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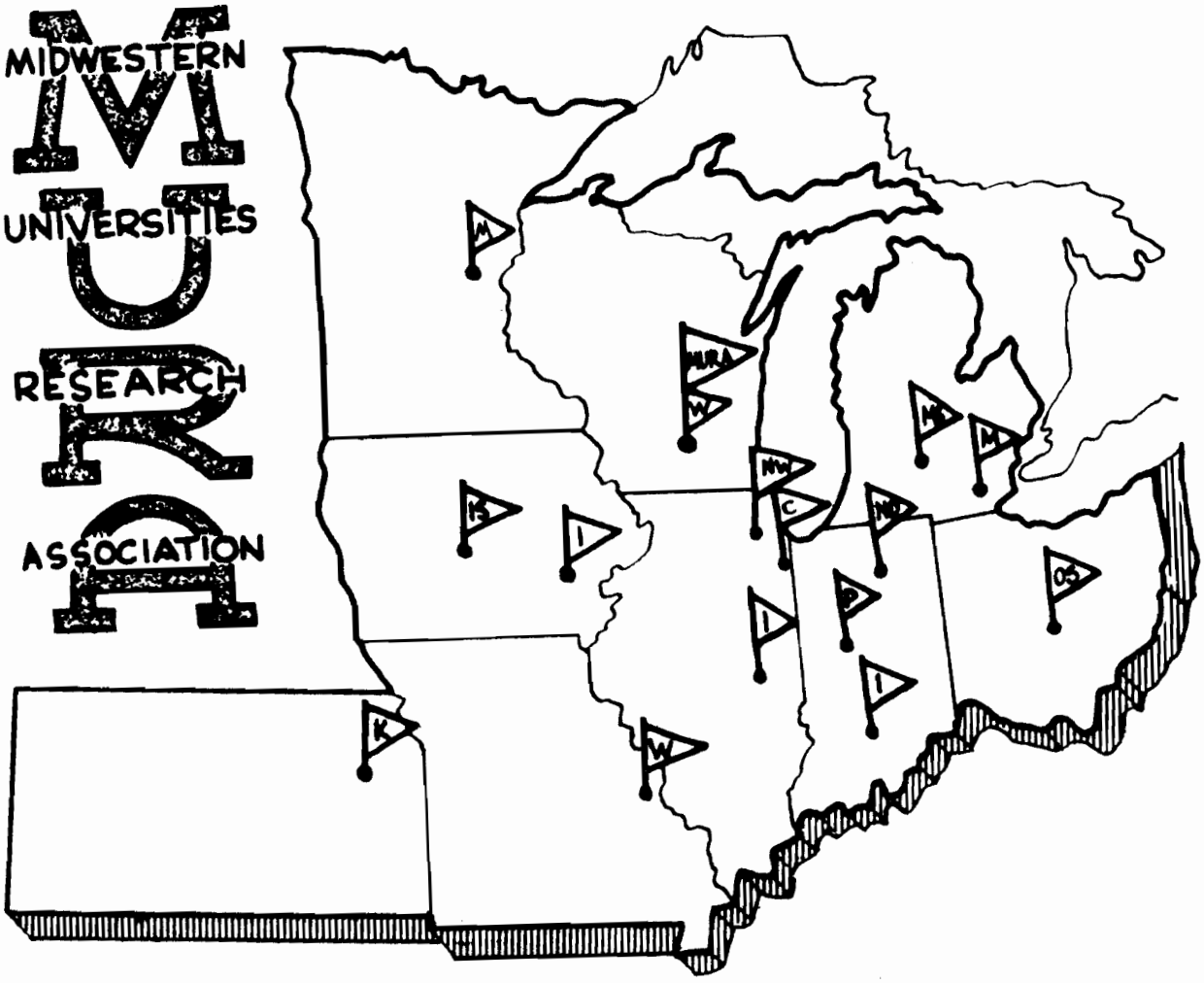

THE NON-LINEAR COUPLING RESONANCE $2\nu_y - \nu_x = 1$

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THE NON-LINEAR COUPLING RESONANCE $2\nu_y - \nu_x = 1$

L. Jackson Laslett**

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ABSTRACT

Computational results, obtained with the DUCK-ANSWER program and pertaining to the $2\nu_y - \nu_x = 1$ resonance, are reported for two examples of coupled Hamiltonian differential equations. Each of the examples contains a term involving $x \cdot y$ (with a periodic coefficient) in the y-equation. The width of the resonance appears in each case to be roughly proportional to the first power of the x-amplitude. Results are also presented to show the effect, on the y-motion, of traversing the $2\nu_y - \nu_x = 1$ resonance, at various rates.

A rough analytic examination of the second set of equations is also given. Comparisons with the computational results suggest the theory to be semi-quantitatively valid.

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I. MOTIVATION:

The character of the solutions to non-linear coupled differential equations, for oscillation frequencies in the neighborhood of certain non-linear coupling resonances, has been reported previously in a number of MURA reports[1†] and by members of the Harwell group.[2] Recently there has been interest in the $2\nu_y - \nu_x = 1$ resonance, since (i) the proposed operating point for the ANL 12.5 Gev accelerator[3] lies close to this resonance and (ii) it may be necessary to traverse this resonance when employing the Hammer-Bureau method[4, 5] of beam extraction from a conventional betatron or synchrotron.

Two systems of coupled differential equations have, accordingly, been studied by means of the MURA IBM 704 DUCK-ANSWER[6] computational program. Although neither of these systems may represent closely the physical situations mentioned above, it was felt that the results would be of interest as illustrative of effects attributable to the $2\nu_y - \nu_x = 1$ resonance. Attention has been focused on the growth of y-amplitude (axial-amplitude) rather than on the possible eventual "turn-over" of the y-growth, axial limitations of aperture frequently making turn-over of somewhat secondary interest. To simplify the study, only those non-linear terms were introduced which would be required to give a Hamiltonian system of equations capable of responding directly to the resonance in question and first-derivative terms were omitted.

A guide to the magnitude of the y-oscillation amplitude was obtained by computing the quantity[7]

$$K_y \equiv \left[y^2 + (\zeta_y / \xi_y)\, p_y^2\right]^{1/2},$$

which should be an invariant for small-amplitude oscillations.

† References are given in Section VI.

II. THE EQUATIONS STUDIED

The equations employed in this study were the following:

SET I.

$$d^2x/d\theta^2 + (0.536 - 1.8 \cos 8\theta - 0.075 \cos \theta)\, x = -(0.025 \cos \theta)\, y^2$$

$$d^2y/d\theta^2 + (-160\, S_2 + 1.2 \cos 8\theta + 0.050 \cos \theta)\, y = -2\,(0.025 \cos \theta)\, x \cdot y\,,$$

which wereput into a form suitable for use of the DUCK-ANSWER program by the transformation $4\theta = \tau$:

$$d^2x/d\tau^2 = 10\,(-0.00335 + 0.01125 \cos 2\tau + 0.00046875 \cos \frac{6\tau}{24})\, x + (-0.0015625 \cos \frac{6\tau}{24})\, y^2$$

$$d^2y/d\tau^2 = 10\,(S_2 - 0.0075 \cos 2\tau - 0.0003125 \cos \frac{6\tau}{24})\, y + 2\,(-0.0015625 \cos \frac{6\tau}{24})\, x \cdot y\,.$$

The constant coefficient S_2 was adjusted to obtain small-amplitude y-oscillation frequencies located as desired in the neighborhood of the $2\nu_y - \nu_x = 1$ resonance.

SET II.

$$d^2x/d\theta^2 + (-2.5\, S_1 - 0.063 \cos \theta)\, x = (-0.0825 - 0.105 \cos \theta)\,(x^2 - y^2)$$

$$d^2y/d\theta^2 + (-2.5\, S_2 + 0.063 \cos \theta)\, y = 2\,(0.0825 + 0.105 \cos \theta)\, x \cdot y,$$

which were transformed by the substitution $\theta = 2\tau$ to obtain the working equations:

$$d^2x/d\tau^2 = 10\,(S_1 + 0.0252 \cos 2\tau)\, x + (-0.33 - 0.42 \cos 2\tau)\,(x^2 - y^2)$$

$$d^2y/d\tau^2 = 10\,(S_2 - 0.0252 \cos 2\tau)\, y + 2\,(0.33 + 0.42 \cos 2\tau)\, x \cdot y\,.$$

In this case the constant coefficients S_1 and S_2 were adjusted together, in concordance with the relation $S_1 + S_2 = -0.4036$, to obtain desired operating points in the neighborhood of the $2\nu_y - \nu_x = 1$ resonance

In connection with this set of equations it was also of interest to traverse the resonance "dynamically"--i.e., during the course of a run. This could be accomplished by introducing, in effect, a secular change in the coefficients S_1 and S_2. Specifically, the factors $S_1 + 0.0252 \cos 2\tau$ and $S_2 - 0.0252 \cos 2\tau$ in the last equations were then supplemented, respectively, by

$$B_1 \cos\left(\frac{4\tau}{16384} + \frac{\pi}{2}\right) \text{ and } B_2 \cos\left(\frac{4\tau}{16384} + \frac{\pi}{2}\right),$$

where the coefficients B_1 and B_2 are related to the rate-of-change of the "field-index," n, substantially by

$$B_1 \cong -3300 \; dn/d\theta \text{ and } B_2 \cong +3300 \; dn/d\theta .$$

The location of the working points, in relation to the $2\nu_y - \nu_x = 1$ resonance line, for these two sets of equations is indicated in Fig. 1. Inferences drawn from the computational results for the amount of non-linearity introduced in these equations may, of course, be re-interpreted for other magnitudes of non-linearity (of the same form) by "scaling" the dependent variables--i.e., by use of the transformation $x = \alpha X$, $y = \alpha Y$, which has the effect of increasing the relative amount of non-linearity by the factor α.

III. RESULTS FOR THE EQUATIONS OF SET I:

A. The Oscillation Frequencies:

The frequencies of small-amplitude oscillations were determined for the equations of Set II (for various values of the parameter S_2) by preliminary orientation runs in which the non-linear (coupling) terms were suppressed. The results are shown below in Table I.

TABLE I

Oscillation Frequencies for Equations of Set I

$\nu_x = 0.7483$

S_2	$-160\ S_2$	$2\nu_y - \nu_x$
-0.005235	0.8376	1.0938
-0.005075	0.8120	1.0664
-0.004995	0.7992	1.0515
-0.004915	0.7864	1.0379
-0.004835	0.7736	1.0235
-0.004755	0.7608	1.0090
-0.004675	0.7480	0.9943
-0.004595	0.7352	0.9796
-0.004515	0.7224	0.9646
-0.004355	0.6968	0.9344
-0.004195	0.6712	0.9036

B. The Examination of y-Growth:

For each of the frequencies listed in Table I, runs were made with a small initial y-amplitude (0.001) and various initial x-amplitudes, in an effort to find y-growth characteristic of the coupling resonance. When such growth was observed, "the lapse-rate" denoting the rate of exponential growth was measurable from a semi-logarithmic plot of K_y vs τ/π and could be conveniently expressed as decades per $\Delta\tau = \pi$. Since $\tau = 4\theta$, the lapse-rate so determined may also be regarded as expressed in "decades per octant."

