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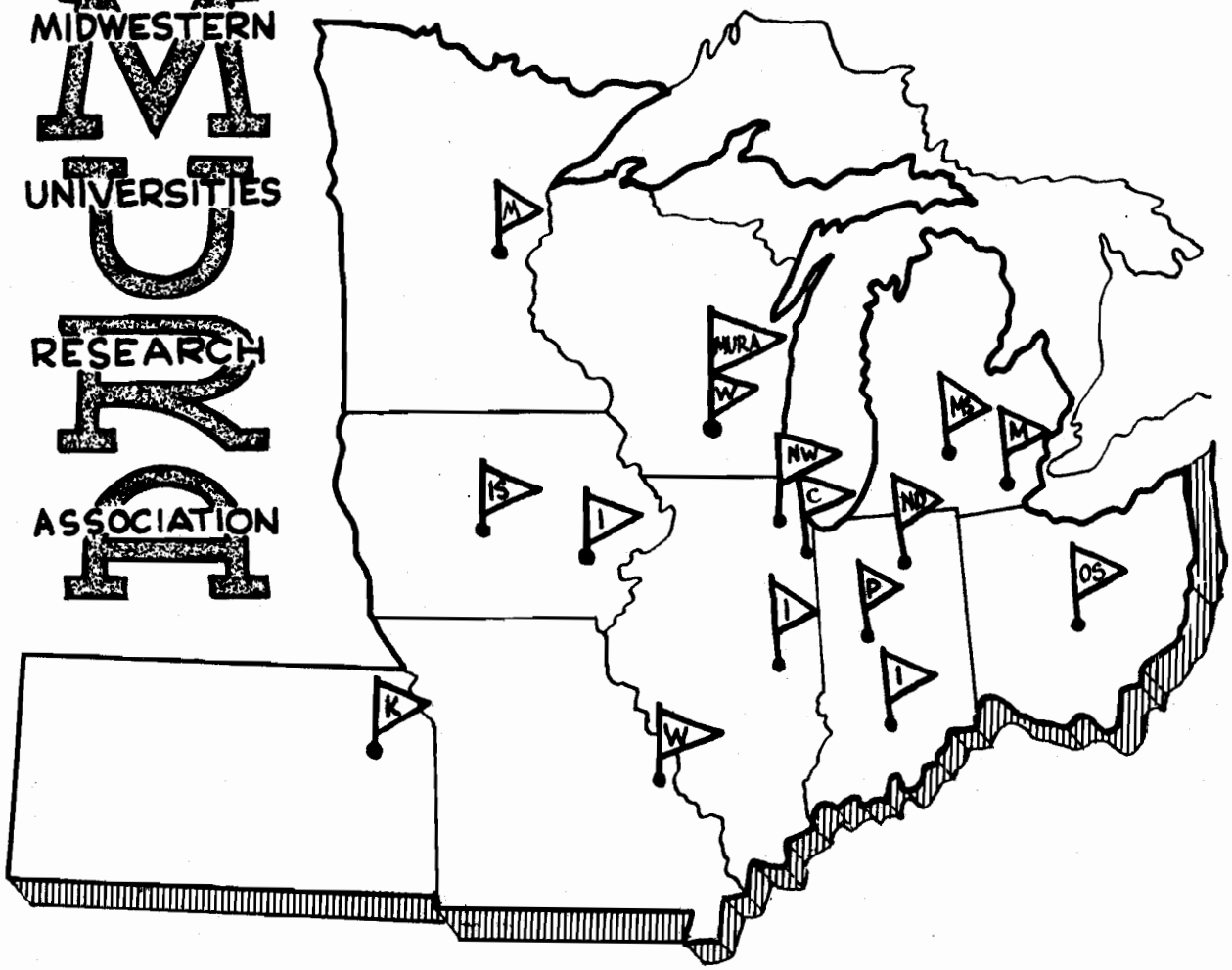

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ON THE INSTABILITY OF A COASTING BEAM**

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ABSTRACT

The time-dependent Boltzmann equation is used to study longitudinal motion of particles, in a circulating beam of particles. The equation is linearized about the stationary solution corresponding to a coasting beam of almost monoenergetic particles, to determine the effect of small density fluctuations. An integral equation is obtained, and solved, leading to a dispersion relation quite similar to that well known in the theory of plasma oscillations. It is shown that if the accelerator is operating above the transition energy, then the motion can be unstable. Expressions are given for the rate of growth of longitudinal waves, and it is shown that if the unperturbed beam satisfies certain criteria,then the motion is either stable or it grows at such a slow rate as to be of no consequence. Numerical examples for MURA particle accelerators show that essentially there is stability.

