$.

TABLE V

COORDINATES OF UNSTABLE FIXED POINTS,
AS CALCULATED FROM THE EXPRESSIONS OF TABLE III
AND AS OBTAINED FROM COMPUTER RESULTS

$\nu/N = 0.3275$

t	From expressions of Table III		From Computer	
	v	p	v	p
0, mod. π	0 $\pm$ 0.10284	- 0.07778 0.04191	0 $\pm$ 0.10295	- 0.07793 0.0419$\dot{5}$
$3\pi/4$, mod. π	- 0.12009_5 0.05854	0 $\mp$ 0.06812	- 0.12021 0.05861	0 $\pm$ 0.06825

As is to be expected, the agreement in this case, with $\nu/N = 0.3275$, is considerably better than for the case $\nu/N = 0.3$ for which the results were described previously in Table IV.

As was mentioned earlier, it is of interest to examine the analytic results, in comparison with computer data, for various values of ν/N. The results of such a comparison, for $t = 0 \pmod{\pi}$ and ν/N in the range 0.30 to 0.36 are summarized below in Table VI,* in which the formulas used to obtain the theoretical results are those of Table III. The data are presented graphically in Figs. 1 through 3, and the percentage of error in the theoretical results is shown in Fig. 4.

A detailed numerical examination of the computer data summarized in Table VI (forming, for example, such quantities as

$$\frac{1}{\nu/N - 1/3}\left[\frac{p}{128\,(\nu/N)^2\,(\nu/N - 1/3)\,\eta_1} - 1\right]$$

and

$$\frac{1}{\nu/N - 1/3}\left[\frac{p}{128\,(\nu/N)^2\,(\nu/N - 1/3)} - 1\right]$$

for the various values of ν/N employed and noting that these quantities respectively approach 7/4 and 21/4 as $\nu/N \rightarrow 1/3$)suggests that the theory has, in fact, been carried correctly through terms of second order in $\nu/N - 1/3$. The correctness of this conclusion may, in fact, be immediately apparent from the second order dependence of the <u>relative</u> error on $\nu/N - 1/3$ in the graphs of Fig. 4.

*I am indebted to Mr. Igor Sviatoslavsky for assistance in performing some of the calculations necessary in the processing of these data.

TABLE VI

COORDINATES OF UNSTABLE FIXED POINTS,
AS CALCULATED FROM THE EXPRESSIONS OF TABLE III AND AS OBTAINED FROM COMPUTER RESULTS

$t = 0 \pmod{77}$

ν/N	Fixed Point on Symmetry Axis			Fixed Points to right and left of Symmetry Axis					
	p formula	p computer	Error %	± v formula	± v computer	Error %	p formula	p computer	Error %
0.300	-0.33461	-0.35065	-4.57	±0.48297	±0.49430	-2.29	+ 0.23849	+ 0.24398	-2.25
0.305	-0.29897	-0.30971	-3.47	±0.42445	±0.43205	-1.76	+ 0.20384	+ 0.20731	-1.67
0.310	-0.25895	-0.26554	-2.48	±0.36171	±0.36638	-1.27	+ 0.16850	+ 0.17049	-1.17
0.315	-0.21416	-0.21768	-1.62	±0.29439	±0.29689	-0.84	+ 0.13262	+ 0.13361	-0.74
0.320	-0.16408	-0.16558	-0.91	±0.22202	±0.22310	-0.48	+ 0.09639	+ 0.09678	-0.40
0.3225	-0.13688	-0.13774	-0.62	±0.18379	±0.18440	-0.33	+ 0.07822	+ 0.07843	-0.27
0.325	-0.10815	-0.10856	-0.38	±0.14409	±0.14439	-0.21	+ 0.06004	+ 0.06014	-0.16
0.3275	-0.07778	-0.07793	-0.20	±0.10284	±0.10295	-0.11	+ 0.04191	+ 0.04195	-0.08
0.33	-0.04568	-0.04571	-0.08	±0.05994	±0.05997	-0.05	+ 0.02386	+ 0.02387	-0.04
0.3325	-0.01174	-0.01176	---	±0.01529	±0.01531	---	+ 0.00594	+ 0.00595	---
0.340	+0.10237	+ 0.10269	-0.30	±0.13038	±0.13060	-0.17	-0.04649	-0.04654	-0.10
0.345	+0.19025	+0.19226	-1.04	±0.23878	±0.24027	-0.62	-0.07963	-0.07987	-0.31
0.350	+0.28943	+0.29655	-2.40	±0.35808	±0.36341	-1.47	-0.11042	-0.11106	-0.57
0.355	+0.40211	+0.42186	-4.68	±0.49047	±0.50549	-2.97	-0.13781	-0.13883	-0.74
0.360	+0.53130	+0.58071	-8.51	±0.63906	±0.67734	-5.65	-0.1603	-0.16070	(-0.25)

(c) The intercept: The intercept of the separatrix on the symmetry axis, for which formulas have been given in Table III, is somewhat more tedious to determine computationally than the location of the fixed points. Computational estimates of the intercept have been obtained, however, for $\nu/N = 0.3$ at $t = 0$ (mod. π) and at $t = 3\pi/4$ (mod. π). The comparison of the theoretical and computational intercepts for these cases is given in Table VII.

TABLE VII

LOCATION OF THE INTERCEPT ON THE AXIS OF SYMMETRY, $\nu/N = 0.3$

t (mod. π)	LOCATION OF INTERCEPT From Table III	From Computer	Relative Error %
0	p = 0.1886	0.191	2
$3\pi/4$	v = 0.3024	0.308	2

E. REFERENCES

1. Cf. L. J. Laslett and A. M. Sessler, MURA-248 (1957).

2. Jürgen Moser, Nach. Gött. Akad. (Math.-Phys. Kl.) Nr. 6, 87-120 (1955).

3. MURA General Conference (Feb. 28, 1959)--presentation of ideas previously suggested by K. Symon in internal discussions.