The results of these runs are summarized in Table II and portrayed in the form of an altitude chart in Fig. 2. In the figure each notch corresponds

to a lapse-rate of 0.005 decades per octant As mentioned in Section II, the results could be re-interpreted for other strengths of the non-linearity by suitable scaling of the dependent variables.

TABLE II

Lapse-Rate for Equations of Set 1

Lapse-rates are given in decades per octant

$\nu_x = 0.7483$

S_2	$2\nu_y - \nu_x$	x_o			
		7.0	4.4	2.2	0.7
-0.005235	1.094	Throb-factor 5.8			
-0.005075	1.066	0.010_1	Throb-factor 4.37		
-0.004995	1.052		0.0062		
-0.004915	1.038			0.0028	
-0.004835	1.024	0.0141	0.0089	0.0043	
-0.004755	1.009	0.0143		0.0048	
-0.004675	0.994	0.0144	0.0091	0.0048	0
-0.004595	0.980	0.0136		0.003	
-0.004515	0.965	0.0123	0.0065	0	
-0.004355	0.934	0.0071_5	0		
-0.004195	0.904	0			

IV. RESULTS FOR THE EQUATIONS OF SET II:

A. Results with no Secular Change

As with the equations of Set 1 the frequencies of small-amplitude oscillations were determined for the equations of Set II by short orientation runs with the non-linear terms suppressed. With the non-linear terms

present a search was made to find y-growth, again using a small initial y-amplitude (0.00001) and various initial x-amplitudes. When y-growth was seen to be present, it was followed through a few decades--in every case through more than one decade save for those runs with x_o equal to 0.01 or to 0.005--and the lapse-rate determined. For the equations of Set II, in which $\theta = 2\tau$, the lapse-rate is conveniently expressed in decades per $\Delta\tau = \pi$ or, equivalently, as decades per revolution.

The results, giving the lapse-rate for various values of the parameters S_1 and S_2 together with the associated frequencies for small-amplitude oscillations, are listed in Table III. The lapse-rates are also shown in Fig. 3 in the form of an altitude chart, with each notch corresponding to a lapse-rate of 0.02 decades per revolution.

B. Results with Secular Change:

As remarked in Section II, the motion characterized by the equations of Set II could be caused to traverse the $2\nu_y - \nu_x = 1$ resonance by introduction of the terms B_1 or $2\cos\left(\frac{4\tau}{16384} + \frac{\pi}{2}\right)$, with $B_1 = -B_2$. These terms, in effect, are equivalent to a slow (substantially linear) secular change of the coefficients S_1 and S_2 and produce a change of the small-amplitude oscillation frequencies simulating a linear change of the field-index, n:

$$dn/d\theta \cong B_2/3300 .$$

The values of S_1 and S_2 actually used throughout this series were -0.1618 and -0.2418, respectively, corresponding to initial oscillation frequencies $\nu_x = 0.6334$ and $\nu_y = 0.7765$, with $2\nu_y - \nu_x = 0.9196$ [Table III].

TABLE

OSCILLATION FREQUENCIES AND LAPSE-RATES FOR EQUATIONS OF SET II

Lapse-Rates are given in decades per revolution

$S_1 + S_2 = -0.4036$

S_1	S_2	ν_x	ν_y	$2\nu_y - \nu_x$	x_0 = 0.50	0.25	0.10	0.050	0.025	0.010	0.005
0.1618	-0.2418	0.6334	0.7765	0.9196							
0.1525	-0.2511	0.6143	0.7914	0.9685	throb-factor 2.5 or 2.6						
0.1505	-0.2531	0.6101	0.7946	0.9791	0.0175						
0.1495	-0.2541	0.6080	0.7962	0.9844		throb-factor 2.5					
0.1485	-0.2551	0.6058	0.7977	0.9896	0.0287	0.0086					
0.1475	-0.2561	0.6037	0.7993	0.9948		0.0141					
0.1470	-0.2566	0.6027	0.8001	0.9975			0.0052				
0.1465	-0.2571	0.6016	0.8009	1.0001	0.0306	0.0156	0.0062	0.0031	0.0016	0.0006_2	0.00038
0.1460	-0.2576	0.6005	0.8017	1.0028			0.0046				
0.1455	-0.2581	0.5994	0.8024	1.0054		0.0127					
0.1445	-0.2591	0.5972	0.8040	1.0107	0.0241	0.0044					
0.1435	-0.2601	0.5950	0.8055	1.0160		0					
0.1425	-0.2611	0.5928	0.8071	1.0213	0						
0.1408	-0.2628	0.5891	0.8097	1.0304							
0.1405	-0.2631	0.5884	0.8102	1.0320	0						
0.1268	-0.2768	0.5561	0.8311	1.1061							
0.1233	-0.2803	0.5469	0.8364	1.1258							
0.1198	-0.2838	0.5370	0.8416	1.1462							
0.1124	-0.2912	0.5000	0.8525	2.2050							

In all the runs made employing this secular change of parameters, the initial y-amplitude was, as before, taken as quite small (0.00001). It would be expected that the factor by which the y-amplitude is increased by traversal of the resonance would depend in a somewhat accidental way upon the phase with which the oscillations enter the region of instability--in most of the work reported here the initial amplitude of x-oscillation was obtained by taking $x_o = 0.50$, $p_{x_o} \equiv [dx/d\tau]_o = 0$ or $x_o = 0$, $p_{x_o} = 0.51$ (each corresponding to an initial amplitude 0.50), or by $x_o = 0.25$, $p_{x_o} = 0$ or $x_o = 0$, $p_{x_o} = 0.255$ (corresponding to an initial amplitude 0.25). The rates of secular change which were employed are listed in Table IV.

TABLE IV

Values of the Coefficients B_1 and B_2, Introduced to Represent a Secular Change of Frequency, and the Corresponding Rate-of-Change of Field-Index

B_1	B_2	Approx. $dn/d\theta$
-0.0990	0.0990	0.000 030
-0.1452	0.1452	0.000 044
-0.2145	0.2145	0.000 065
-0.3168	0.3168	0.000 096
-0.462	0.462	0.000 140
-0.66	0.66	0.000 200
-0.99	0.99	0.000 300

The results of such runs are shown in Figs. 4-10. Although traversal of the resonance $2\nu_y - \nu_x = 1$ is seen to have a material effect on the amplitude of the y-oscillations, normally increasing the amplitude by a sub-

stantial factor, the magnitude of the effect is seen to depend considerably upon the phase (of the x-oscillations in this case) at the start of the run and in some cases a decrease of y-amplitude is seen to result [Figs. 9 and 10]. In some of the runs, specifically those with the more rapid secular changes, the computations continued for a sufficient number of revolutions to carry the operating point to the neighborhood of $\sigma_x = \pi (\nu_x = 1/2)$--in such cases, of course, the x-motion would be expected to experience instability and, through coupling with the y-motion, exert a pronounced influence on the latter. In an auxiliary investigation,[8] however, no resonances leading to y-growth were detected in the interval between $2\nu_y - \nu_x = 1$ and $\nu_x = 1/2$ for the simple equations of Set II (as was to be expected).

Since, as noted above, the effect on the y-amplitude of traversing the coupling resonance will necessarily depend markedly upon the initial phases of the oscillatory motion, the results depicted in Fig. 7 [$B_1 = -0.3168$, $B_2 = 0.3168$; $dn/d\theta \cong 0.000\,096$] were supplemented by sixty additional runs to give what it was hoped would be a representative selection of initial phases for both the x- and the y-motion. As before, the initial values corresponded to an initial x-amplitude of either 0.50 or 0.25. From the results of this survey (summarized in Appendix I), it was felt that the following factors represent a fair estimate of the amount of growth which may be obtained with this rate of traversal of the $2\nu_y = \nu_x = 1$ resonance (cf. Fig. 13):

For an initial x-amplitude of 0.50, growth by a factor 18 or 1.25 decade;

For an initial x-amplitude of 0.25, growth by a factor 3.3 or 0.52 decade.

V. APPROXIMATE ANALYTIC TREATMENT:

A. The Case of No Secular Change:

It may be of interest to attempt an analytic treatment of the equations of Set II along the lines previously employed[9] in examination of other coupling resonances, although the accuracy of such theoretical results may suffer in the present instance because the oscillation frequencies are sufficiently high that both σ_x and σ_y lie rather close to π. The method[10] basically assumes the x-motion to be prescribed, unaffected by coupling with the relatively small y-motion, and this solution when substituted into the y-equation thus gives a differential equation linear in the single dependent variable y.

Since we are here attempting no more than an approximate treatment of the $2\nu_y - \nu_x = 1$ resonance, it apparently is sufficient to employ a simplified form of the y-equation

$$\frac{d^2y}{d\theta^2} + \left[\nu_y^2 + (d/2)(\cos\theta)\ x\right] y = 0,$$

where $d = -0.42$ in the computations reported above (Section IV). If a simple representation of the x-motion,

$$x = A_x \cos \nu_x \theta ,$$

is now employed, one obtains

$$\frac{d^2y}{d\theta^2} + \left[\nu_y^2 + (A_x\, d/2)(\cos \nu_x \theta)(\cos\theta)\right] y = 0$$

or

$$\frac{d^2y}{d\theta^2} + \left[\nu_y^2 + (A_x\, d/4)\cos(1 + \nu_x)\theta + (A_x\, d/4)\cos(1 - \nu_x)\theta\right] y = 0.$$

For purposes of studying the $2\nu_y - \nu_x = 1$ resonance, we may ignore the last term in the coefficient of y and consider the simple Mathieu equation

$$\frac{d^2y}{d\theta^2} + \left[\nu_y^2 + (A_x\, d/4) \cos (1 + \nu_x)\, \theta\right] y = 0.$$

(1.) This last equation has, as is well known, the stability boundaries [cf. Ref. 9, Appendix IB, C]:

$$-\left|\frac{A_x d}{2}\right| < (2\,\nu_y)^2 - (1 + \nu_x)^2 < \left|\frac{A_x d}{2}\right| ,$$

leading to a full-width for the resonance which may be conveniently expressed as

$$\Delta\left[2\,\nu_y - \nu_x - 1\right] = \frac{|A_x d|}{2\,\nu_y + \nu_x + 1} .$$

Numerically, for the problem at hand, this becomes

$$W = \frac{0.42}{3.2} A_x$$

$$= 0.131\, A_x ,$$

where W denotes the full-width of the resonance in units of $2\,\nu_y - \nu_x$.