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I. INTRODUCTION

The theory of collective particle behavior in accelerators is still in a rather rudimentary state. The static effects of cooperative phenomena have been treated in a crude approximation insofar as they affect transverse particle motion,[1] and in a more rigorous manner for the one-dimensional case of longitudinal motion.[2] Extensive work has been done on intense (self-constricted) beams[3] but this is not the phenomenon of interest in most particle accelerators. Some speculation has been made[4] as to certain dynamic effects but in general no extensive treatment has been given of non-static collective effects. It is the purpose of this paper to study the simplest dynamical motion we are aware of, by a method which is capable of extension to more complicated situations.

We consider the longitudinal motion of particles in an unneutralized coasting beam in a circular accelerator. In Section II the situation is discussed physically using elementary mathematics, but the method is not easily extended to other than monoenergetic beams. In Section III (which is independent of Section II) the time-dependent Boltzmann equation is reduced to an integral equation, which is then used to yield a dispersion relation. In Section IV this dispersion relation is used to study the motion for various unpertrubed beams. Numerical examples are given in Section V for two MURA accelerators which are seen to be stable for practical purposes.

In future reports by Dr. Symon and the authors, the methods of this report will be used to investigate the effects of neutralizing background charges and intersecting beams. In this way the "two-stream amplification" process[4] (as it is relevant to particle accelerators) will be studied.

II. PHYSICAL ARGUMENTS

For a monoenergetic beam of particles circulating in an accelerator, which is operated above the transition energy, it is easy to see that there is a longitudinal instability, a phenomenon first pointed out by J. C. Maxwell. Consider such a beam which is initially, as a function of azimuth, uniform in density. Suppose at one azimuth there is a small increase in density. Particles just behind this perturbation (relative to the sense of rotation of the particles), are subject to repulsive space charge forces which cause the particles to lose energy. Since the accelerator is operating above the transition energy, these particles which lose energy gain in frequency (by slipping to an orbit of slightly smaller radius) and so move ahead into the region of the perturbation thus increasing the perturbation. The same argument works for particles ahead of the perturbation which slip backwards into the perturbation. These extra particles moving into the region of the perturbation cause increased space charge forces which thus cause other particles to be subject to the same effect and the initially uniform beam collapses at an exponential rate to one azimuthal position.

It should be observed in the above argument that it is repulsive space charge forces which cause particles to move together. This apparent paradox is due to the unusual relation between frequency and energy in an accelerator operating above the transition energy--effectively the particles may be thought of as having a negative mass.

The above argument will be made quantitative in this section and a growth rate for the instability derived.[5] It should be observed that if the

unperturbed beam has an energy spread, then particles in the beam have different frequencies so that the effect of a perturbation is somewhat compensated. This very important effect is beyond the method used in this section, but is discussed in Sections III and IV.

The kinetic energy of a particle is assumed to be of the form

$$T = T_0 - \alpha(\varphi - \omega), \tag{1}$$

where φ is the actual angular velocity of the particle and ω is the mean angular velocity of the beam. Let F be the force acting of the particle, so that

$$\frac{dT}{dt} = -\alpha\ddot{\varphi} = Fc, \tag{2}$$

where c is the velocity of light which is here taken to be the velocity of the particles.