4. F. T. Cole, MURA/FTC-3 (1956). In the present work the sign of b_1 (θ) has been changed from that appearing in Dr. Cole's report--see footnote p. 6 of MURA-263.

5. E. D. Courant and H. S. Snyder, Annals of Physics 3, No. 1, 1-48 (January, 1958)--Section 4a, esp. eqns. (4.4) and (4.5), p. 18.

6. This simplification is in the spirit of the "smooth approximation"--K. R. Symon, KRS (MURA)-1 (1954); K. R. Symon, et al., Phys. Rev. 103, 1837 (1956), Appendix A.

7. J. N. Snyder, DUCK-ANSWER (I. B. M. Program 75), MURA-237 (1957).

8. J. N. Snyder, FORANAL (I. B. M. Program 52), MURA-228 (1957); John McNall, DUCKNALL (I. B. M. Program 219), MURA-438 (1958).

9. This consideration concerning the complex conjugate feature of the new variables has been pointed out by H. Meier, MURA-387 (1958).

10. The preferential retention of the constant part of the coefficient of J^2 appears to be consistent with ideas emphasized by P. Sturrock, Annals of Physics 3, No. 2, 113-189 (February, 1958). I am indebted to Mr. Meier, Dr. Cole, and Dr. Symon for discussions concerning this feature of the procedure.

APPENDIX A

EXPRESSION OF Ω_1 EXPLICITLY IN TERMS OF ξ_1, $\bar{\xi}_1$

An iterative solution of eqn. (22b) for ξ_0 (ξ_1 , $\bar{\xi}_1$) leads to

$$\begin{aligned}
\xi_0 &= \xi_1 + (i/48)(N/\nu)^{3/2}\left[\Phi_1\xi_1^2 + 2\Phi_2\,\xi_1\bar{\xi}_1 + 3\Phi_3\,\bar{\xi}_1^2\right] \\
&\quad - (1/1152)(N/\nu)^3\left[(\Phi_1^2+\psi_1)\,\xi_1^3 + (3\Phi_1\Phi_2 + 2\psi_2)\,\xi_1^2\bar{\xi}_1 + (3\Phi_1\Phi_3 + 2\Phi_2^2 + 3\psi_3)\,\xi_1\bar{\xi}_1^2 + (3\Phi_2\Phi_3 + 4\psi_4)\,\bar{\xi}_1^3\right] \\
&= \xi_1 + (i/48)(N/\nu)^{3/2}\left[\Phi_1\xi_1^2 + 2\Phi_2\,\xi_1\bar{\xi}_1 + 3\Phi_3\,\bar{\xi}_1^2\right] \\
&\quad + (1/1152)(N/\nu)^3\left[\left((3/2)\Phi_0\Phi_2 - (1/2)\Phi_1^2\right)\xi_1^3 + \left((9/2)\Phi_0\Phi_3 - (1/2)\Phi_1\Phi_2\right)\xi_1^2\bar{\xi}_1 + \left((3/2)\Phi_1\Phi_3 - (1/2)\Phi_2^2\right)\xi_1\bar{\xi}_1^2\right], \qquad \text{(A1)}
\end{aligned}$$

in which the cubic term has been simplified by elimination of ψ_1 , $\cdots$ through use of eqns. (2b-e). Solution of eqn. (22a) for $\bar{\xi}_0$ (ξ_1 , $\bar{\xi}_1$) similarly gives

$$\begin{aligned}
\bar{\xi}_0 &= \bar{\xi}_1 - (i/48)(N/\nu)^{3/2}\left[3\Phi_0\xi_1^2 + 2\Phi_1\,\xi_1\bar{\xi}_1 + \Phi_2\,\bar{\xi}_1^2\right] \\
&\quad + (1/1152)(N/\nu)^3\left[(3\Phi_0\Phi_1 + 4\psi_0)\,\xi_1^3 + (6\Phi_0\Phi_2 + \Phi_1^2 + 3\psi_1)\,\xi_1^2\bar{\xi}_1 + (9\Phi_0\Phi_3 + 2\Phi_1\Phi_2 + 2\psi_2)\,\xi_1\bar{\xi}_1^2 + (3\Phi_1\Phi_3 + \psi_3)\,\bar{\xi}_1^3\right] \\
&= \bar{\xi}_1 - (i/48)(N/\nu)^{3/2}\left[3\Phi_0\,\xi_1^2 + 2\Phi_1\,\xi_1\bar{\xi}_1 + \Phi_2\,\bar{\xi}_1^2\right] \\
&\quad + (1/1152)(N/\nu)^3\left[\left((3/2)\Phi_0\Phi_1 - (1/2)\Phi_1^2\right)\xi_1^2\bar{\xi}_1 + \left((9/2)\Phi_0\Phi_3 - (1/2)\Phi_1\Phi_2\right)\xi_1\bar{\xi}_1^2 + \left((3/2)\Phi_1\Phi_3 - (1/2)\Phi_2^2\right)\bar{\xi}_1^3\right]. \qquad \text{(A2)}
\end{aligned}$$

It may be noted that, since $\Phi_2 = \Phi_1^*$ and $\Phi_3 = \Phi_0^*$ [eqns. (20c, d)], eqns. (A1, 2) are consistent with the statement that ξ_1 , $\bar{\xi}_1$ form a complex conjugate pair to this order.