We may compare this theoretical width with that estimated from the computational results of Section IV A, as is done in Table V below.

TABLE V

Comparison of the Theoretical and Computational Width for the Resonance $2\,\nu_y - \nu_x = 1$

The Table gives the widths in units of $2\,\nu_y - \nu_x$.

$d = -0.42$ $\qquad \nu_x \cong 0.6016,\ \nu_y \cong 0.8008$

A_x	0.50	0.25	0.10
$W_{theor.}$	0.066	0.033	0.013
$W_{obs.}$	0.051	0.029	0.012

(2.) The lapse-rate characterizing y-growth in the unstable region

may also be estimated for the Mathieu equation cited earlier, by reference to methods used previously [cf. Ref. 9, Appendix IV]. One obtains

$$\mu = \frac{|d|}{8} \frac{\sqrt{A_x^2 - A_{thr.}^2}}{1 + \nu_x} \doteq \frac{1}{4} \sqrt{W^2 - 4q^2} \text{ nepers/radian of } \theta ,$$

where q denotes $2\nu_y - \nu_x - 1$ and $A_{thr.}$ the threshold amplitude,

$$\mu_{Max.} = \frac{|d|}{8} \frac{A_x}{1 + \nu_x} \doteq \frac{W}{4} \text{ nepers/radian of } \theta .$$

If, for convenience, we convert these results to decades per revolution (through multiplication by $2\pi \log e \doteq 2.72875$) and insert the appropriate constants for the problem at hand (when required), we obtain

$$\mu = 0.089 \sqrt{A_x^2 - A_{thr.}^2} \doteq 0.68 \sqrt{W^2 - 4q^2} \text{ decades/revolution of } \theta$$

and

$$\mu_{Max.} = 0.089\, A_x \doteq 0.68\, W \text{ decades/revolution of } \theta .$$

The formula for $\mu_{Max.}$ may be compared with the computational results, summarized in Table III, for $2\nu_y - \nu_x = 1.0001$, which corresponds closely to the resonant condition and for which the lapse-rates attain nearly their maximum values. This comparison is given in Table VI.

TABLE VI

Comparison of Theoretical and Observed Lapse-Rates.
Lapse-rates are given in decades/revolution.

$S_1 = -0.1465$, $S_2 = -0.2571$ $\nu_x = 0.6016$, $\nu_y = 0.8009$, $2\nu_y - \nu_x = 1.0001$

A_x	0.50	0.25	0.10	0.050	0.025	0.010	0.005
μ Calculated from A_x	0.045	0.022	0.0089	0.0045	0.0022	0.0009	0.0004
μ Calculated from $W_{obs.}$	0.035	0.020	0.0082				
$\mu_{obs.}$ from Computer	0.0306	0.0156	0.0062	0.0031	0.0016	0.0006	0.0004

From the comparisons shown we infer that the simple theory outlined in this section appears to provide a semi-quantitative account of the effects of the resonance, in the absence of secular change, although the widths for the resonance and the associated lapse-rates appear to be somewhat greater than observed from the computational results.

B. Effect of Traversal of Resonance:

It is tempting to employ the foregoing theoretical results to estimate the possible increase of y-amplitude when traversing the $2\nu_y - \nu_x = 1$ resonance. The results of such an attempt certainly cannot be expected to be of high accuracy, in part because of the approximate character of the preceding analysis and in part because of a certain amount of adiabatic amplitude-change (which we shall ignore) before reaching the resonance, but perhaps primarily because the situation with secular change is in a sense different and the net effect upon the y-motion will certainly (as we have seen) depend markedly on the phases of the respective oscillations.

From the results of the preceding sub-section, we estimate the growth of y-amplitude which can result from traversal of the $2\nu_y - \nu_x = 1$ resonance to be, if the ascending exponential solution dominates,

$$\text{Growth} = \int \frac{\sqrt{W^2 - 4q^2}}{4}\, d\theta \qquad \text{nepers, with the integral taken through the resonance}$$

$$= \frac{1}{4\left|\frac{dq}{d\theta}\right|} \int_{-W/2}^{W/2} \sqrt{W^2 - 4q^2}\, dq$$

$$= \frac{W^2}{4\left|\frac{dq}{d\theta}\right|} \int_0^1 \sqrt{1 - \xi^2}\, d\xi$$

$$= \frac{\pi}{16} \frac{W^2}{\left|\frac{dq}{d\theta}\right|} \qquad \text{nepers,}$$

where, as before, q denotes $2\nu_y - \nu_x - 1$. From the observed dependence of ν_x and ν_y on the parameters S_1 or S_2 (Table III), and from the rate at which the coefficients B_1 and B_2 in effect modify S_1 and S_2, one finds for the equations of Set II (with $B_1 = -B_2$),

$$\left|\frac{dq}{d\theta}\right| \cong \frac{5.3}{8192} \left| B_1 \right|$$

and

$$\text{Growth} = \frac{\pi}{16} \frac{8192}{5.3} \frac{W^2}{|B_1|} \text{ nepers}$$

$$= 303.5 \frac{W^2}{|B_1|} \text{ nepers}$$

$$= 132 \frac{W^2}{|B_1|} \text{ decades,}$$

W being the full-width of the resonance, for the x-amplitude under consideration, measured in units of $2\nu_y - \nu_x$.

In particular, for the case $B_1 = -0.3168$, $B_2 = 0.3168$,

$$\text{Growth} = 416\ W^2 \text{ decades.}$$

If we employ the observed widths of the resonance (Table V) for the x-amplitudes 0.50 and 0.25, we then expect

For $x_o = 0.50$, Growth of 1.08 decades (factor 12);

For $x_o = 0.25$, Growth of 0.35 decades (factor 2.2).

[If the theoretical values of W were employed, the expected growth would be somewhat larger--1.81 and 0.45 decades, or factors of 65 and 2.8, respectively.] As noted in Section IV B, the corresponding figures estimated from actual computational runs (32 runs for each x-amplitude) were

For $x_o = 0.50$, Growth of 1.25 decades (factor 18);

For $x_o = 0.25$, Growth of 0.52 decades (factor 3.3).

VI. REFERENCES AND NOTES:

[1]For example, MURA-263, 295, 319, 320, 365 and 379.

[2]See, for example, A. E. R. E. T/R 2342.

[3]Argonne National Laboratory, Particle Accelerator Division Summary Report ANL-5630 (April-September 1956)--Section III.

[4]C. L. Hammer and A. J. Bureau, Rev. Sci. Inst. 26, 594, 598 (June, 1955).

[5]C. L. Hammer and L. J. Laslett, Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, Switzerland (September, 1958)--Paper A/Conf. 15/P/726.

[6]J. N. Snyder, Internal MURA Reports (IBM Programs 75 and 77) 237 and 238 (February-March, 1957).

[7]L. Jackson Laslett, MURA Report 206 (November 21, 1956).

[8]In an investigation to see whether any resonances leading to y-growth could be detected for the equations of Set II in the interval between $2\nu_y - \nu_x = 1$ and $\nu_x = 1/2$, the initial x-amplitude was determined by taking $x_o = 0.50$ and, as before, the value $y_o = 10^{-5}$ was employed. The values used for the constants S_1, S_2, and estimates of the corresponding small-amplitude oscillation frequencies are listed below $\left[S_1 + S_2 = -0.4036\right]$.

S_1	S_2	ν_x(est.)	ν_y (est.)
-0.1380	-0.2656	0.583	0.814
-0.1355	-0.2681	0.577	0.818
-0.1330	-0.2706	0.571	0.822
-0.1305	-0.2731	0.575	0.825_5
-0.1280	-0.2756	0.559	0.829
-0.1255	-0.2781	0.553	0.833
-0.1230	-0.2806	0.546	0.837
-0.1205	-0.2831	0.539	0.840_5
-0.1180	-0.2856	0.531	0.844
-0.1155	-0.2881	0.52_3	0.848
-0.1130	-0.2906	0.51_0	0.852

As expected, with the coupling terms employed in the equations of Set II, no evidence of any coupling resonance was seen in this interval. For the two runs for which the coefficients were those listed in the last two lines of the preceding Table, however, x-instability for the x-amplitude employed (0.50) rapidly became apparent, attributable to the proximity to the $\sigma_x = \pi$ ($\nu_x = 1/2$) resonance (Figs. 11 and 12).

[9]Esp. L. Jackson Laslett and A. M. Sessler, MURA-263 (May 6, 1957).

[10]The procedure in principle thus parallels that suggested by W. Walkinshaw for analysis of the $2\nu_y - \nu_x = 0$ resonance--W. Walkinshaw, "A Spiral Ridged Bevatron," A. E. R. E., Harwell (1956).

APPENDIX I

Data Illustrating Growth in Traversal of Resonance

$dn/d\theta = 0.000\,096$

$\lvert A_x \rvert = 0.50$								
x_o	p_{x_o}	y_o	p_{y_o}	Growth Factor For Successive Maxima and Minima After Traversing Resonance				Geom. Mean of Middle Two Factors
.5	0	.00001	0	17.657	8.920	15.646	9.703	11.81
.354	.361			10.838	5.774	9.706	6.135	7.49
0	.51			3.146	1.661	2.709	1.769	2.12
-.354	.361			7.537	3.774	6.475	4.249	4.94
-.5	0			18.725	9.598	16.635	9.694	4.00
-.354	-.361			22.034	11.883	19.607	11.970	15.26
0	-.51			23.547	12.456	20.180	13.902	15.85
.354	-.361			23.390	11.906	20.104	12.619	15.47
.5	0	.00000 7101	.0000 11999	23.606	11.923	20.911	12.965	15.79
.354	.361			23.937	12.762	21.408	13.533	16.53
0	.51			18.128	9.586	14.962	10.585	11.98
-.354	.361			10.203	5.084	8.800	5.753	6.69
-.5	0			0.378	0.183	0.325	0.235	0.24
-.354	-.361			7.552	4.016	6.708	4.121	5.19
0	-.51			14.729	7.826	12.671	8.713	9.96
.354	-.361			22.651	11.539	19.476	12.238	14.99
.5	0	0	.0000 16999	15.724	7.927	13.917	8.638	10.50
.354	.361			23.303	12.424	20.849	13.178	16.09
0	.51			23.565	12.468	20.145	13.760	15.85
-.354	.361			22.182	11.056	19.161	12.517	14.55
-.5	0			18.154	9.301	16.108	9.388	12.24
-.354	-.361			10.980	5.916	9.773	5.969	7.60
0	-.51			3.044	1.621	2.639	1.810	2.07
.354	-.361			8.504	4.365	7.331	4.656	5.66
.5	0	-.00000 7101	.0000 11999	2.136	1.068	1.883	1.178	1.42
.354	.361			8.469	4.519	7.622	4.816	5.87
0	.51			14.636	7.756	12.558	8.265	9.87
-.354	.361			21.196	10.570	18.300	11.961	13.91
-.5	0			26.199	13.425	23.245	13.547	17.67 ←
-.354	-.361			23.184	12.523	20.631	12.587	16.07
0	-.51			18.088	9.560	15.494	10.675	12.17
.354	-.361			10.497	5.346	9.025	5.681	6.95

