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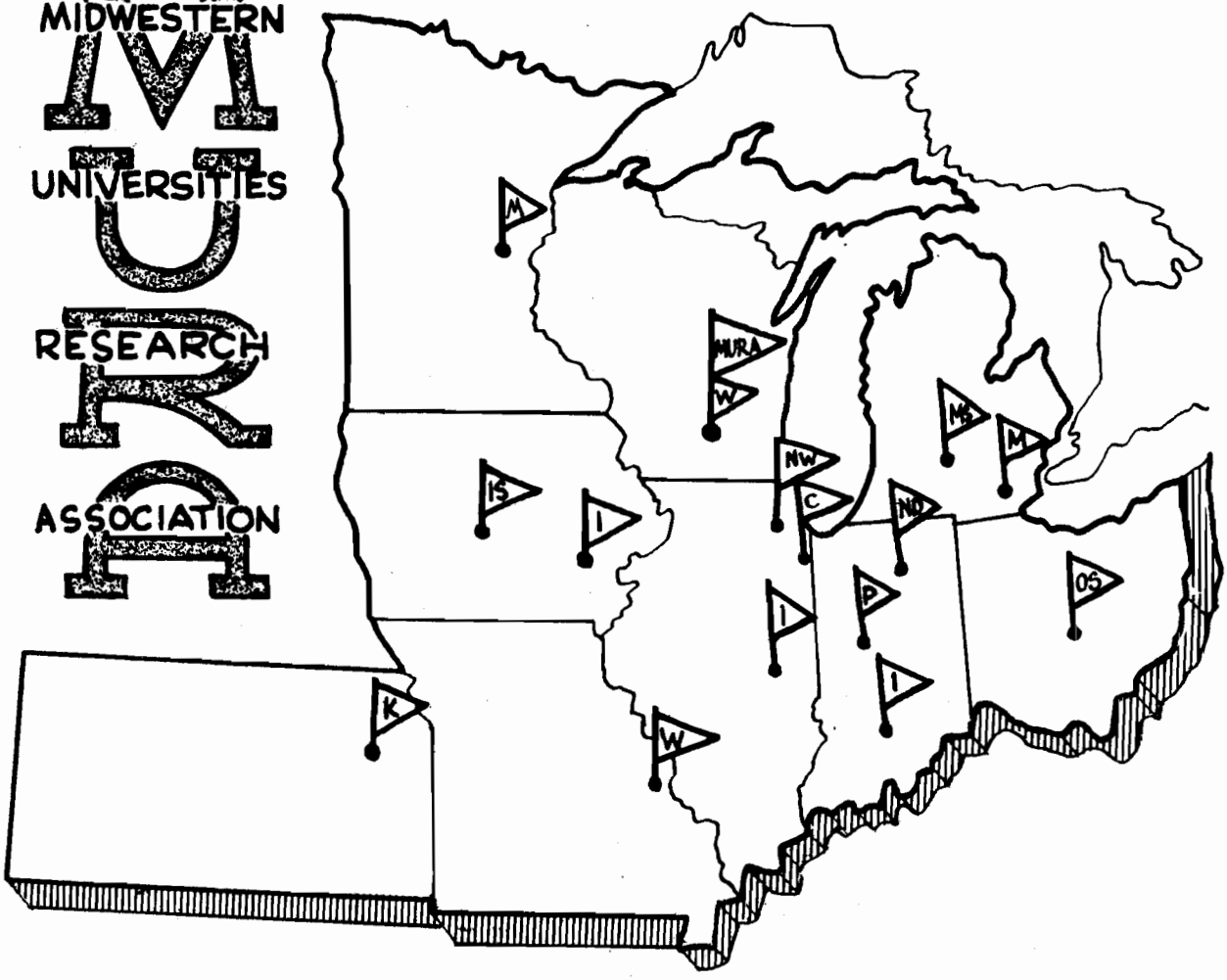

ON THE SPIRAL ORBIT SPECTROMETER

L. Jackson Laslett

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ABSTRACT

The characteristics of orbits in the median plane of a spiral orbit spectrometer are briefly examined from the viewpoint of phase-plots similar to those used in accelerator theory. The characteristics of the spiral orbit spectrometer may be suggestive of injection methods which would prove useful in accelerator design.

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I. INTRODUCTION

The ingenious spiral orbit spectrometer has been described[1-7,†] and analyzed[1, 3-5] in a series of published papers and its experimental use reported[8, 9] for the study of μ^+ meson decay. The instrument employs an axially-symmetric magnetic field characterized by a vector potential ($A\hat{e}_\theta$) having a stationary value at a radius (r_o) such that Br at that radius is equal to the magnetic rigidity of the particles. Particles with this magnetic rigidity, or momentum, emitted from a source on the axis then describe orbits which approach (asymptotically) a circle of radius r_o while particles of lower momentum do not reach this radius and particles of larger momentum cross the circle quickly. The field-configuration thus appears well suited for the selection, with good resolution and large solid angle, of particles of the selected momentum -- particularly if a directional detector is used.