Forming the product of eqns. (A1, 2) then leads, through fourth order terms, to

$$\begin{aligned}
\xi_0\bar{\xi}_0 &= \xi_1\bar{\xi}_1 + (i/48)(N/\nu)^{3/2}\left[-3\Phi_0\xi_1^3 - \Phi_1\xi_1^2\bar{\xi}_1 + \Phi_2\,\xi_1\bar{\xi}_1^2 + 3\Phi_3\,\bar{\xi}_1^3\right] \\
&\quad + (1/768)(N/\nu)^3\left[\Phi_0\Phi_1\xi_1^4 + 4\Phi_0\Phi_2\,\xi_1^3\bar{\xi}_1 + (9\Phi_0\Phi_3 + \Phi_1\Phi_2)\,\xi_1^2\bar{\xi}_1^2 + 4\Phi_1\Phi_3\,\xi_1\bar{\xi}_1^3 + \Phi_2\Phi_3\,\bar{\xi}_1^4\right]. \qquad \text{(A3)}
\end{aligned}$$

In addition

$$\xi_0+\bar{\xi}_0 = \xi_1+\bar{\xi}_1 + (i/48)(N/\nu)^{3/2}\left[(-3\Phi_0+\Phi_1)\xi_1^2+(-2\Phi_1+2\Phi_2)\xi_1\bar{\xi}_1+(-\Phi_2+3\Phi_3)\bar{\xi}_1^2\right.$$

$$+ \text{ terms of third order,} \tag{A4}$$

so that

$$(\xi_0+\bar{\xi}_0)^3 = \xi_1^3 + 3\xi_1^2\bar{\xi}_1 + 3\xi_1\bar{\xi}_1^2 + \bar{\xi}_1^3$$
$$+(i/16)(N/\nu)^{3/2}\left[(-3\Phi_0+\Phi_1)\xi_1^4+(-6\Phi_0+2\Phi_2)\xi_1^3\bar{\xi}_1+(-3\Phi_0-3\Phi_1+3\Phi_2+3\Phi_3)\xi_1^2\bar{\xi}_1^2+(-2\Phi_1+6\Phi_3)\xi_1\bar{\xi}_1^3+(-\Phi_2+3\Phi_3)\bar{\xi}_1^4\right]. \tag{A5}$$

Finally, the expression for $\partial F_1/\partial t$, which appears as a function ξ_0, $\bar{\xi}_1$ in eqn. (23), assumes the form

$$\partial F/\partial t = -(i/48)(N/\nu)^{3/2}\left[\Phi_0'\xi_1^3+\Phi_1'\xi_1^2\bar{\xi}_1+\Phi_2'\xi_1\bar{\xi}_1^2+\Phi_3'\bar{\xi}_1^3\right]$$
$$+(1/2304)(N/\nu)^3\left[(3\Phi_0'\Phi_1+2\Psi_0')\xi_1^4+(6\Phi_0'\Phi_2+2\Phi_1'\Phi_1+2\Psi_1')\xi_1^3\bar{\xi}_1\right.$$
$$\left.+(9\Phi_0'\Phi_3+4\Phi_1'\Phi_2+\Phi_2'\Phi_1+2\Psi_2')\xi_1^2\bar{\xi}_1^2+(6\Phi_1'\Phi_3+2\Phi_2'\Phi_2+2\Psi_3')\xi_1\bar{\xi}_1^3+(3\Phi_2'\Phi_3+2\Psi_4')\bar{\xi}_1^4\right]$$
$$= -(i/48)(N/\nu)^{3/2}\left[\Phi_0'\xi_1^3+\Phi_1'\xi_1^2\bar{\xi}_1+\Phi_2'\xi_1\bar{\xi}_1^2+\Phi_3'\bar{\xi}_1^3\right]$$
$$+(1/1536)(N/\nu)^3\left[(\Phi_0'\Phi_1-\Phi_0\Phi_1')\xi_1^4+2(\Phi_0'\Phi_2-\Phi_0\Phi_2')\xi_1^3\bar{\xi}_1+(3\Phi_0'\Phi_3+\Phi_1'\Phi_2-\Phi_1\Phi_2'-3\Phi_0\Phi_3')\xi_1^2\bar{\xi}_1^2\right.$$
$$\left.+2(\Phi_1'\Phi_3-\Phi_1\Phi_3')\xi_1\bar{\xi}_1^3+(\Phi_2'\Phi_3-\Phi_2\Phi_3')\bar{\xi}_1^4\right] \tag{A6}$$

in terms of the variables ξ_1, $\bar{\xi}_1$.

Accordingly, the Hamiltonian Ω_1 becomes

$$
\begin{aligned}
\Omega_1 &= \Omega_0 + \partial F_1/\partial t \\
&= -i\,(2\nu/N)\,\xi_0\bar{\xi}_0 - (1/96)(N/\nu)^{3/2}(e^{2it} - e^{-2it})(\xi_0 + \bar{\xi}_0)^3 + \partial F_1/\partial t \\
&= -i\,(2\nu/N)\,\xi_1\bar{\xi}_1
\end{aligned}
$$