$|A_x| = 0.25$

x_o	p_{x_o}	y_o	p_{y_o}	Growth Factor For Successive Maxima and Minima After Traversing Resonance				Geom. Mean of Middle Two Factors
.25	0	.00001	0	3.899	2.759	3.681	2.843	3.187
.177	.1805			4.010	2.857	3.745	2.945	3.271
0	.255			3.615	2.497	3.291	2.590	2.867
-.177	.1805			2.625	1.851	2.459	1.949	2.133
-.25	0			1.338	0.922	1.234	0.960	1.067
-.177	-.1805			0.561	0.419	0.541	0.436	0.476
0	-.255			2.045	1.478	1.917	1.510	1.683
.177	-.1805			3.280	2.293	2.988	2.409	2.618
.25	0	.00000 7101	.0000 11999	2.031	1.402	1.877	1.510	1.622
.177	.1805			3.285	2.348	3.070	2.412	2.685
0	.255			3.969	2.802	3.667	2.846	3.205
-.177	.1805			4.074	2.840	3.774	3.020	3.274
-.25	0			3.579	2.491	3.314	2.601	2.873
-.177	-.1805			2.591	1.808	2.395	1.823	2.081
0	-.255			1.310	0.905	1.191	0.945	1.038
.177	-.1805			0.526	0.381	0.513	0.405	0.442
.25	0	0	.0000 16999	1.331	0.921	1.229	0.934	1.064
.177	.1805			0.590	0.442	0.567	0.455	0.501
0	.255			1.999	1.461	1.915	1.472	1.673
-.177	.1805			3.259	2.255	2.986	2.409	2.595
-.25	0			3.975	2.761	3.669	2.881	3.183
-.177	-.1805			4.007	2.863	3.754	2.843	3.278 ←
0	-.255			3.565	2.491	3.285	2.607	2.861
.177	-.1805			2.686	1.833	2.435	1.958	2.113
.25	0	-.00000 7101	.0000 11999	3.568	2.529	3.361	2.569	2.915
.177	.1805			2.584	1.810	2.406	1.885	2.087
0	.255			1.341	0.908	1.204	0.980	1.046
-.177	.1805			0.314	0.491	0.369	0.483	0.426
-.25	0			2.025	1.440	1.900	1.493	1.654
-.177	-.1805			3.204	2.325	3.052	2.322	2.664
0	-.255			3.931	2.753	3.626	2.892	3.159
.177	-.1805			4.153	2.849	3.768	3.038	3.276

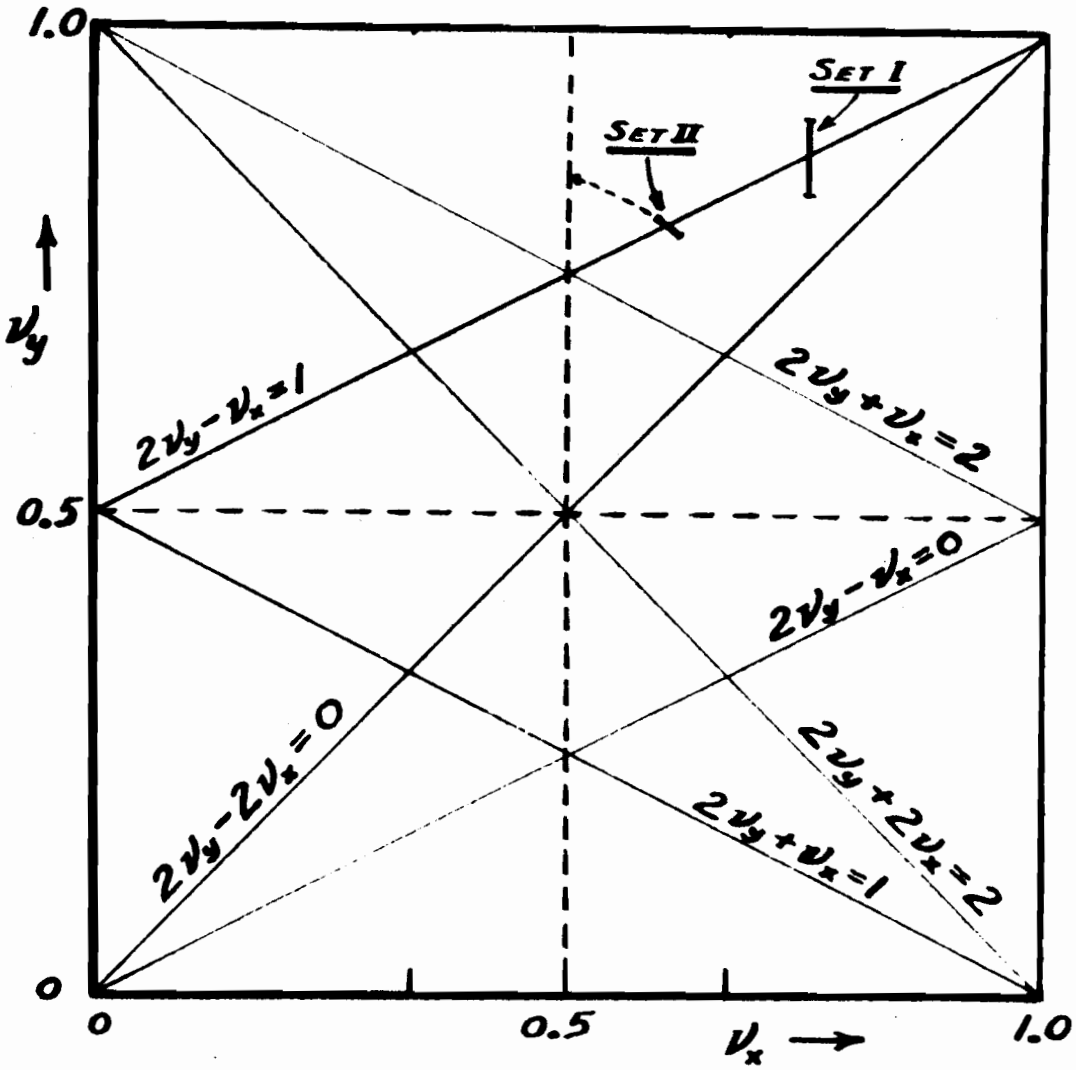


Fig. 1. Frequency diagram, showing location of the working points used in computational study of the equations of Sets I and II.

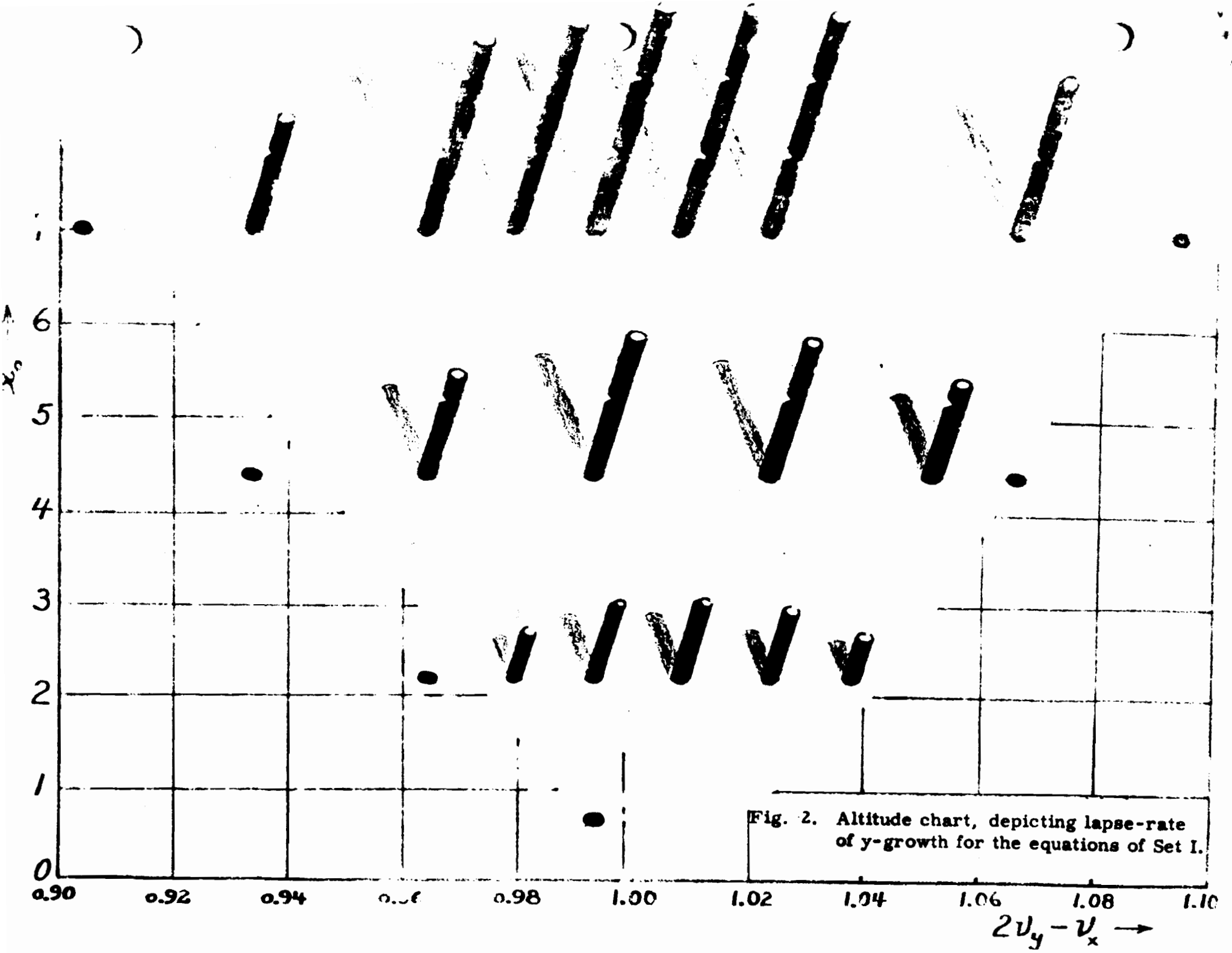


Fig. 2. Altitude chart, depicting lapse-rate of y-growth for the equations of Set I.