Let x be the angular displacement ($\varphi - \omega t$) of a particle in the unbunched beam; we use x to label the particles. Let y be the actual displacement of the particle in the bunched beam. Then if ρ_0 is the density in the unbunched beam and ρ is the density in the bunched beam, we have

$$\rho(y)\,dy = \rho_0\,dx\,. \tag{3}$$

One can regard y as a function of x and t, or x as a function of y and t. Thus

$$\rho = \rho_0 \frac{\partial}{\partial y} x(y,t) \tag{4}$$

$$\frac{\partial \rho}{\partial y} = \rho_0 \frac{\partial^2 x}{\partial y^2}\,. \tag{5}$$

Assume that the force F is proportional to the charge density gradient, or that

$$F = -k \frac{\partial \rho}{\partial y} = -k\rho_0 \frac{\partial^2 x}{\partial y^2} . \tag{6}$$

But $\ddot{\varphi} = \frac{\partial^2 y}{\partial t^2}$, so Eq. (2) becomes:

$$\alpha \frac{\partial^2 y}{\partial t^2} = k c \rho_0 \frac{\partial^2 x}{\partial y^2} . \tag{7}$$

Now, from $x \equiv x\left[y(x, t), t\right]$, we obtain by differentiation,

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial x} = 1 . \tag{8}$$

Differentiating again, one can show that

$$\frac{\partial^2 x}{\partial y^2} \left(\frac{\partial y}{\partial x}\right)^2 + \frac{\partial x}{\partial y} \frac{\partial^2 y}{\partial x^2} = 0 , \tag{9}$$

so that

$$\frac{\partial^2 x}{\partial y^2} = -\left(\frac{\partial y}{\partial x}\right)^{-3} \frac{\partial^2 y}{\partial x^2} , \tag{10}$$

which allows the equation of motion (Eq. (7)) to be written in the form

$$\alpha \left(\frac{\partial y}{\partial x}\right)^3 \frac{\partial^2 y}{\partial t^2} + k c \rho_0 \frac{\partial^2 y}{\partial x^2} = 0 . \tag{11}$$

Equivalently, using Eq. (4)

$$\alpha \frac{\partial^2 y}{\partial t^2} + k c \frac{\rho^3}{\rho_0^2} \frac{\partial^2 y}{\partial x^2} = 0 . \tag{12}$$

Taking the partial derivative with respect to x, ignoring terms quadratic in the disturbance, and writing the equation in terms of ρ, one obtains

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{k c}{\alpha} \frac{\rho^3}{\rho_0^2} \frac{\partial^2 \rho}{\partial x^2} = 0 . \tag{13}$$

If ρ is very nearly uniform, then $\rho^3/\rho_0^2 \approx \rho_0$, and letting

$$\gamma^2 = \frac{k c \rho_0}{\alpha} \quad , \tag{14}$$

we obtain

$$\frac{\partial^2 \rho}{\partial t^2} + \gamma^2 \frac{\partial^2 \rho}{\partial x^2} = 0. \tag{15}$$

From this it is clear that if $\frac{\partial^2 \rho}{\partial x^2} < 0$, then $\frac{\partial^2 \rho}{\partial t^2} > 0$.

If we have a sinusoidal distribution corresponding to n waves about the accelerator, then the solution to Eq. (15) is

$$\rho = \mathrm{Re} \sin n (x + i \gamma t), \tag{16}$$

corresponding to a characteristic growth time which for n = 1 is just $T = 1/\gamma$.

Using the notation of the next section, we have

$$\begin{aligned} \alpha &= \left[-2\pi \frac{df}{dE} \right]^{-1} \\ k &= \frac{e^2 g}{2\pi \gamma^2 R^2} \\ \rho_0 &= \frac{N}{2\pi} \\ f &= \frac{c}{2\pi R} \quad , \end{aligned} \tag{17}$$

so that

$$T = \frac{1}{\gamma} = \left[\frac{\alpha}{k c \rho_0} \right]^{1/2},$$

or

$$T = \left[\frac{\gamma^2 R}{e^2 g N f \left| \frac{df}{dE} \right|} \right]^{1/2}, \tag{18}$$

in agreement with the result to be obtained in Section IV for a monoenergetic beam.