$$
-\frac{1}{48}\left(\frac{N}{\nu}\right)^{3/2}\left\{
\begin{aligned}
&\left[i\Phi_0' + 6(\nu/N)\Phi_0 + (1/2)(e^{2it} - e^{-2it})\right]\xi_1^3 \\
&+\left[i\Phi_1' + 2(\nu/N)\Phi_1 + (3/2)(e^{2it} - e^{-2it})\right]\xi_1^2\bar{\xi}_1 \\
&+\left[i\Phi_2' - 2(\nu/N)\Phi_2 + (3/2)(e^{2it} - e^{-2it})\right]\xi_1\bar{\xi}_1^2 \\
&+\left[i\Phi_3' - 6(\nu/N)\Phi_3 + (1/2)(e^{2it} - e^{-2it})\right]\bar{\xi}_1^3
\end{aligned}
\right\}
$$

$$
-\frac{i}{1536}\left(\frac{N}{\nu}\right)^{3}\left\{
\begin{aligned}
&\left(\begin{aligned}
&\left[i\Phi_0' + 6(\nu/N)\Phi_0 + (1/2)(e^{2it} - e^{-2it})\right]\Phi_1 \\
&-\left[i\Phi_1' + 2(\nu/N)\Phi_1 + (3/2)(e^{2it} - e^{-2it})\right]\Phi_0 - (1/2)(3\Phi_0 - \Phi_1)(e^{2it} - e^{-2it})
\end{aligned}\right)\xi_1^4 \\
&+\left(\begin{aligned}
&2\left[i\Phi_0' + 6(\nu/N)\Phi_0 + (1/2)(e^{2it} - e^{-2it})\right]\Phi_2 \\
&-2\left[i\Phi_2' - 2(\nu/N)\Phi_2 + (3/2)(e^{2it} - e^{-2it})\right]\Phi_0 - (3\Phi_0 - \Phi_2)(e^{2it} - e^{-2it})
\end{aligned}\right)\xi_1^3\bar{\xi}_1 \\
&+\left(\begin{aligned}
&3\left[i\Phi_0' + 6(\nu/N)\Phi_0 + (1/2)(e^{2it} - e^{-2it})\right]\Phi_3 \\
&+\left[i\Phi_1' + 2(\nu/N)\Phi_1 + (3/2)(e^{2it} - e^{-2it})\right]\Phi_2 \\
&-\left[i\Phi_2' - 2(\nu/N)\Phi_2 + (3/2)(e^{2it} - e^{-2it})\right]\Phi_1 \\
&-3\left[i\Phi_3' - 6(\nu/N)\Phi_3 + (1/2)(e^{2it} - e^{-2it})\right]\Phi_0 - (3/2)(\Phi_0 + \Phi_1 - \Phi_2 - \Phi_3)(e^{2it} - e^{-2it})
\end{aligned}\right)\xi_1^2\bar{\xi}_1^2 \\
&+\left(\begin{aligned}
&2\left[i\Phi_1' + 2(\nu/N)\Phi_1 + (3/2)(e^{2it} - e^{-2it})\right]\Phi_3 \\
&-2\left[i\Phi_3' - 6(\nu/N)\Phi_3 + (1/2)(e^{2it} - e^{-2it})\right]\Phi_1 - (\Phi_1 - 3\Phi_3)(e^{2it} - e^{-2it})
\end{aligned}\right)\xi_1\bar{\xi}_1^3 \\
&+\left(\begin{aligned}
&\left[i\Phi_2' - 2(\nu/N)\Phi_2 + (3/2)(e^{2it} - e^{-2it})\right]\Phi_3 \\
&-\left[i\Phi_3' - 6(\nu/N)\Phi_3 + (1/2)(e^{2it} - e^{-2it})\right]\Phi_2 - (1/2)(\Phi_2 - 3\Phi_3)(e^{2it} - e^{-2it})
\end{aligned}\right)\bar{\xi}_1^4
\end{aligned}
\right\},
$$

which, by virtue of eqns. (19a-d), reduces to*

$$\Omega_1 = -i(2\nu/N)\xi_1\bar{\xi}_1 - (1/96)(N/\nu)^{3/2}\left[e^{2it}\xi_1^3 - e^{-2it}\bar{\xi}_1^3\right]$$

$$-\frac{i}{3072}\left(\frac{N}{\nu}\right)^3 \left\{ \begin{aligned} & \left[-3\Phi_0(e^{2it}-e^{-2it}) + \Phi_1(2e^{2it}-e^{-2it})\right]\xi_1^4 \\ & + \left[-6\Phi_0(e^{2it}-e^{-2it}) + 2\Phi_2(2e^{2it}-e^{-2it})\right]\xi_1^3\bar{\xi}_1 \\ & + \left[-3\Phi_0(e^{2it}-2e^{-2it}) - 3\Phi_1(e^{2it}-e^{-2it}) \right. \\ & \qquad \left. + 3\Phi_2(e^{2it}-e^{-2it}) + 3\Phi_3(2e^{2it}-e^{-2it})\right]\xi_1^2\bar{\xi}_1^2 \\ & + \left[-2\Phi_1(e^{2it}-2e^{-2it}) + 6\Phi_3(e^{2it}-e^{-2it})\right]\xi_1\bar{\xi}_1^3 \\ & + \left[-\Phi_2(e^{2it}-2e^{-2it}) + 3\Phi_3(e^{2it}-e^{-2it})\right]\bar{\xi}_1^4 \end{aligned} \right\}. \tag{A7}$$

With respect to the quartic terms, use will be made in particular[10] of that part of the coefficient of $\xi_1^2\bar{\xi}_1^2$ which is independent of t --this specific contribution to Ω_1 is

$$-\frac{i}{2048}\left(\frac{N}{\nu}\right)^3\left[6\,\frac{\nu/N}{1-(\nu/N)^2} - \frac{1}{1+3\nu/N}\right]\xi_1^2\bar{\xi}_1^2 ,$$

as is readily found by use of eqns. (20a-d) for the functions Φ_0, Φ_1, Φ_2, and Φ_3.