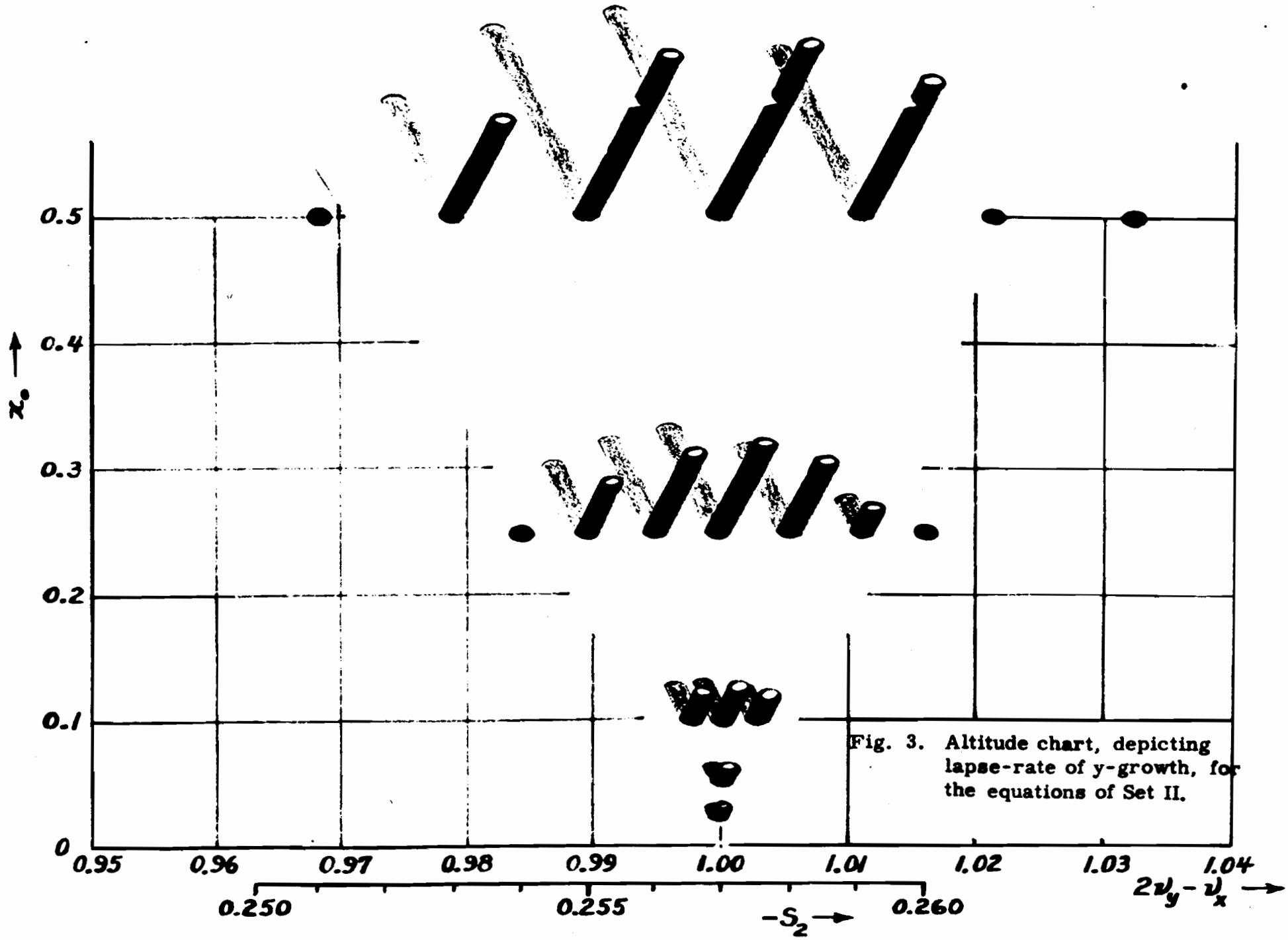


Fig. 3. Altitude chart, depicting lapse-rate of y-growth, for the equations of Set II.

Fig. 4. Variation of y-amplitude during traversal of resonance.

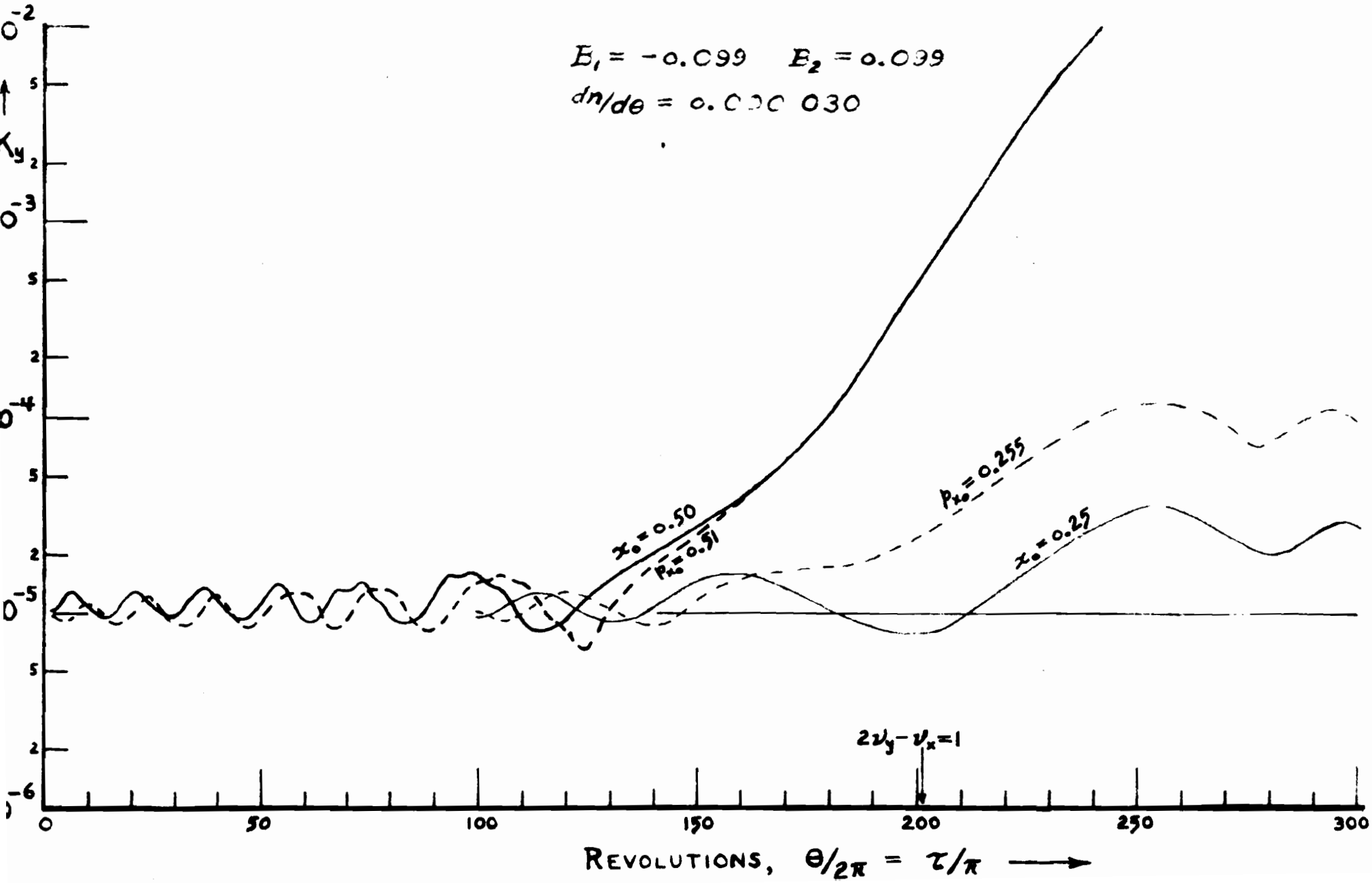


Fig. 5. Variation of y-amplitude during traversal of resonance.

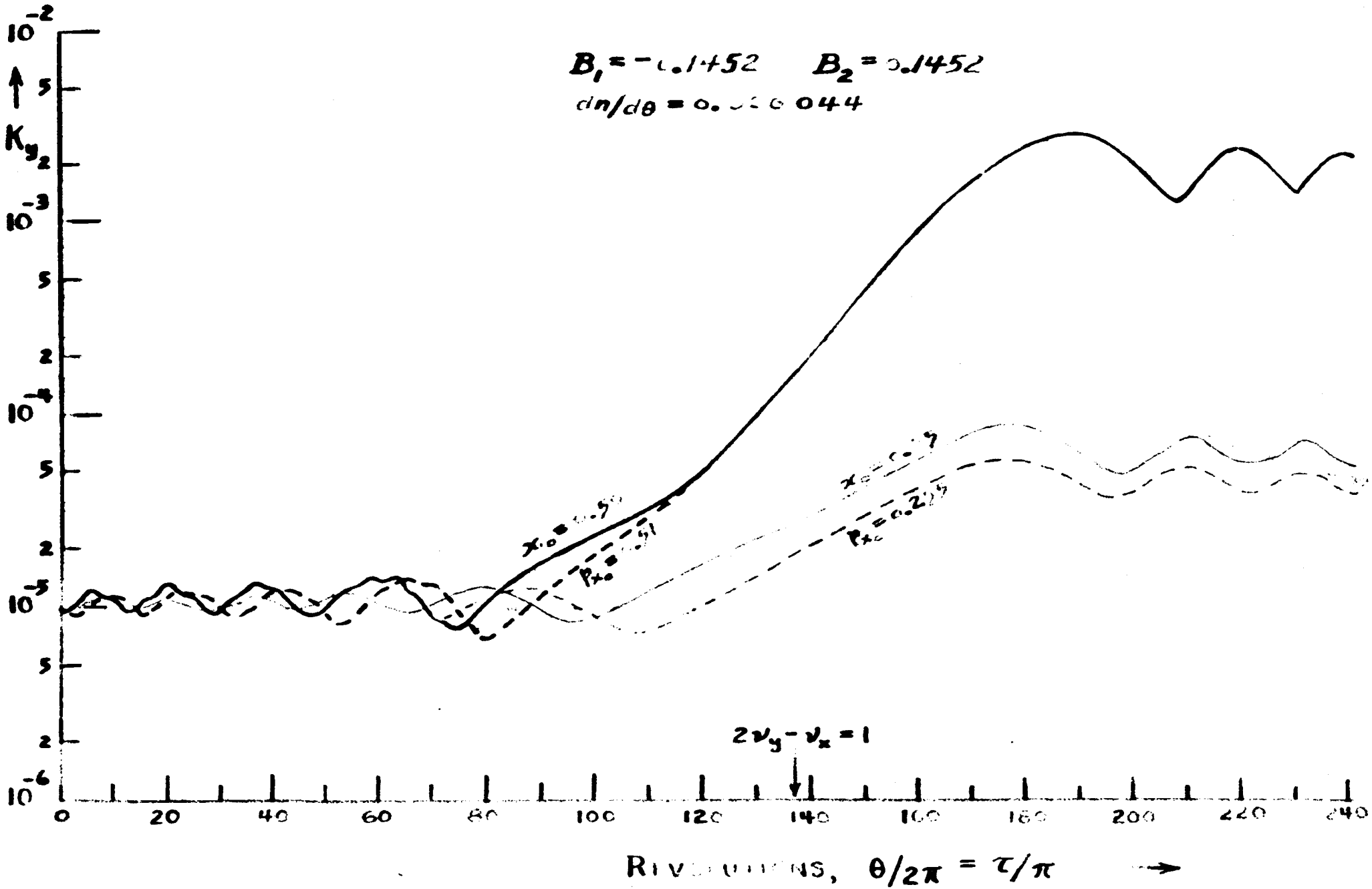


Fig. 6. Variation of y-amplitude durin traversal of resonance.