Although a source on the axis may not be realized exactly in practice and the particles which are emitted with initial conditions suitable for approaching the circle of convergence thus (even assuming the mechanical momentum to be correct) in a sense constitute a set of measure zero, the orbit characteristics may be of interest (beyond the spectrometer application) in suggesting effective means for injection into particle accelerators. The spectrometer characteristics have been calculated in some detail in the references cited (esp. ref. 3), but it may be useful here to describe the radial motion (in the median plane) briefly in a way which parallels the viewpoint frequently adopted for the examination of orbits

† References are given in Section IV.

in particle accelerators. Radial phase-plane plots may then be examined, in analogy with the procedure used in accelerator design. The magnetic field, as a function of radius, is normally bell-shaped and axial focusing may be expected [see, F. ex., eq. (43) of ref. 3.]

II. THE ORBITS IN THE MEDIAN PLANE

Employing polar coordinates (r, θ) the trajectories in the median plane may be obtained from the "space Lagrangian" (principle of Least Action)

$$L = \sqrt{r^2 + r'^2} + \frac{e}{p} r A(r) \qquad [\text{emu or MKS}] \tag{1}$$

where $A(r)$ represents the vector potential, e and p the charge and mechanical momentum of the particle, and a prime denotes differentiation with respect to θ. It is convenient to normalize the argument of the vector potential so that it may be expressed in terms of a normalized function $a(x)$ as follows:

$$A(r) = -\frac{p_o}{e} a(r/r_o), \tag{2}$$

where $a(1) = 1$ (3a)

and $a'(1) = 0.$ (3b)

[Thus at $x \equiv r/r_o = 1$ the vector potential is stationary and, at this point, $|Br| = |p_o/e|$; hence a possible orbit of a particle with mechanical momentum p_o is the circle $r = r_o$.]

We thus write

$$L = \sqrt{r^2 + r'^2} - \frac{1}{1+\epsilon} r\, a(r/r_o)$$

$$= r_o\left\{\sqrt{x^2 + x'^2} - \frac{1}{1+\epsilon} x\, a(x)\right\}, \tag{4}$$

where $x \equiv r/r_o$ and $\epsilon \equiv p/p_o - 1$. One may then employ in what follows the

simple Lagrangian

$$\mathcal{L} = \sqrt{x^2 + x'^2} - \frac{1}{1+\epsilon}\, x\, a(x). \tag{5}$$

From the Lagrangian (5) one obtains

$$P = \frac{\partial \mathcal{L}}{\partial x'} = x' / \sqrt{x^2 + x'^2} = r' / \sqrt{r^2 + r'^2} = \cos\alpha, \text{ or} \tag{6a}$$

$$x' = Px / \sqrt{1-P^2} \tag{6b}$$

$$\sqrt{1-P^2} = x / \sqrt{x^2 + x'^2} = r / \sqrt{r^2 + r'^2} = \sin\alpha, \tag{6c}$$

where α denotes the angle between the direction of motion and the radius vector;

$$P' = + \frac{\partial \mathcal{L}}{\partial x} = x / \sqrt{x^2 + x'^2} - \frac{1}{1+\epsilon} \frac{\partial}{\partial x}\left[x\, a(x)\right]. \tag{7}$$

The corresponding Hamiltonian is

$$\begin{aligned} \mathcal{H} &= Px' - \mathcal{L} \\ &= -x\sqrt{1-P^2} + \frac{1}{1+\epsilon}\, x\, a(x), \end{aligned} \tag{8}$$

and will be a constant of the motion. Again from $\mathcal{H}$ the equations for the trajectory may be obtained:

$$x' = \frac{\partial \mathcal{H}}{\partial P} = Px / \sqrt{1-P^2} \tag{9a}$$

$$P' = -\frac{\partial H}{\partial x} = \sqrt{1-P^2} - \frac{1}{1+\epsilon} \frac{\partial}{\partial x}\left[x\, a(x)\right], \tag{9b}$$

as before.

The geometrical interpretation of P, the canonical momentum conjugate to x, as the cosine of the angle between the direction of motion and the radius vector is noted; one also sees [from (9b)] that one can have P identically zero ($x' \equiv o$, corresponding to motion on a circle) at x=1 for ϵ=0, since $\left\{\partial/\partial x \left[x\, a(x)\right]\right\}_{x=1} = 1$.

For a specific illustration of the features of the trajectories, as described by the foregoing equations, one may consider a bell-shaped magnetic field for which the vector potential has the simple form

$$a(x) = \frac{2x}{1 + x^2}, \tag{10}$$

for which, as desired, $a(1) = 1$ and $a'(1) = 0$. The general nature of the median-plane magnetic field, B, implied by this vector potential is indicated in Table I.

TABLE I

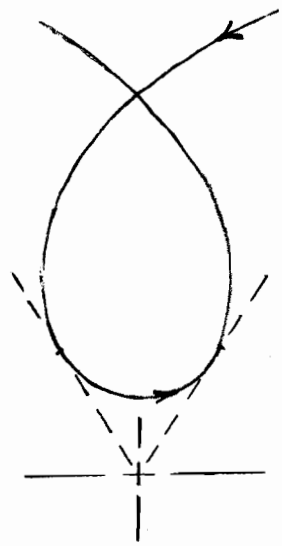
CHARACTER OF MAGNETIC FIELD DESCRIBED BY $a(x) = 2x/(1+x^2)$

Radius x	Field $-(er_o/p_o)\,B$	Field Times Radius $-(e/p_o)\,rB = -x\left[(er_o/p_o)\,B\right]$
0	4	0
0.2955 9774	3.382 9758	1
1	1	1

It is noted that the radii represented by each of the last two lines of this table correspond to possible circular motion of a particle with mechanical momentum p_o.