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*It was to effect this specific reduction of the cubic term that the quantities $\Phi_0, \cdots \Phi_3$ were required to satisfy eqns. (19a-d).

APPENDIX B

EXPRESSION OF K_1 EXPLICITLY IN TERMS OF γ_1, J_1

The Hamiltonian K_1 as given by eqn. (53), with K_o represented by eqn. (50) and the dynamical variables by eqns. (52a, b), may be expressed

$$K_1 = 2(\nu/N)J_1 + (1/48)(N/\nu)^{1/2}J_1^{3/2}\left[3\frac{\cos(\gamma_o - 2t)}{1-\nu/N} + 3\frac{\cos(\gamma_o + 2t)}{1+\nu/N} - 3\frac{\cos(3\gamma_o + 2t)}{1+3\nu/N}\right]$$

$$+ (1/48)(N/\nu)^{3/2}\left\{J_1^{3/2} + (3/64)(N/\nu)^{3/2}J_1^2\left[\frac{\cos(\gamma_o - 2t)}{1-\nu/N} + \frac{\cos(\gamma_o + 2t)}{1+\nu/N} - \frac{\cos(3\gamma_o + 2t)}{1+3\nu/N}\right]\right\}\times$$

$$\times\left\{3\cos(\gamma_o - 2t) - 3\cos(\gamma_o + 2t) + \cos(3\gamma_o + 2t) - \cos(3\gamma_o - 2t)\right\}$$

$$+ (1/48)(N/\nu)^{3/2}J_1^{3/2}\left[-3\frac{\cos(\gamma_o - 2t)}{1-\nu/N} + 3\frac{\cos(\gamma_o + 2t)}{1+\nu/N} - \frac{\cos(3\gamma_o + 2t)}{1+3\nu/N}\right]$$

$$= 2(\nu/N)J_1 + (1/48)(N/\nu)^{3/2}J_1^{3/2}\left\{\begin{array}{l}\left[3\frac{\cos(\gamma_o - 2t)}{1-\nu/N} + 3\frac{\cos(\gamma_o + 2t)}{1+\nu/N} - 3\frac{\cos(3\gamma_o + 2t)}{1+3\nu/N}\right]\cdot\left(\frac{\nu}{N}\right) \\ + 3\cos(\gamma_o - 2t) - 3\cos(\gamma_o + 2t) + \cos(3\gamma_o + 2t) - \cos(3\gamma_o - 2t) \\ -3\frac{\cos(\gamma_o - 2t)}{1-\nu/N} + 3\frac{\cos(\gamma_o + 2t)}{1+\nu/N} - \frac{\cos(3\gamma_o + 2t)}{1+3\nu/N}\end{array}\right\}$$

$$+ (1/1024)(N/\nu)^{3/2}J_1^2\left[\begin{array}{l}\frac{3}{2}\frac{1}{1-\nu/N} - \frac{3}{2}\frac{1}{1+\nu/N} - \frac{1}{2}\frac{1}{1+3\nu/N} \\ \text{plus terms of argument} \\ 4t,\ 2\gamma_o,\ 4\gamma_o,\ 6\gamma_o,\ 2\gamma_o \pm 4t,\ 4\gamma_o \pm 4t,\ 6\gamma_o + 4t\end{array}\right] . \qquad \text{(B1)}$$

By the nature of the transformation, as determined by the selected generating function G_1, the coefficient of $J_1^{3/2}$ is such that a considerable cancellation is seen to be possible. Those terms in the coefficient of J_1^2 which involve t and/or γ_0 will be ignored, since, to the order to which the analysis is to be carried, they will not contribute t-independent terms to the Hamiltonian which results from the final transformation.[10]

In view of the remarks just made, K_1 is taken to be effectively

$$K_1 = 2\,(\nu/N)\,J_1 - (1/48)\,(N/\nu)^{3/2}\,J_1^{3/2}\cos(3\gamma_0 - 2t)$$
$$+ (1/2048)\,(N/\nu)^3\,J_1^2\left[\frac{6\,\nu/N}{1-\nu^2/N^2} - \frac{1}{1+3\,\nu/N}\right] . \tag{B2}$$

Since [cf. eqn. (52b)] the variable γ_0 differs from γ_1 by terms of order $J_1^{1/2}$, we may expect that substitution for γ_0 in the second term of eqn. (B2) will contribute additional terms to the coefficient of J_1^2; this substitution, however, will not introduce terms other than those of the form which already have been ignored in the coefficient of J_1^2 and we therefore write, finally,

$$K_1 = 2\,(\nu/N)\,J_1 - (1/48)\,(N/\nu)^{3/2}J_1^{3/2}\cos(3\gamma_1 - 2t)$$
$$+ (1/2048)\,(N/\nu)^3\,J_1^2\left[\frac{6\,\nu/N}{1-\nu^2/N^2} - \frac{1}{1+3\,\nu/N}\right] . \tag{B3}$$

The last factor appearing in the J_1^2 term will be recognized as the parameter denoted by α in the text [eqn. (25)].