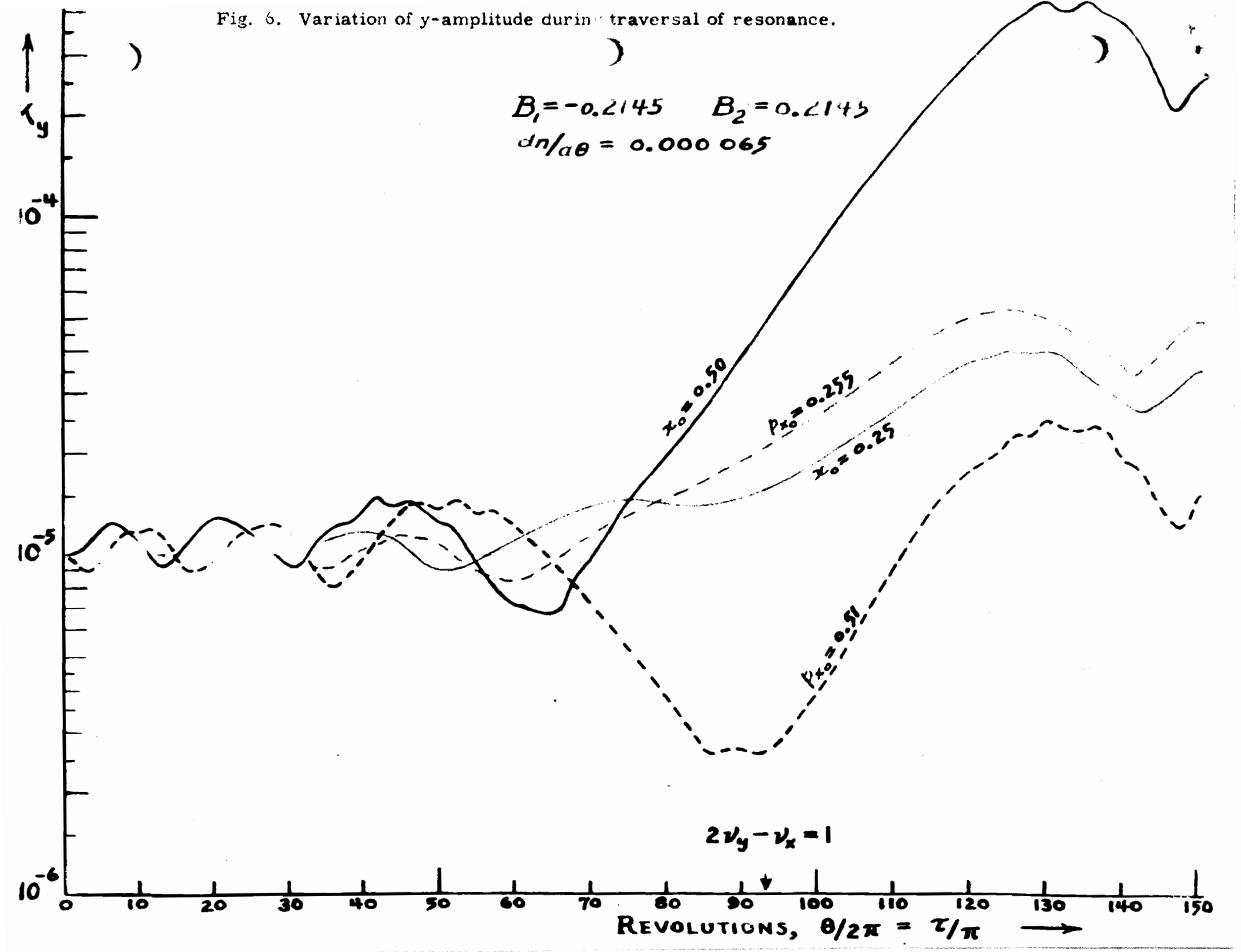


Fig. 7. Variation of y-amplitude during traversal of resonance.

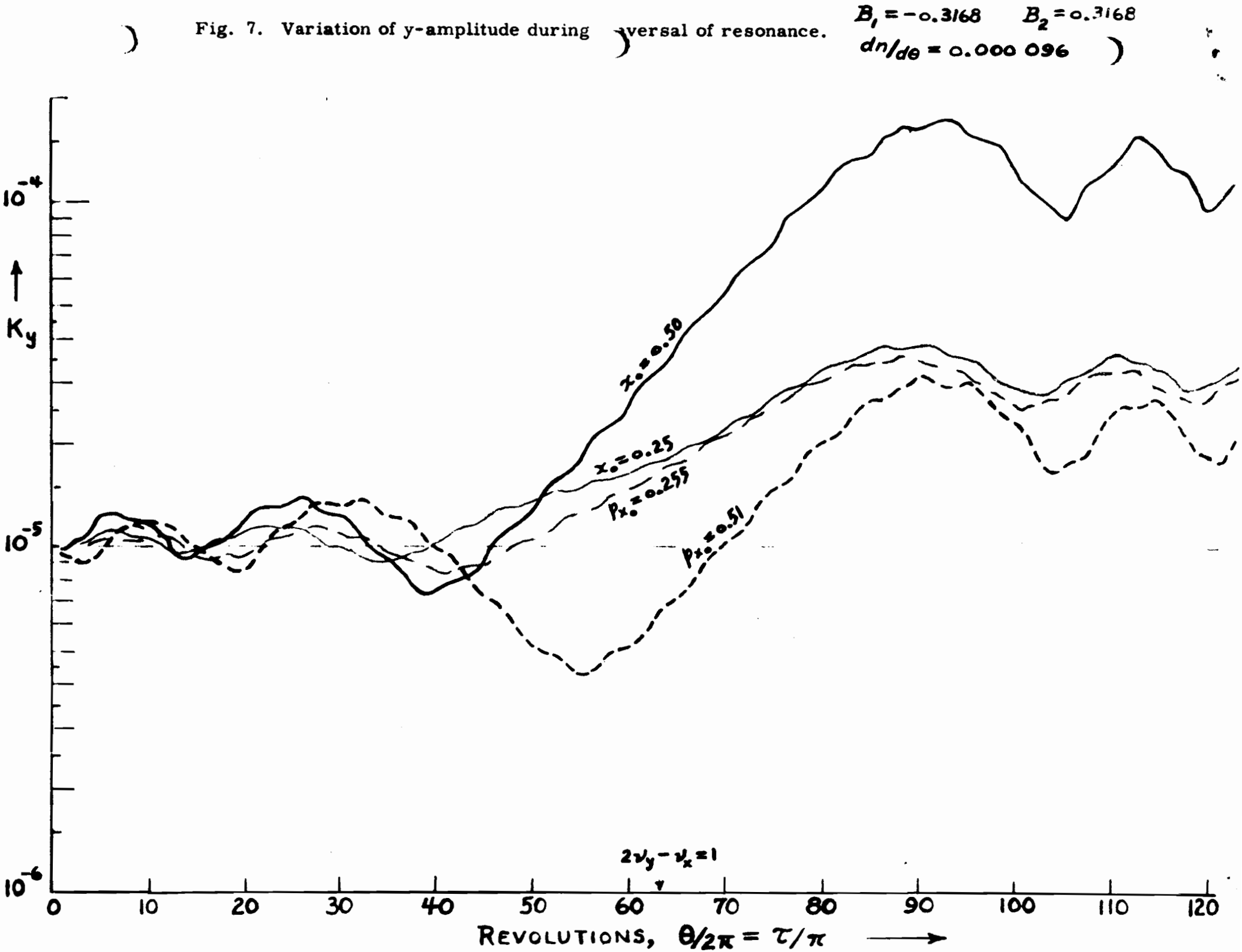


Fig. 8. Variation of y-amplitude during traversal of resonance.