The invariant phase curves, in the x, P-plane, are given by $\mathcal{H}$ = constant. With a(x) as given by eq. (10), such phase curves are illustrated[10] in Fig. 1 for $\epsilon = 0$ $(p = p_o)$. It is noted that the axis of the spectrometer $(x = 0)$ corresponds to $\mathcal{H} = 0$ and that the curve $\mathcal{H} = 0$ passes through the point (1, 0), so that particles emitted from the axis with $p = p_o$ may approach the circle $r = r_o$, albeit requiring a logarithmically infinite time (as we shall see) to reach this radius. The dotted lines in Fig. 1 which connect branches of phase curves for

which $h > o$ evidently correspond to retrograde motion, for which the motion in time is backward in θ and $\sqrt{1 - P^2} = \sin\alpha$ should be taken as negative.



The situation for particles of somewhat larger momentum ($\epsilon > 0$) is illustrated in Fig. 2 and for particles of momentum smaller than p_o ($\epsilon < 0$) is shown in Fig. 3.

III. CORRELATION OF PHASE POINTS WITH θ OR t

The progress of the motion along the phase curves, such as those shown in Fig. 1, may be indicated by noting values of θ along such a curve. The progress of θ is given by

$$\Delta\theta = \int_{x_1}^{x} \frac{dx}{x'}$$

$$= \int_{x_1}^{x} \frac{\sqrt{1-P^2}}{Px}\, dx \qquad (11)$$

[cf eq. (15) of ref. 3], where P(x) is given in terms of the parameter $\mathcal{H}$ by eq. (8). Near the circle of convergence (1, 0) for $\epsilon = 0$ and $\mathcal{H} = 0$ the quantity P approaches zero in such a way that θ becomes infinite as x approaches unity; thus, for $\epsilon = 0$ and $\mathcal{H} = 0$,

$$P = \frac{1 - x^2}{1 + x^2}$$

$$\theta(x) - \theta(0) = \int_0^x \frac{2}{1 - x^2} \, dx$$

$$= \ln \frac{1 + x}{1 - x} = 2 \tanh^{-1} x. \qquad (12)$$

For other phase curves the progress of θ may perhaps be most conveniently found by numerical integration, aided by analytic integration of asymptotic forms applicable in the neighborhood of $P = 0$.

The progress may also be noted in terms of time, by noting

$$\Delta t = \frac{r_o}{v} \int_{x_1}^{x} \frac{dx}{P} .$$

Again for $\epsilon = 0$ and $\mathcal{H} = 0$, a logarithmic infinity is obtained in approaching the point (1, 0); specifically with the form of field considered here and for $\epsilon = 0$, $\mathcal{H} = 0$,

$$t(x) - t(0) = \frac{r_o}{v} \int_{x_1}^{x} \frac{1 + x^2}{1 - x^2} \, dx$$

$$= \frac{r_o}{v} \left[\ln \frac{1 + x}{1 - x} - x \right] = \frac{r_o}{v} \left[2 \tanh^{-1} x - 1 \right]. \qquad (13)$$

In Fig. 4 the curves $\mathcal{H} = 0$ and $\mathcal{H} = -.025$ of Fig. 1 ($\epsilon = 0$) have been approximately labeled with values of θ, fixing arbitrarily the relative positions of the points $\theta = 0$ on the two curves. As θ, the independent variable of our formulation, increases, phase points located between these two curves will progress as indicated and one may expect the area occupied by such points to be conserved. The progress of points with <u>time</u>, however, may be of somewhat greater interest and in Fig. 5 an attempt is made to attach time labels to the curves $\mathcal{H} = 0$ and $\mathcal{H} = -.025$ of Fig. 1. It does not appear that the region occupied

by points in an x, P phase plot, when observed at a time common to all such particles, should be conserved. From Fig. 5 it is at any rate apparant that a certain accumulation of points in the neighborhood of the equilibrium circle ($x = 1$) is obtained.

IV. REFERENCES AND NOTES

1. G. Miyamoto, Proc. Phys. - Math. Soc. Japan [Nihon Sugaku - Buturigakukai Kaisi] 17, 557 (1943) -- in Japanese. The present writer is indebted to Dr. R. Sagane for his courtesy in providing reprints of this and some of the following references, with English translations of references 1 and 3.

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4. M. Sakai, Jour. Phys. Soc. Japan 5, 178 (1950).

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8. R. Sagane and P. C. Giles, Phys. Rev. 81, 653 (A) (February 15, 1951). This reference is an abstract of a paper presented before the Los Angeles meeting of the American Physical Society (December 29, 1950).

9. R. Sagane, W. L. Gardner, and H. W. Hubbard, Phys. Rev. 82, 557 (L) (May 15, 1951).

10. The writer is happy to acknowledge the assistance of Mr. Bob Clark, of the MURA Laboratory, in preparing the figures for this report.