FIG. 1

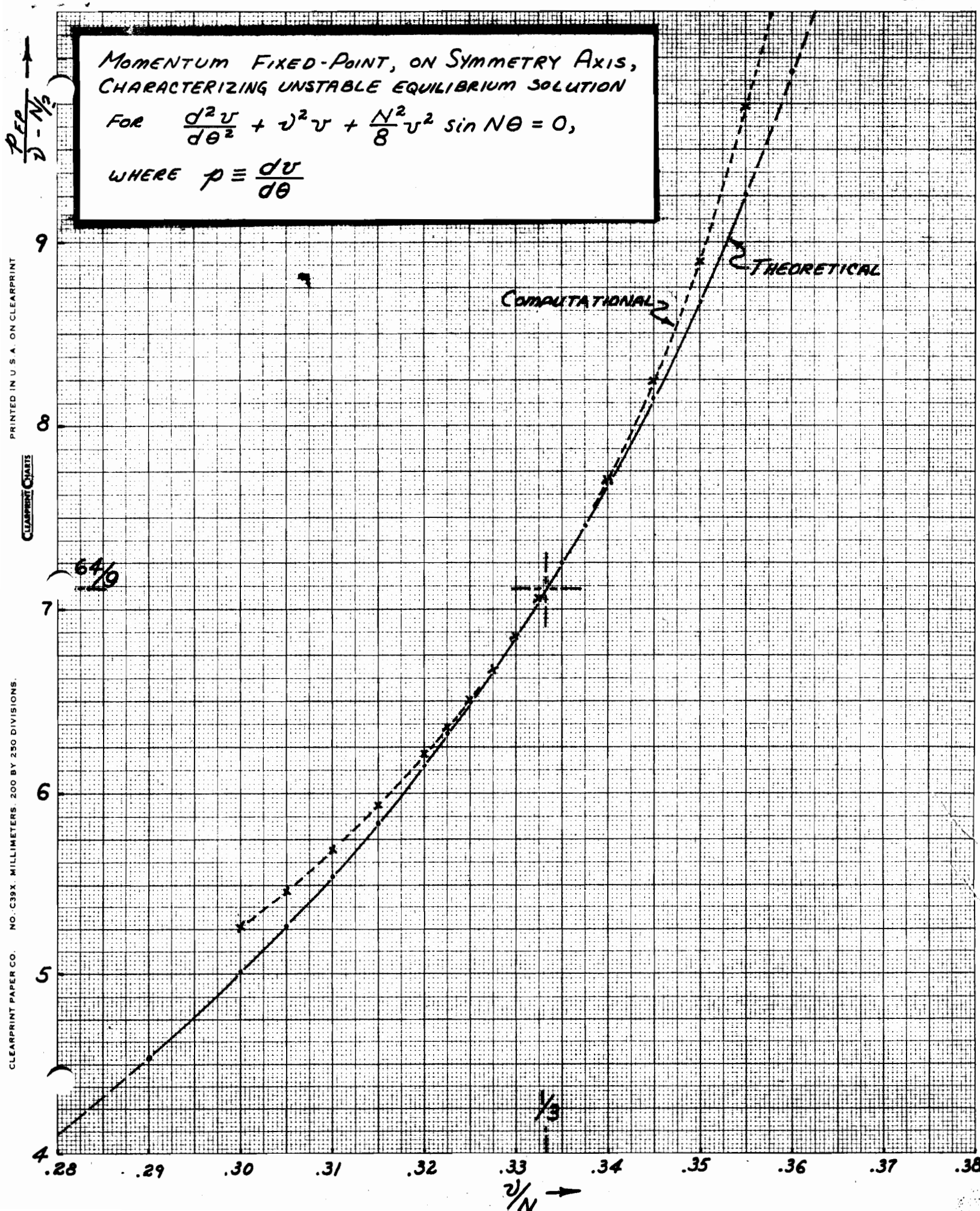

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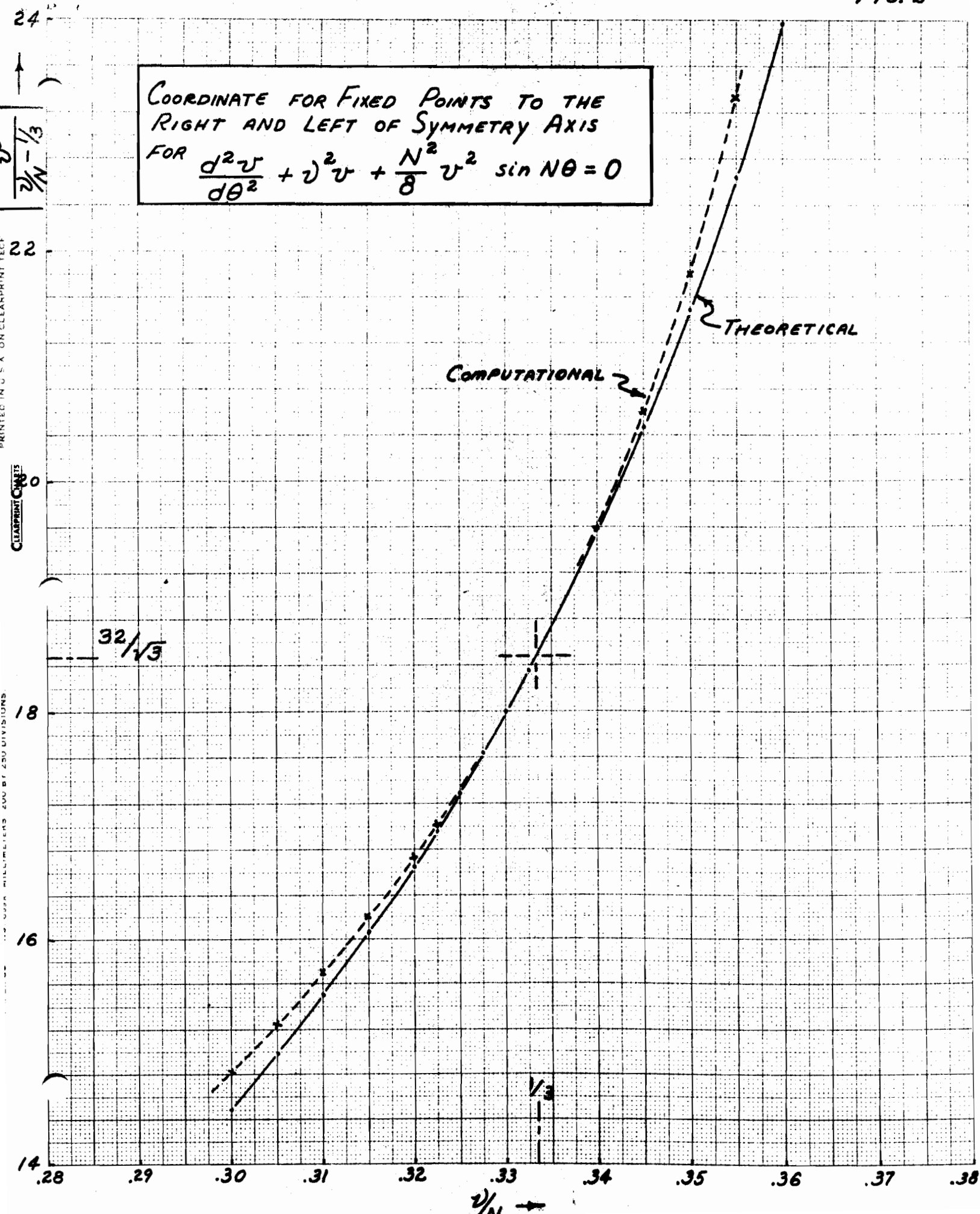
FIG. 2