III. GENERAL FORMULATION

Let the distribution function be $\psi(\varphi, \dot{\varphi}, t)$, where φ is the azimuthal angle of a particle as measured in a stationary coordinate system, and $\dot{\varphi}$ is related to the energy of the particle. Note that following the usual convention, we are working in a coordinate-velocity space rather than in terms of canonical coordinates and momenta.[6] We make the approximation that Liouville's Theorem is valid in μ-space[7] so that ψ satisfies the following Boltzmann equation:

$$\frac{\partial \psi}{\partial t} = -\dot{\varphi} \frac{\partial \psi}{\partial \varphi} - \frac{d\dot{\varphi}}{dt} \frac{\partial \psi}{\partial \dot{\varphi}} . \tag{19}$$

A stationary solution to this equation, since there are no oscillators (i.e. forces tending to charge $\dot{\varphi}$ in time), is given by the arbitrary function $\psi_0(\dot{\varphi})$. This describes the energy spread in the stationary, azimuthally uniform stacked beam. Let us look for small variations from this solution by letting

$$\psi(\varphi, \dot{\varphi}, t) = \psi_0(\dot{\varphi}) + \psi_1(\varphi, \dot{\varphi}, t). \tag{20}$$

Inserting this into Eq. (19) and linearizing the resulting equation, we obtain

$$\frac{\partial \psi_1}{\partial t} = -\dot{\varphi} \frac{\partial \psi_1}{\partial \varphi} - \frac{d\dot{\varphi}}{dt} \frac{\partial \psi_0}{\partial \dot{\varphi}} , \tag{21}$$

where $\frac{d\dot{\varphi}}{dt}$, which characterizes the force of space charge, is linear in ψ_1. Note that the term $\frac{d\dot{\varphi}}{dt} \frac{\partial \psi_1}{\partial \dot{\varphi}}$ vanishes if only the ψ_0 contribution to $\frac{d\dot{\varphi}}{dt}$ is included (no longitudinal fields from a uniform charge distribution), and is otherwise of higher order.

We need now to evaluate $\frac{d\dot{\varphi}}{dt}$, after which Eq. (21) will be an integral equation for ψ_1. Clearly[8]

$$\frac{d\dot{\varphi}}{dt} = 2\pi \frac{df}{dt}, \tag{22}$$

where f is the particle frequency. We may write $\frac{df}{dt} = \frac{df}{dE}\frac{dE}{dt}$, and

$$\frac{dE}{dt} = 2\pi R f e \mathcal{E}, \tag{23}$$

where R is the radius of the orbit, E is the particle energy, and $\mathcal{E}$ is the electric field due to space charge (or any other sources) taken as positive in the direction of particle motion. Combining this with Eq. (21), we obtain

$$\frac{\partial \psi_1}{\partial t} = -\dot{\varphi}\frac{\partial \psi_1}{\partial \varphi} - 4\pi^2 R\left(f\frac{\partial f}{\partial E}\right)_s e\mathcal{E}\frac{\partial \psi_0}{\partial \dot{\varphi}}, \tag{24}$$

where $f\frac{\partial f}{\partial E}$ may be evaluated for a typical particle in the stationary distribution $\psi_0(\dot{\varphi})$ and hence, is labeled with the subscript s. The electric field $\mathcal{E}$ may be evaluated very easily in the case that the wavelength of the perturbation is large compared to the gap G of the accelerator tank. In this case the electric field may be taken to depend only upon the gradient of the charge distribution at the azimuth in question. Thus by an extension[9] of previously given arguments,[2]

$$\mathcal{E} = \frac{-eg}{2\pi\gamma^2 R^2}\frac{\partial}{\partial\varphi}\int \psi_1(\varphi, \dot{\varphi}, t)\, d\dot{\varphi}, \tag{25}$$

where $\gamma = E/E_0$, and where g is the geometrical factor which depends logarithmically on the ratio of the gap G, to the radius a of the coasting beam, and is given by[2]

$$g = 1 + 2\log_e \frac{2G}{\pi a}. \tag{26}$$

Inserting this into Eq. (24), we obtain

$$\frac{\partial \psi_1}{\partial t} = -\dot{\varphi}\frac{\partial \psi_1}{\partial \varphi} + \frac{4\pi^2 e^2 q}{2\pi \gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s \frac{\partial \psi_0}{\partial \dot{\varphi}} \frac{\partial}{\partial \varphi}\int \psi_1 \, d\dot{\varphi} , \tag{27}$$

which is an integral-differential equation for ψ_1.