11. It has been pointed out by Dr. D. Judd (private communication, 1959) that the "space Hamiltonian" which we have employed is essentially $-p_\theta$; i.e., the negative of the canonical momentum conjugate to θ, which of course is a constant of the motion in the present problem. This process of changing from the independent variable t of a true Hamiltonian to the variable θ (which may be demonstrated generally

by use of Hamilton's principle), and the distinction between observation of phase points projected in the r, p_r-plane at a common θ or at a common time, is illustrated by the following artificial example:

Consider the Hamiltonian

$$H(x, \theta; p_x, p_\theta; t) = \frac{1}{x}\left(p_\theta + \frac{1}{2} p_x^2\right).$$

For this Hamiltonian the equations of motion in time are, of course,

$$\dot{\theta} = \partial H/\partial p_\theta = \frac{1}{x}; \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0, \quad \text{or } p_\theta = \text{const.};$$

$$\dot{x} = \partial H/\partial p_x = p_x/x; \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = \frac{p_\theta + \frac{1}{2} p_x^2}{x^2} = \frac{H}{x}; \text{ and } H = \text{const.}$$

From these equations it follows that the derivatives with respect to θ are

$$x' = p_x$$

$$p_x' = H, \qquad \text{with solutions} \quad p_x = p_{x_o} + H\theta$$

$$x = x_o + p_{x_o}\theta + \frac{1}{2} H\theta^2,$$

and the functional determinant, $\partial(x, p_x)/\partial(x_o, p_{xo})$, with the partial derivatives evaluated with θ held constant is, of course, unity, independent of θ:

$$\frac{\partial(x, p_x)}{\partial(x_o, p_{xo})} \equiv \begin{vmatrix} \frac{\partial x}{\partial x_o} & \frac{\partial p_x}{\partial x_o} \\ \frac{\partial x}{\partial p_{x_o}} & \frac{\partial p_x}{\partial p_{x_o}} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \theta & 1 \end{vmatrix} = 1.$$

The time, t, is given in terms of θ by

$$dt = x\, d\theta = \left(x_o + p_{x_o}\theta + \frac{1}{2} H\theta^2\right) d\theta,$$

$$t - t_o = \left(x_o\theta + \frac{1}{2} p_{x_o}\theta^2 + \frac{1}{6} H\theta^3\right).$$

In concordance with Judd's observations, the equations for x' and p_x' may be obtained by writing

$$\mathcal{H}(x,\ t;\ p_x,\ -H;\ \theta\ ) = -p_\theta$$
$$= -xH + \frac{1}{2}\ p_x^2\ ,$$

from which

$$\frac{\partial x}{\partial \theta} = \frac{\partial \mathcal{H}}{\partial p_x} = p_x \qquad \frac{\partial p_x}{\partial \theta} = \frac{\partial \mathcal{H}}{\partial x} = H$$

$$\frac{\partial t}{\partial \theta} = \frac{\partial \mathcal{H}}{\partial(-H)} = x \qquad \frac{\partial(-H)}{\partial \theta} = -\frac{\partial \mathcal{H}}{\partial t} = 0$$

as before.

If, for simplicity, we consider the particular solutions for which H = 0,

$$p_x = p_{x_o} \qquad x = x_o + p_{x_o}\theta\ ,$$

$$\theta = \frac{-x_o + \sqrt{x_o^2 + 2p_{x_o}\ (t - t_o)}}{p_{x_o}}\ , \qquad x = \sqrt{x_o^2 + 2p_{x_o}\ (t - t_o)}\ .$$

If we form the functional determinant $\partial(x,\ p_x)\ /\ \partial(x_o,\ p_{xo})$ from these solutions, performing the partial differentiation with t held fixed, we obtain

$$\frac{\partial(x,\ p_x)}{\partial(x_o,\ p_{x_o})} \equiv \begin{vmatrix} \frac{\partial x}{\partial x_o} & \frac{\partial p_x}{\partial x_o} \\ \frac{\partial x}{\partial p_{x_o}} & \frac{\partial p_x}{\partial p_{x_o}} \end{vmatrix} = \begin{vmatrix} \frac{x_o}{\sqrt{x_o^2 + 2p_{x_o}\ (t - t_o)}} & 0 \\ \frac{t - t_o}{\sqrt{x_o^2 + 2p_{x_o}\ (t - t_o)}} & 1 \end{vmatrix} =$$

$$\frac{x_o}{\sqrt{x_o^2 + 2p_{x_o}\ (t - t_o)}} = \frac{x_o}{x}\ ;$$

this expression clearly will <u>not</u> in general be equal to unity.

The writer is indebted to Dr. B. C. Carlson and Dr. F. T. Cole for helpful general discussions concerning such specialized phase plots and in particular for remarks leading to the following summary:

This example illustrates a general situation of some interest. If one takes a group of particles governed by a Hamiltonian and projects the region occupied by these particles on a subspace of the total phase space, the area occupied by these particles on this subspace may or may not be constant as the motion develops, depending on the way in which the initial conditions of the particle are chosen.

Consider a system of two degrees of freedom governed by the Hamiltonian $H(r, p_r; \theta, p_\theta; t)$, where H is independent of θ, as in the example above. Then p_θ is a constant of the motion and if we consider a group of particles with the same p_θ, but different values of r and p_r and observe the progress of the system in time, the area projected on the $r - p_r$ plane by the particles will be constant in time, since effectively $H = H(r, p_r; t)$. Geometrically, all the phase space points representing the particles lie at all times on the hyperplane p_θ = const. normal to the $r - p_r$ plane in the four-dimensional phase space.