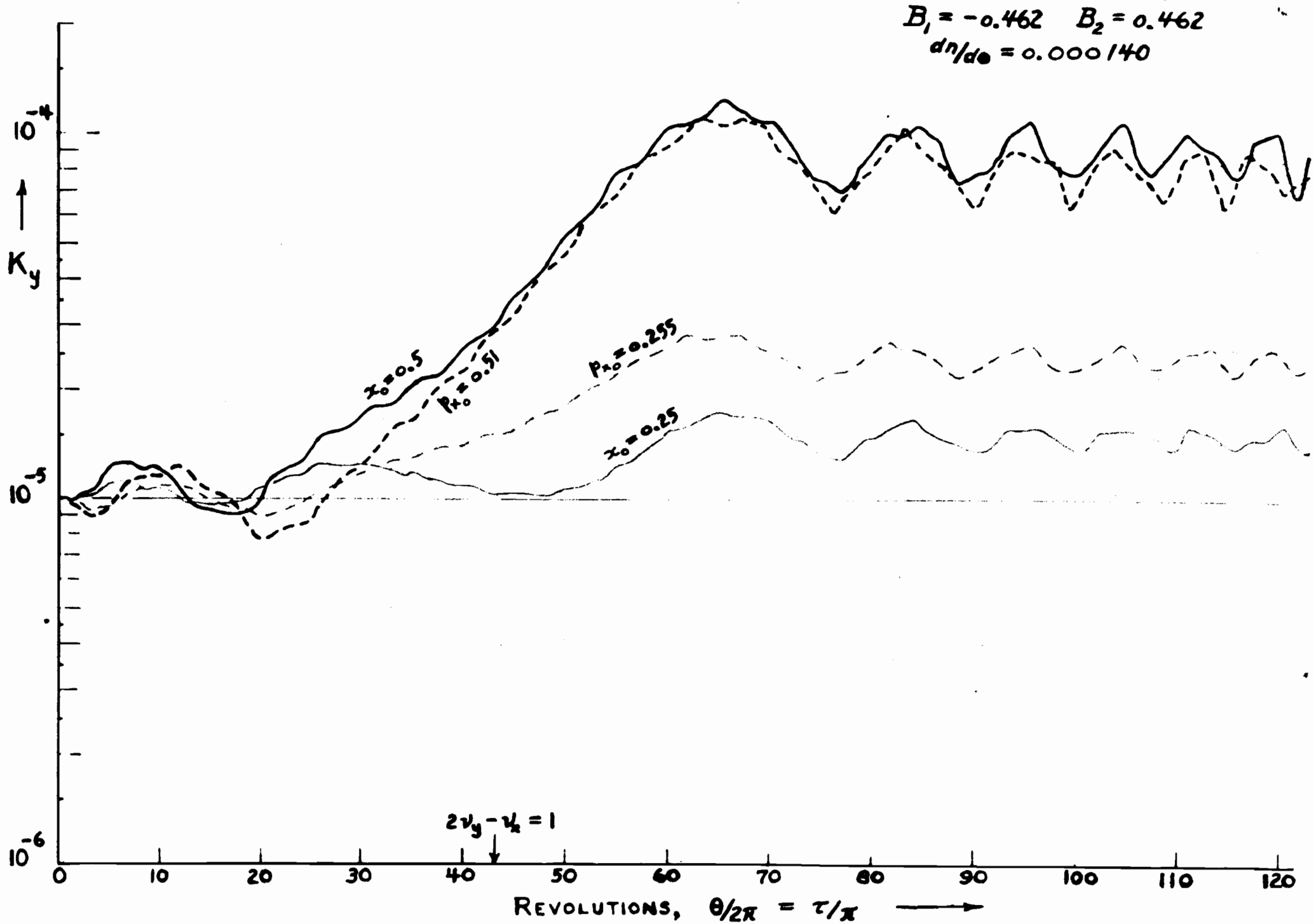


Fig. 9. Variation of y-amplitude during trav al of resonance.

$B_1 = -0.66 \quad B_2 = c\ 66$

$dn/d\theta = 0.000\,200$

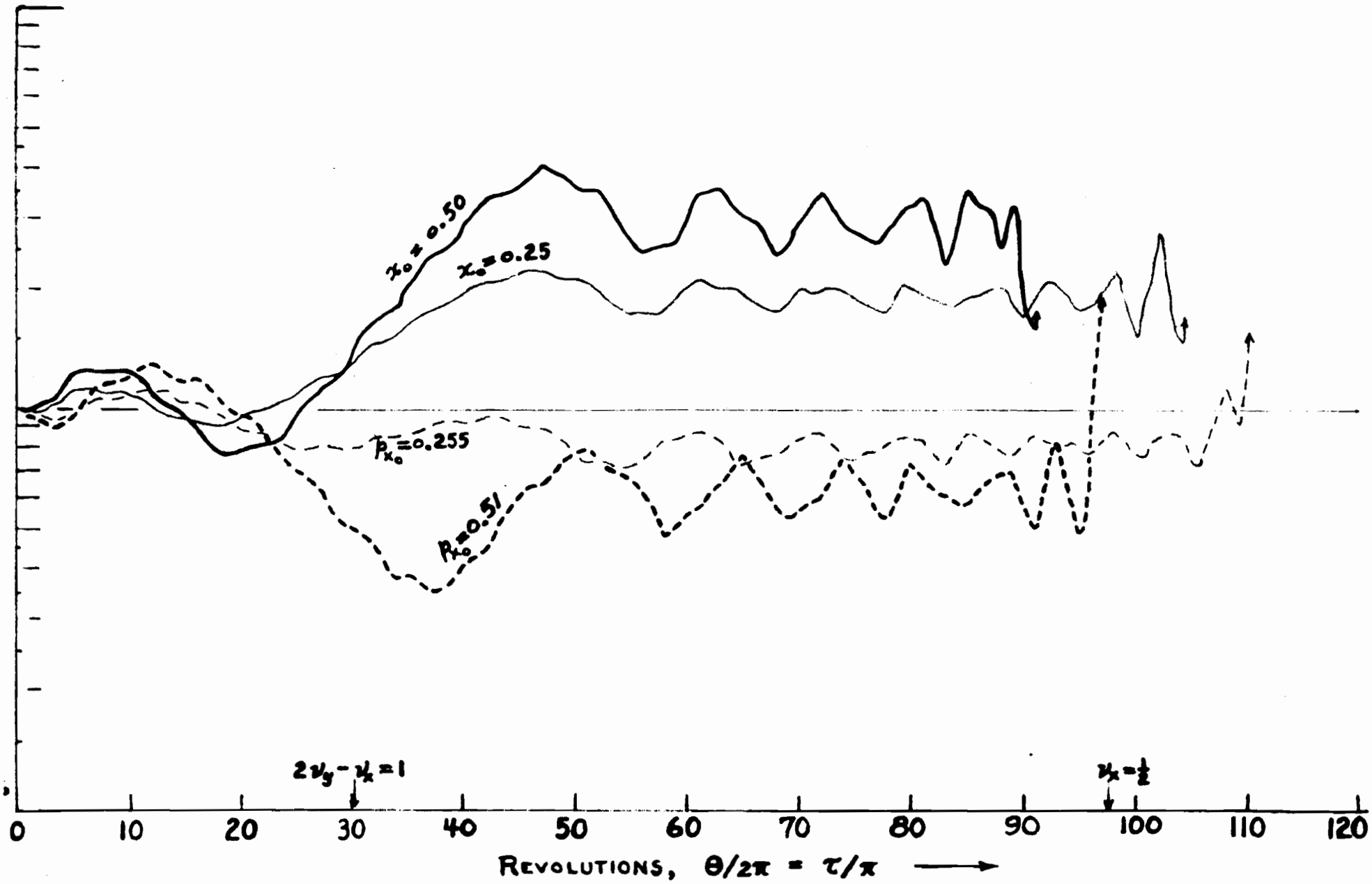
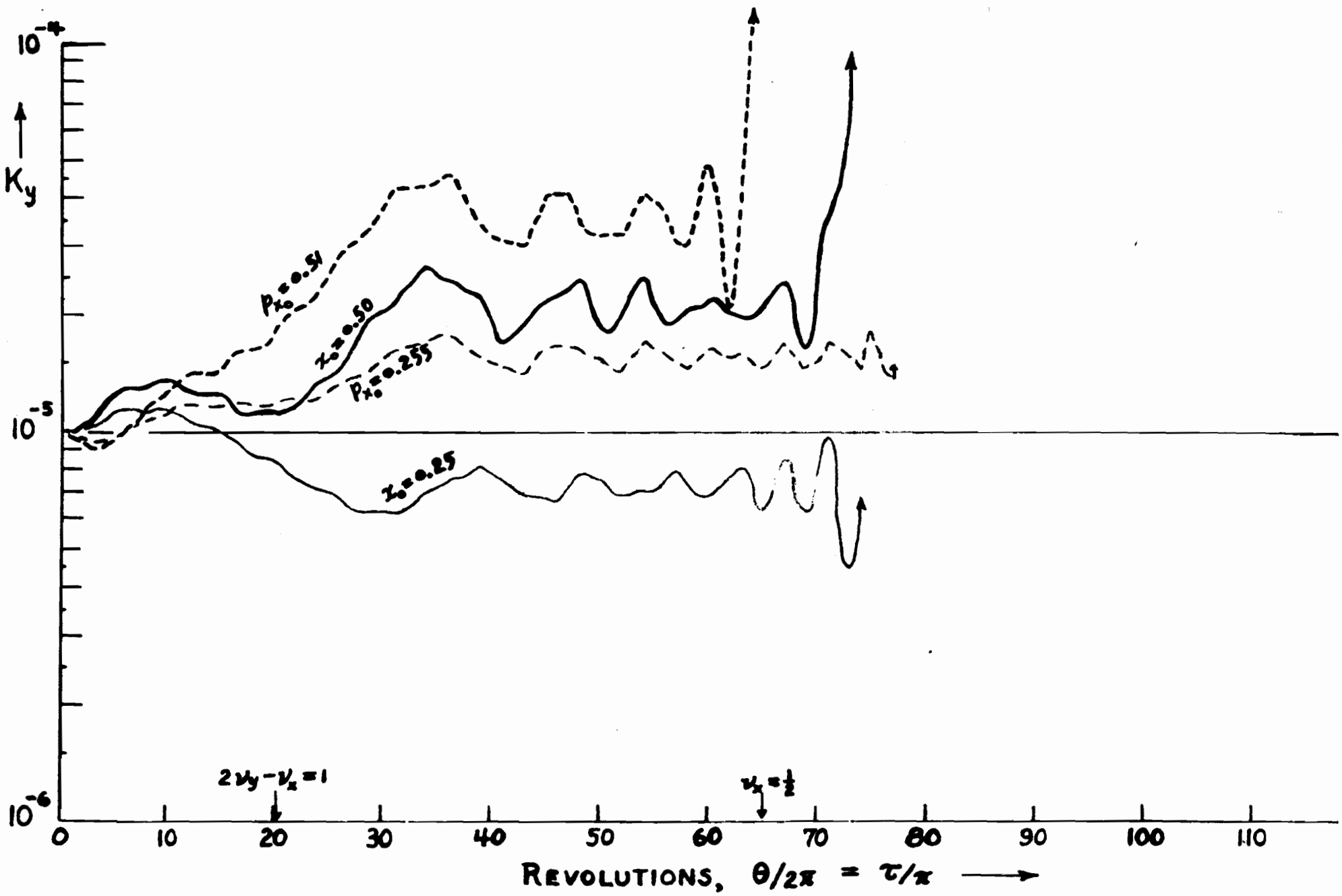


Fig. 10. Variation of y-amplitude during traversal of resonance.