We now seek solutions to Eq. (27) of the form

$$\psi_1(\varphi, \dot{\varphi}, t) = \overline{\psi}_1(\dot{\varphi}) e^{i(n\varphi - \omega t)} , \tag{28}$$

where, since Eq. (27) is linear with real coefficients, we may use a complex solution, meaning always either the real or imaginary part. Inserting Eq. (28) into Eq. (27), we obtain

$$\overline{\psi}_1(\dot{\varphi})(n\dot{\varphi} - \omega) = +\frac{2\pi e^2 q}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s \frac{\partial \psi_0}{\partial \dot{\varphi}} n \int \overline{\psi}_1(\dot{\varphi}) \, d\dot{\varphi} . \tag{29}$$

This integral equation for $\overline{\psi}_1(\dot{\varphi})$ may be solved immediately since the dependence of $\overline{\psi}_1(\dot{\varphi})$ on $\dot{\varphi}$ is explicit. Let

$$\overline{\psi}_1(\dot{\varphi}) = \frac{C \frac{\partial \psi_0(\dot{\varphi})}{\partial \dot{\varphi}}}{n\dot{\varphi} - \omega} , \tag{30}$$

where C is a constant, and insert this into Eq. (29) which yields a self-consistency requirement after cancelling C from both sides of the equation, namely:

$$1 = +\frac{2\pi e^2 q}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s n \int \frac{\frac{\partial \psi_0}{\partial \dot{\varphi}}(\dot{\varphi}) \, d\dot{\varphi}}{n\dot{\varphi} - \omega} . \tag{31}$$

This equation is a dispersion relation between the "wavelength" n and the "frequency" ω.

IV. THE DISPERSION RELATION

The dispersion relation (Eq. (31)) is of a very well known form,[10] and the general theory of the relation between the dispersion relation and the solution to Eq. (24) has been the subject of much study.[11] This theory, and its extension to accelerator problems will form the subject of a future MURA report by Dr. K. R. Symon. We can summarize his conclusions (which agree with those of Twiss[11]) by saying that any unstable motion will be characterized by solutions of the dispersion relation having complex values of ω , for real values of n. Thus only real solutions to Eq. (31), or only roots whose complex part is tolerably small suffices to ensure that the motion is stable for practical purposes. The singularity in Eq. (31) is to be defined by distorting the integral along the real axis so that it always lies below the singularity.

We proceed now by considering a number of cases[12] corresponding to different choices for $\psi_0(\dot{\varphi})$.

Case I.

We approximate the nearly monoenergetic coasting beam with

$$\psi_0(\dot{\varphi}) = \begin{cases} \dfrac{N}{2\pi(2\Delta)} & \dot{\varphi}_A - \Delta < \dot{\varphi} < \dot{\varphi}_A + \Delta \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

corresponding to a beam of N particles uniformly spread in "energy" over an interval of width 2Δ about a mean value $\dot{\varphi}_A$. In this case we may readily perform the integral in Eq. (31) since

$$\frac{\partial \psi_0(\dot{\varphi})}{\partial \dot{\varphi}} = \frac{N}{4\pi\Delta}\left\{-\delta(\dot{\varphi}-\dot{\varphi}_A-\Delta)+\delta(\dot{\varphi}-\dot{\varphi}_A+\Delta)\right\}. \tag{33}$$

Equation (31) becomes

$$1 = \frac{2\pi e^2 g}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s n \frac{N}{4\pi\Delta}\left\{\frac{-1}{n(\dot{\varphi}_A+\Delta)-\omega} + \frac{1}{n(+\dot{\varphi}_A-\Delta)-\omega}\right\}$$

or simply, by solving for ω :

$$\omega = n\left\{\dot{\varphi}_A \pm \left[\frac{e^2 g N}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s + \Delta^2\right]^{1/2}\right\} . \tag{34}$$