If, however, as a second example we choose a group of initial conditions with the same H and differing values of p_θ, the functional dependence of $H(r, p_r; t)$ varies from particle to particle, so that all particles are not governed by the same Hamiltonian. Area in the $r - p_r$ plane is <u>not</u> conserved in time.

The same problem can be viewed with θ as independent variable; the "Hamiltonian" is $-p_\theta = h\,(r,\ p_r;\ t,\ -H;\ \theta)$. If H is independent of t, it is a constant of the motion, and plays the same role as p_θ did when H was the Hamiltonian. A group of particles with the same H, but different values of p_θ (as in the second example) will have the same Hamiltonian $h\,(r,\ p_r,\ \theta)$ governing the motion and the area in the $r - p_r$ plane will be constant in θ. It goes almost without saying that Liouville's Theorem, which is concerned with the total phase space volume, is conserved in all cases.

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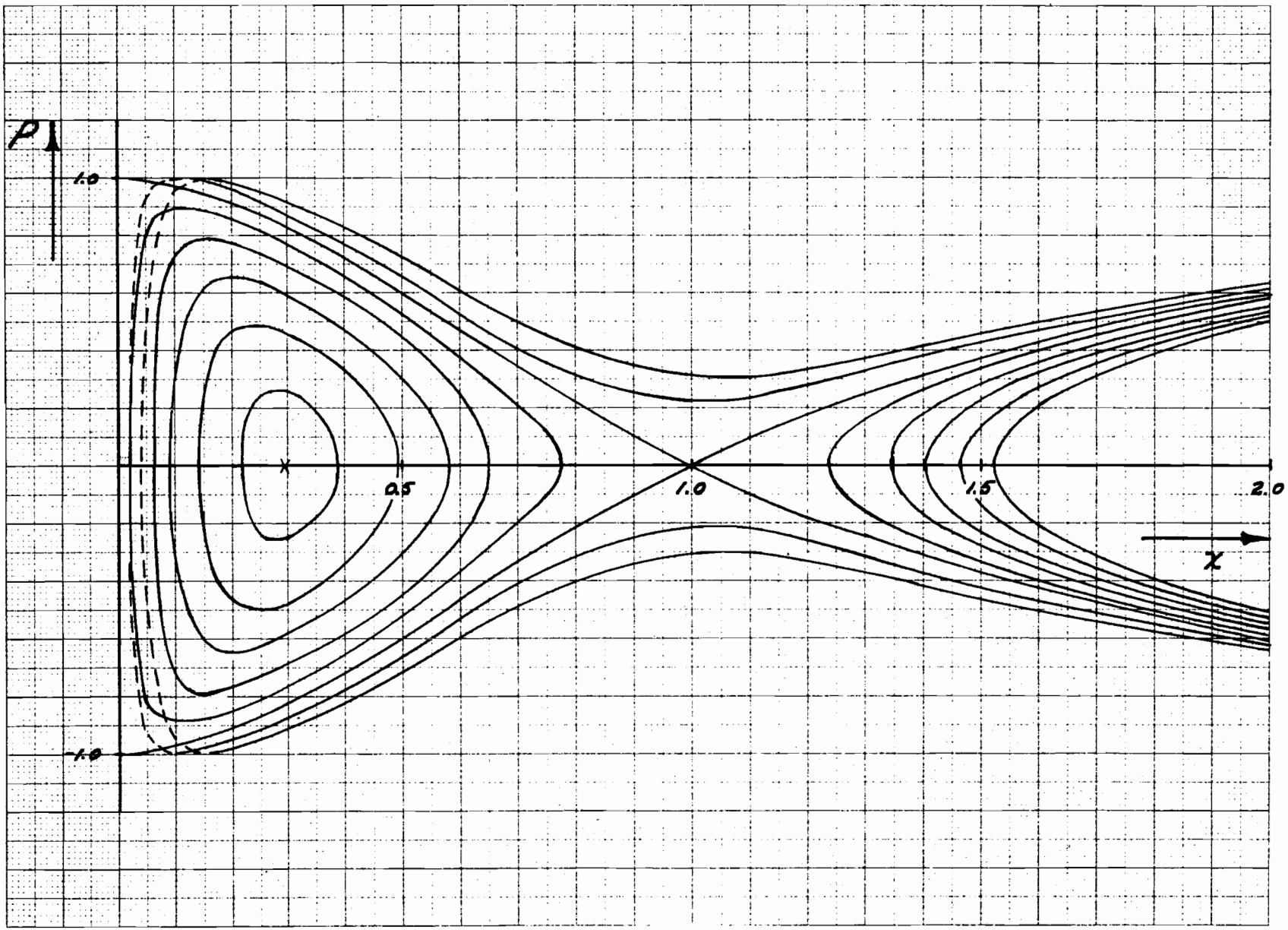

Fig. 1. $\epsilon = 0$.