COORDINATE FOR FIXED POINTS TO THE RIGHT AND LEFT OF SYMMETRY AXIS FOR $\frac{d^2v}{d\theta^2} + \nu^2 v + \frac{N^2}{8} v^2 \sin N\theta = 0$
THEORETICAL
COMPUTATIONAL
$32/\sqrt{3}$
$1/3$
24
22
20
18
16
14
.28
.29
.30
.31
.32
.33
.34
.35
.36
.37
.38
ν/N

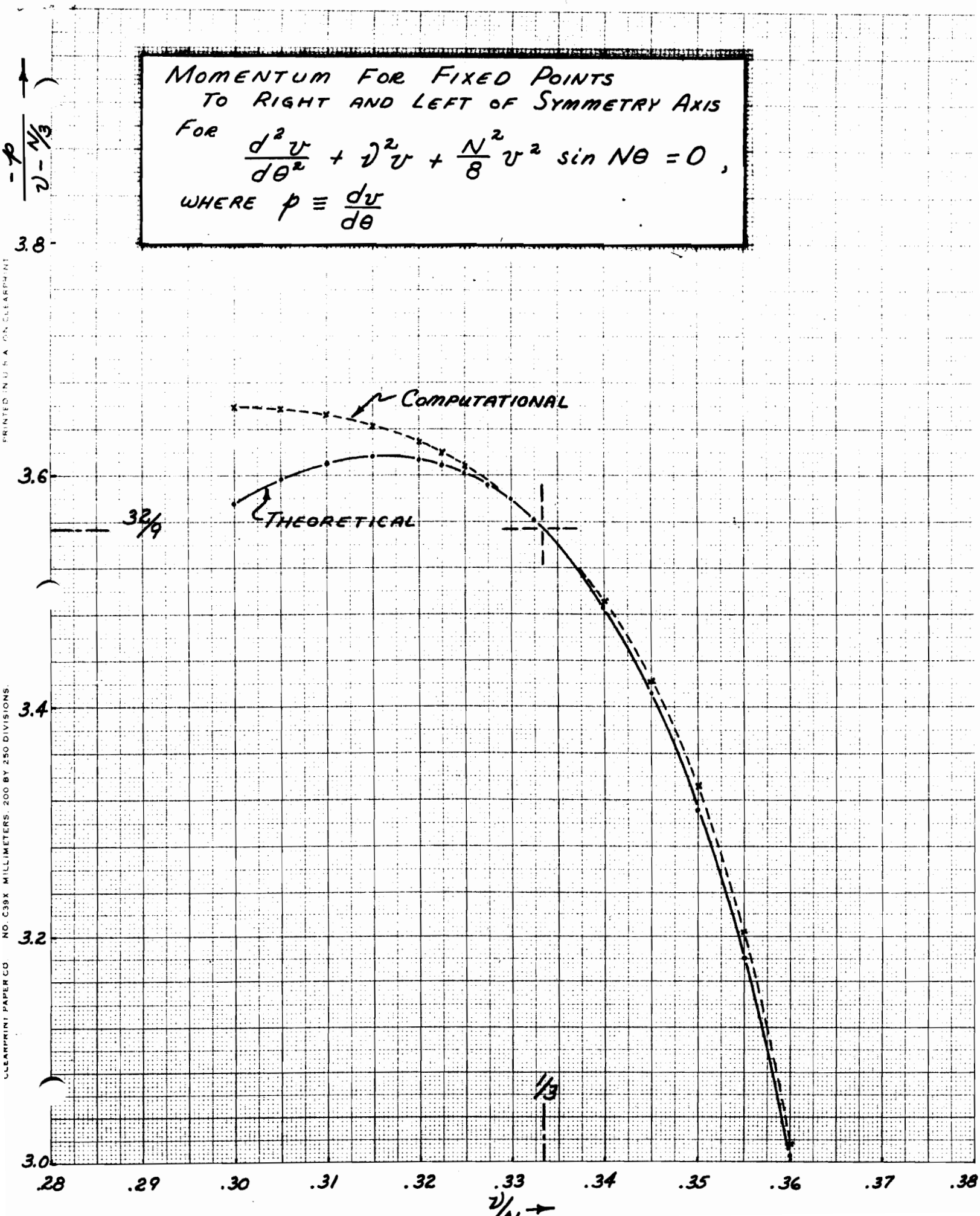

MOMENTUM FOR FIXED POINTS
TO RIGHT AND LEFT OF SYMMETRY AXIS
FOR
$\frac{d^2 v}{d\theta^2} + \nu^2 v + \frac{N^2}{8} v^2 \sin N\theta = 0$,
WHERE $p \equiv \frac{dv}{d\theta}$
$\frac{-p}{\nu - N/3}$
3.8
3.6
3.4
3.2
3.0
COMPUTATIONAL
THEORETICAL