Thus the perturbation moves with the average speed of the beam, $\dot{\varphi}_A$. If Δ is small, then for real n, ω will be imaginary for $\frac{\partial f}{\partial E} < 0$, i.e. for operation above the transition energy. It should be noted that increasing the spread in beam energy (increasing Δ) is always a stabilizing influence. Recalling that the energy spread in the beam ΔE is related to Δ by

$$\Delta E = \frac{2\Delta}{2\pi\left(\frac{\partial f}{\partial E}\right)_s} , \tag{35}$$

we may write Eq. (34) in the more convenient form:

$$\omega = n\left\{\dot{\varphi}_A \pm \left[\frac{e^2 g N}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s + \pi^2\left(\frac{\partial f}{\partial E}\right)_s^2 (\Delta E)^2\right]^{1/2}\right\} . \tag{36}$$

Thus if the initial disturbance has n waves about the accelerator, the perturbation will grow as $e^{t/T}$ where the time to increase by a factor of e, T is given by:

$$T = \frac{1}{n}\left\{+\frac{e^2 g N}{\gamma^2 R}\left(f\left|\frac{\partial f}{\partial E}\right|\right)_s - \pi^2\left(\frac{\partial f}{\partial E}\right)_s^2 (\Delta E)^2\right\}^{-1/2} , \tag{37}$$

assuming that $\frac{\partial f}{\partial E} < 0$, and the bracket is positive. If the bracket is negative, the motion, of course, is stable. Because of the approximation implied by Eq. (25), Eq. (37) is only valid for $n < \frac{2\pi R}{G}$. The characteristic time T is seen to depend on n, and in fact decreases with increasing n, but it should be noted that the critical energy spread ΔE necessary for stability is independent of n. By setting $\Delta E = 0$ and $n = 1$, Eq. (37) may be seen to reduce to the result obtained in Section II, namely Eq. (18).

Case 2.

We approximate the nearly monoenergetic coasting beam with

$$\psi_0(\dot{\varphi}) = \frac{N}{2\pi^2\left[(\dot{\varphi}-\dot{\varphi}_A)^2 + \Delta^2\right]} , \tag{38}$$

corresponding to N particles spread over an interval characterized by Δ , about a mean value $\dot{\varphi}_A$. As in Case 1 $\frac{\partial \psi_0}{\partial \dot{\varphi}}$ is easy to evaluate, and the integral in the dispersion relation may be readily performed by contour integration (or less readily, directly) to obtain:

$$1 = \frac{2\pi e^2 g}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s \frac{N}{2\pi(\dot{\varphi}_A - \frac{\omega}{n} - i\Delta)^2} . \tag{39}$$

This may be solved for ω directly to obtain:

$$\omega = n\left\{\dot{\varphi}_A - i\Delta \pm \left(\frac{e^2 g N}{\gamma^2 R}\left(f\frac{\partial f}{\partial E}\right)_s\right)^{1/2}\right\} . \tag{40}$$

This is similar to Case 1 in that for $\frac{\partial f}{\partial E} > 0$ (operation below the transition energy) the motion is stable, whereas for $\frac{\partial f}{\partial E} < 0$, the motion is unstable unless the beam energy spread

$$\Delta E = \frac{2\Delta}{2\pi\left(\frac{\partial f}{\partial E}\right)_s} \tag{41}$$

is greater than a critical value. Thus the criterion for stability is

$$\Delta E > \frac{1}{\pi\left(\frac{\partial f}{\partial E}\right)_s}\left[\frac{e^2 g N}{\gamma^2 R}\left(f\,\frac{\partial f}{\partial E}\right)_s\right]^{1/2}, \tag{42}$$

which is exactly the same as is implied by Case 1 (Eq. (37)). Note however that Δ E is here defined in terms of the width at half maximum, whereas for Case 1, ΔE is simply the width.

Case 3.