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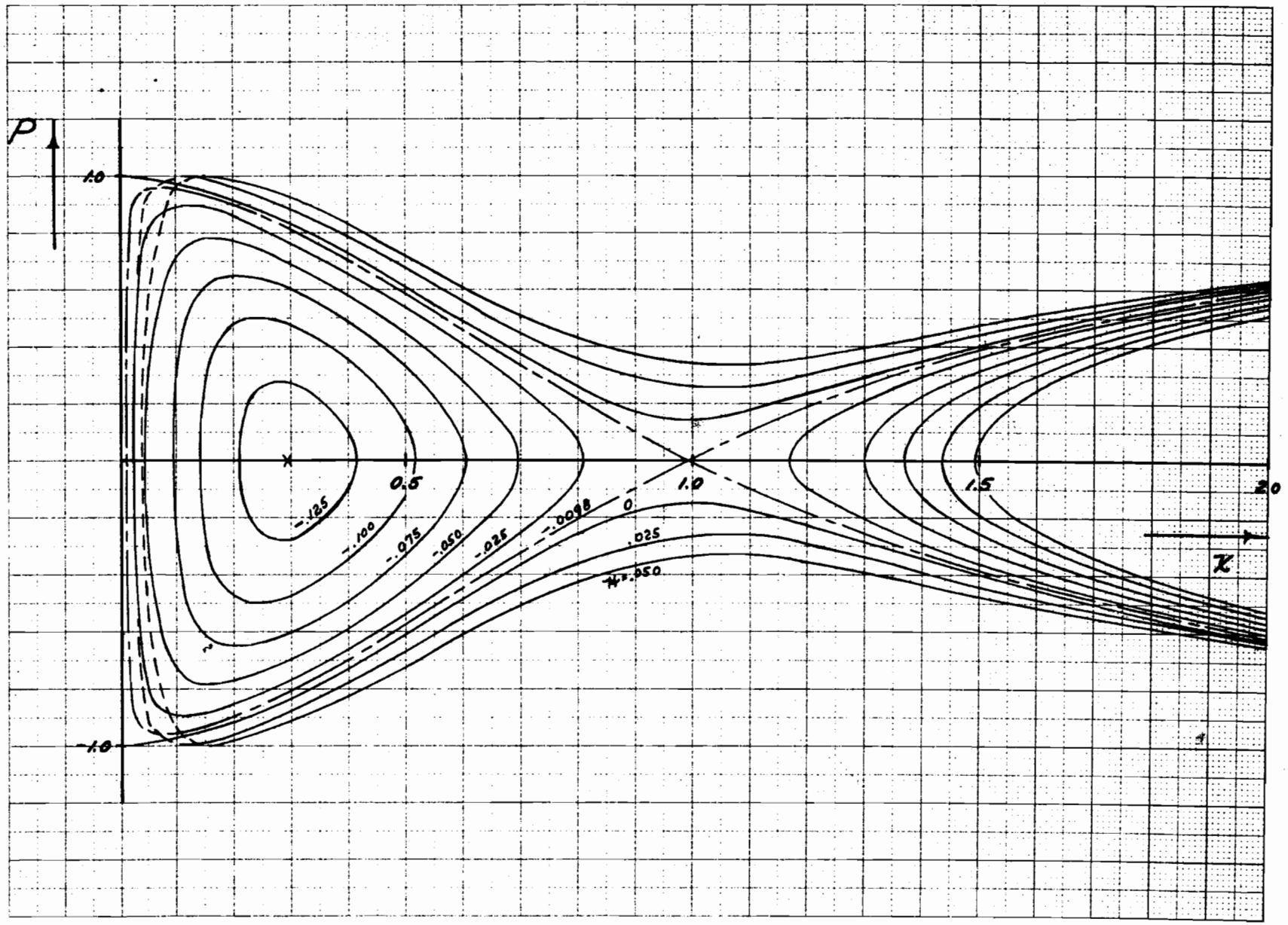