32/9
1/3
.28
.29
.30
.31
.32
.33
.34
.35
.36
.37
.38
ν/N
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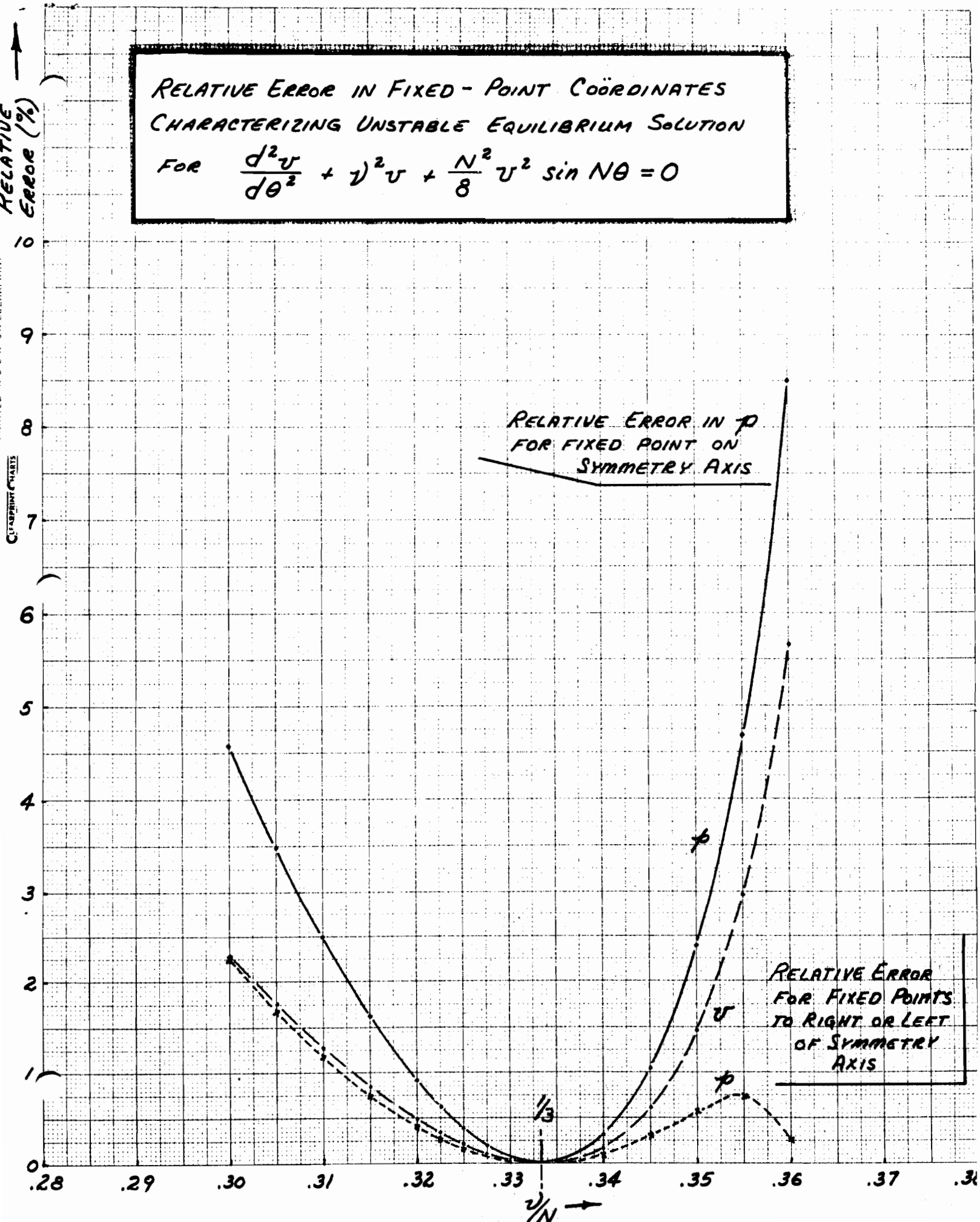
FIG. 4


RELATIVE ERROR IN FIXED-POINT COÖRDINATES
CHARACTERIZING UNSTABLE EQUILIBRIUM SOLUTION
FOR $\frac{d^2v}{d\theta^2} + \nu^2 v + \frac{N^2}{8} v^2 \sin N\theta = 0$
RELATIVE ERROR (%)
RELATIVE ERROR IN ϕ FOR FIXED POINT ON SYMMETRY AXIS
ϕ
v
ϕ
RELATIVE ERROR FOR FIXED POINTS TO RIGHT OR LEFT OF SYMMETRY AXIS
1/3
ν/N
0
1
2
3
4
5
6
7
8
9
10
.28
.29
.30
.31
.32
.33
.34
.35
.36
.37