$B_1 = -0.99$ $B_2 = 0.99$

$dn/d\theta = 0.000\,300$



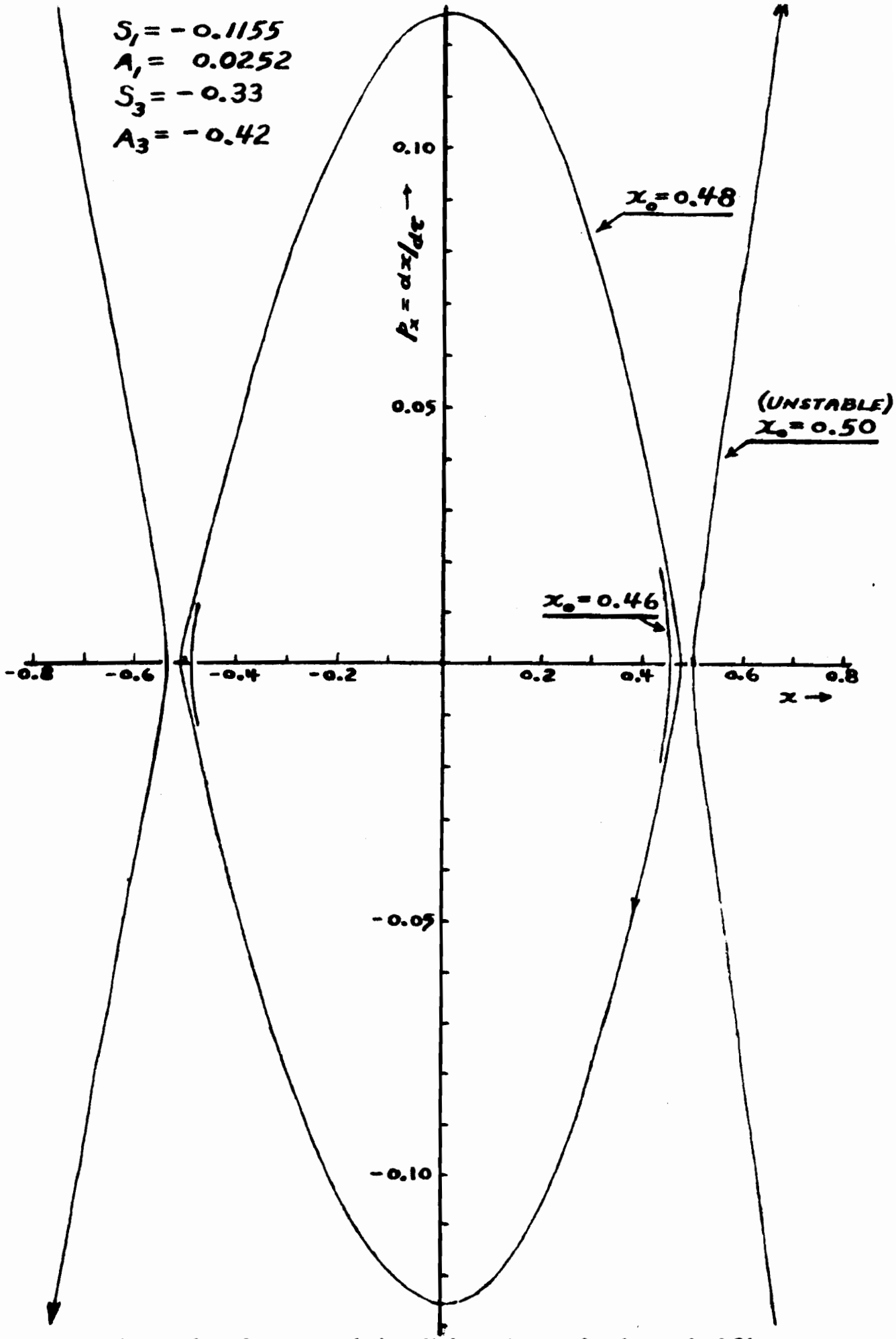


Fig. 11. Phase-plots for uncoupled radial motion at $\theta = 0$, mod. 2π. $\nu_{x_0} \doteq 0.52_3$

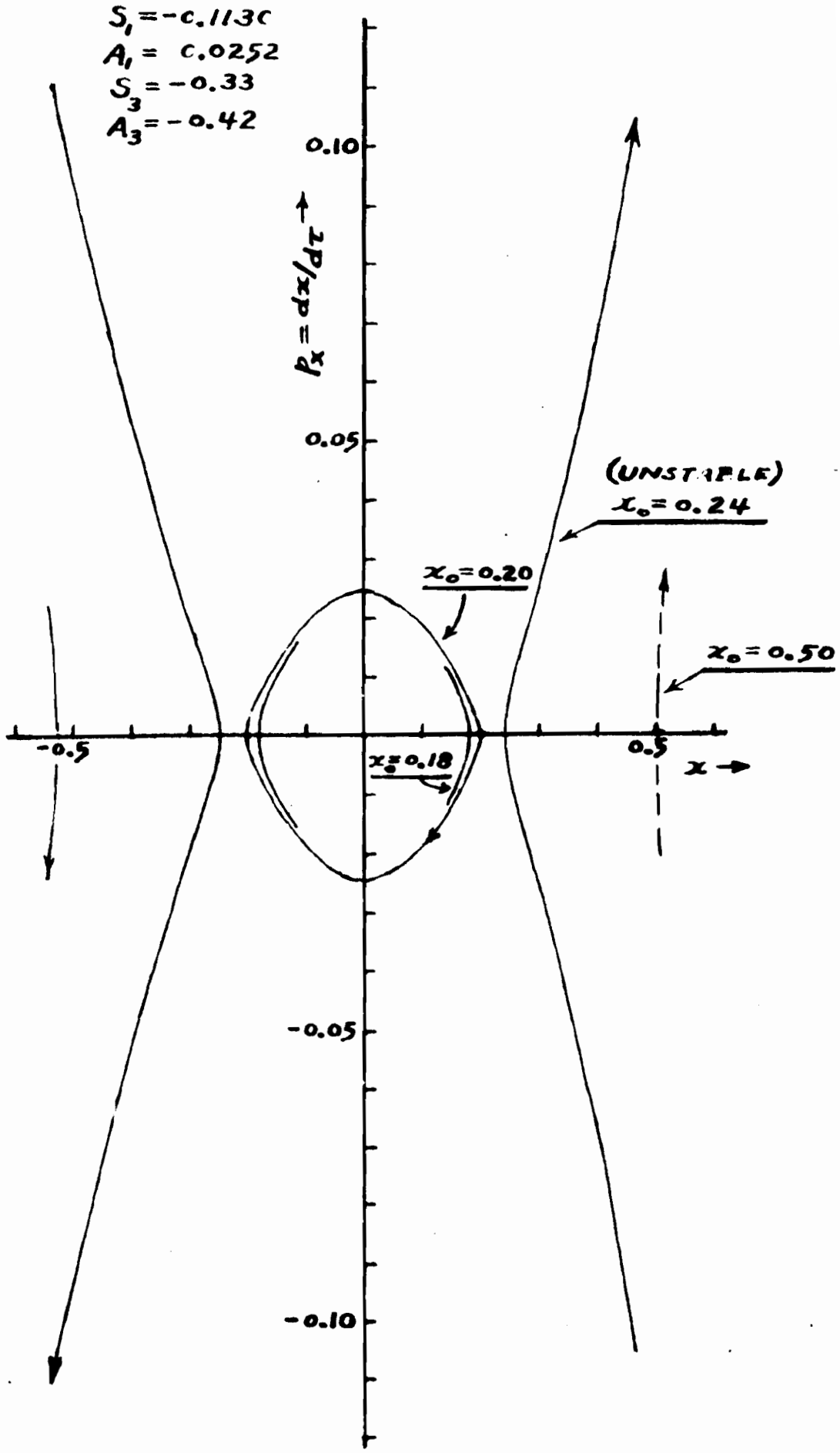


Fig. 12. Phase-plots for uncoupled radial motion at $\theta = 0$, mod. 2π. $\nu_{x_c} \doteq 0.51_0$

Example: $[dn/d\theta = 0.000\,096 \quad x_0 = 0.5]$

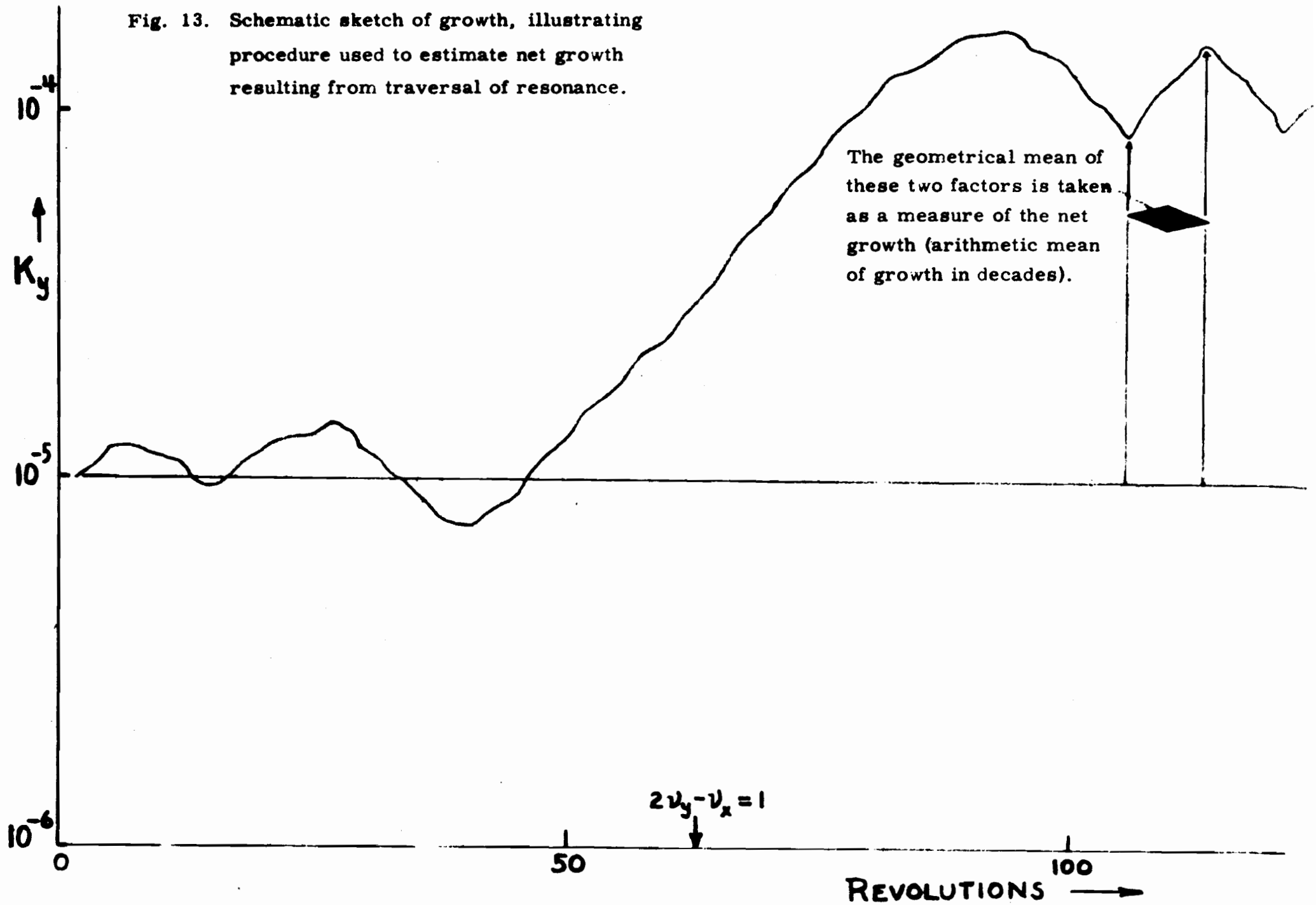


Fig. 13. Schematic sketch of growth, illustrating procedure used to estimate net growth resulting from traversal of resonance.