Fig. 2. $\epsilon = .01$ Separatrix at $x = 0.99$

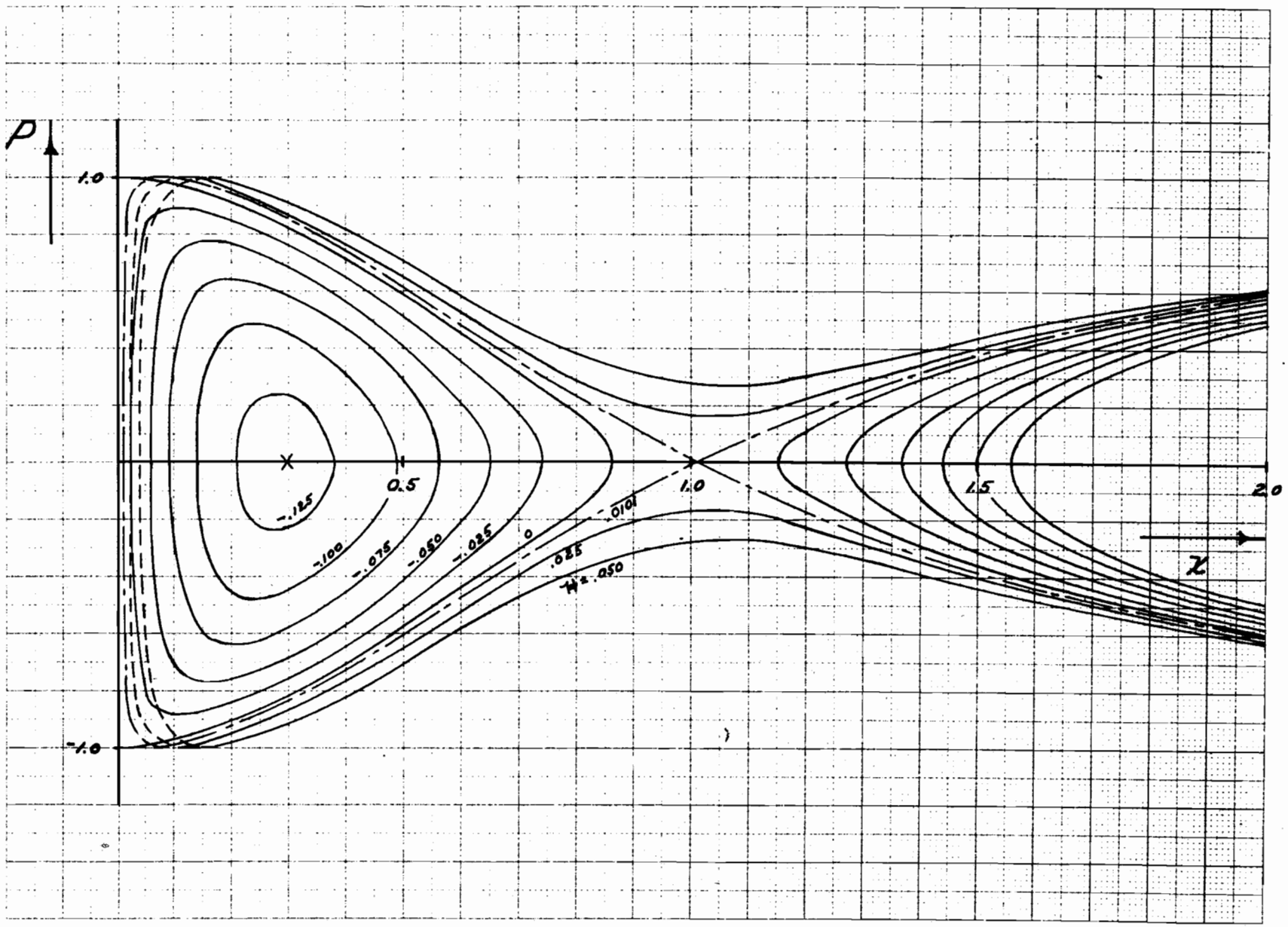

Fig. 3. ϵ =-.01 Separatrix at x = 1.01

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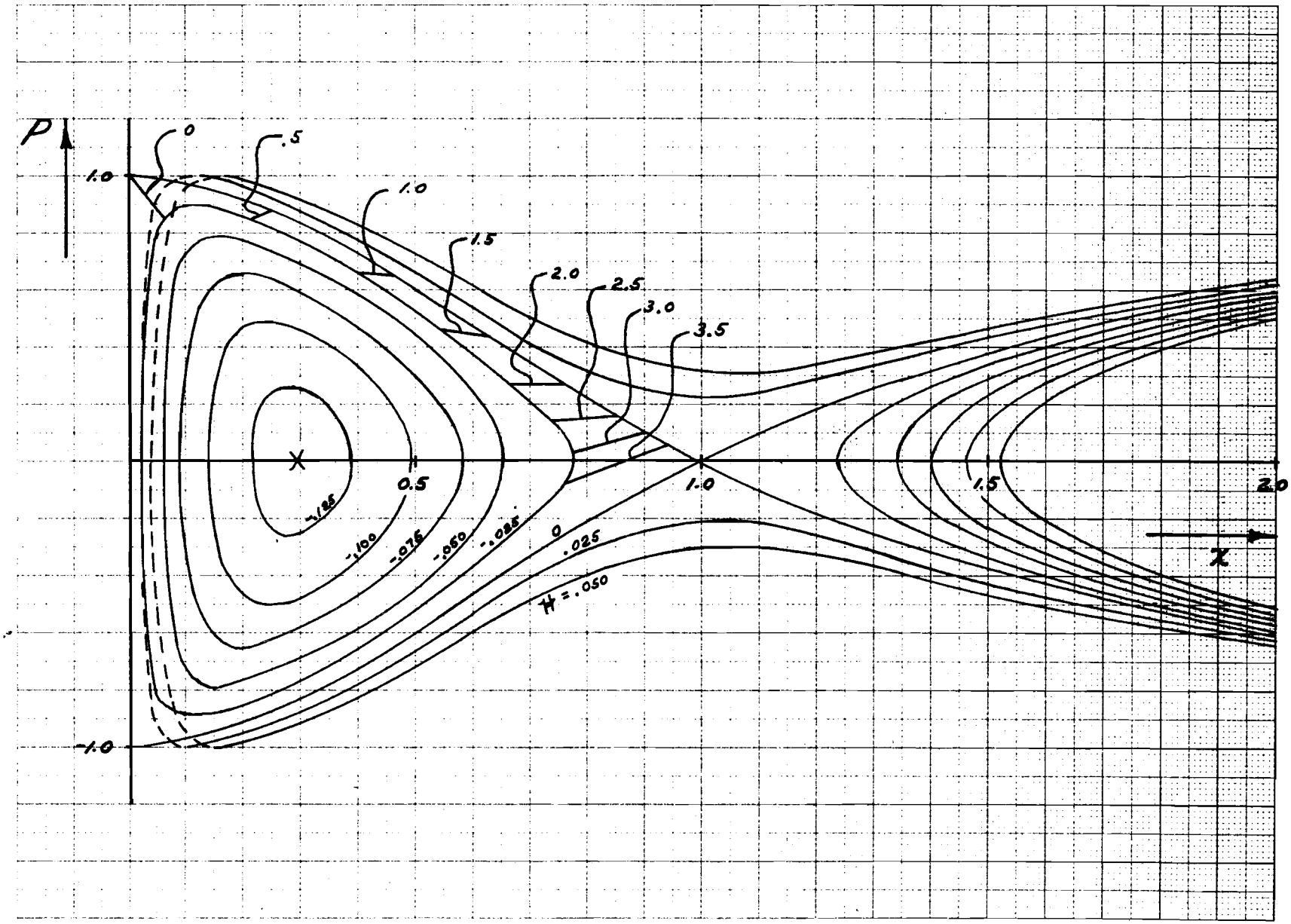

Fig. 4. $\epsilon = 0$. x, P phase plot for $\epsilon = 0$, with the progress of points in θ noted approximately for $\theta = 0, 0.5, \ldots 3.5$. The starting points, or relative values for the origin of θ, are arbitrarily chosen for the two curves $\mathcal{H} = 0$ and $\mathcal{H} = -0.025$.

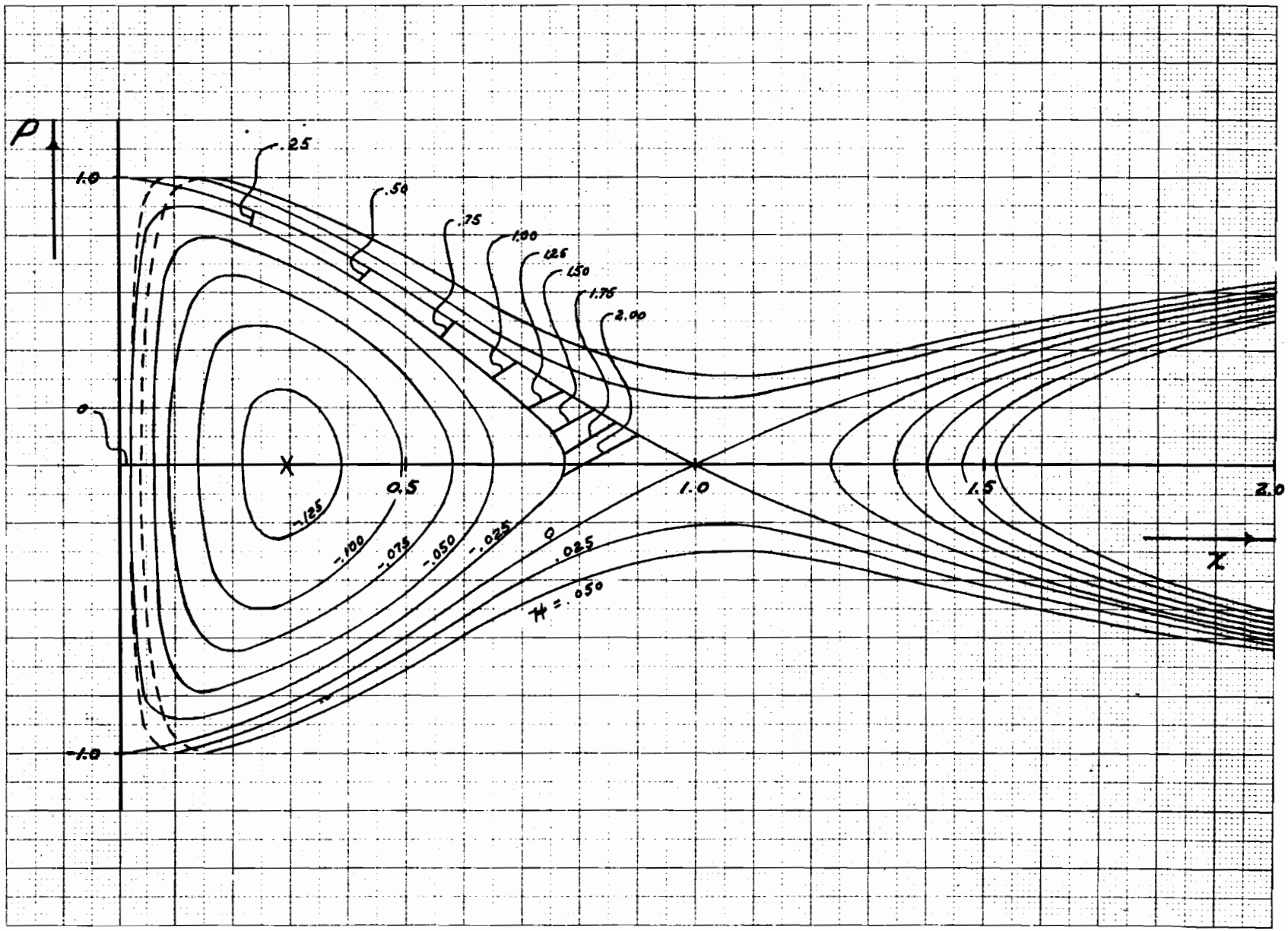

Fig. 5. $\epsilon = 0$. x, P phase plot for $\epsilon = 0$, with the progress of points in time noted approximately for $(v/r_o)t = 0, 0.25, \ldots 2.00$.