We approximate the coasting beam with the triangular distribution

$$\psi_0 = \frac{2N}{\pi\Delta^2}\begin{cases} \dot{\varphi} - (\dot{\varphi}_A - \Delta/2) & \dot{\varphi}_A - \Delta/2 < \dot{\varphi} < \dot{\varphi}_A \\ (\dot{\varphi}_A + \Delta/2) - \dot{\varphi} & \dot{\varphi}_A < \dot{\varphi} < \dot{\varphi}_A + \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

corresponding to N particles spread over an interval Δ . Once again $\frac{\partial \psi_0}{\partial \dot{\varphi}}$ may be easily evaluated and the integral in the dispersion relation evaluated to obtain

$$1 = \frac{2\pi e^2 g}{\gamma^2 R}\left(f\,\frac{\partial f}{\partial E}\right)_s \frac{2N}{\pi\Delta^2}\log\frac{\left(\dot{\varphi}_A - \frac{\omega}{n}\right)^2}{\left(\dot{\varphi}_A - \frac{\Delta}{2} - \frac{\omega}{n}\right)\left(\dot{\varphi}_A + \frac{\Delta}{2} - \frac{\omega}{n}\right)} \tag{44}$$

To solve for ω we must examine both the real and imaginary parts of the right-hand side of Eq. (44). One can readily see, using the well known

properties of the logarithm of a complex number, that the imaginary part is zero along the real axis in the ω- plane excluding the interval $\frac{\omega}{n}$ between $\dot{\varphi}_A + \Delta/2$ and $\dot{\varphi}_A - \Delta/2$. It is also zero along the vertical axis in the ω-plane going through the real point $\omega/n = \dot{\varphi}_A$.

For $\frac{\partial f}{\partial E} > 0$ there are no roots to the real part of the equation along the imaginary axis, and two real roots on the real axis corresponding to values of ω/n greater than $\dot{\varphi}_A + \Delta/2$ and smaller than $\dot{\varphi}_A - \Delta/2$. The motion is thus stable.

For $\frac{\partial f}{\partial E} < 0$ there are no real roots, but there are two roots corresponding to $\omega/n = \dot{\varphi}_A \pm Z i$ where

$$1 = \frac{2\pi e^2 g}{\gamma^2 R} \left(f \frac{\partial f}{\partial E} \right)_s \frac{2N}{\pi \Delta^2} \log \frac{Z^2}{Z^2 + \Delta^2/4} \quad . \tag{45}$$

Thus the motion is unstable independent of the size of Δ , as contrasted with Cases 1 and 2. However, for Δ of the order of magnitude encountered in practice, the lapse rate Z is very small.

Case 4.

We approximate $\psi_0(\dot{\varphi})$ by a rectangular distribution (as in Case 1) plus a small triangular distribution (as in Case 3) on top of it. The result obtained here is:

1. An instability as in Eq. (45) where N is replaced by the number of particles in the triangular distribution. This is thus less serious than in Case 3.

2. A criterion for stability which is essentially the same as Case 1, such that if the criterion is not satisfied, the motion is quite unstable just as in Case 1. Now, however, if the criterion is satisfied, the motion is not completely stable but rather it is slightly unstable. The instability is very weak, however, and can be ignored.

V. NUMERICAL EXAMPLES

We wish to conclude from the cases studied in Section IV, that Cases 1 or 2 suffice to provide a criterion for stability, namely Eq. (42). It should be emphasized that we have not shown that this is a general consequence of the dispersion relation, although we believe this to be true.[13]

Consider then, two numerical examples:

1. The proposal accelerator.[14]

$R = 10^4$ cm. $\qquad N = 800$ amperes

$G = 15$ cm. $\qquad n = 1$

$a = 2$ cm. $\qquad \gamma = 15$

$$\left(f \frac{\partial f}{\partial E}\right)_s = 30 \times 10^{-6} \frac{Mc^2}{sec^2 - Bev} \qquad \Delta E = 300 \text{ Mev}$$

2. The 40 Mev two-way model.[15]

$R = 208$ cm. $\qquad N = 1.5 \times 10^{13}$

$G = 5$ cm. $\qquad n = 1$

$a = 1.5$ cm. $\qquad \gamma = 80$

$$\left(f \frac{\partial f}{\partial E}\right)_s = 1.1 \times 10^{18} \text{ sec}^{-2} \text{ erg}^{-1} \qquad \Delta E = 3.0 \text{ Mev}$$

For the first example, with $\Delta E = 0$, the characteristic time T (Eq. (37) or (18)) is of the order of 8 seconds, while the criterion for

stability (Eq. (42)) demands $\Delta E > 5$ Mev. Thus the normal stacked beam with an energy spread of 300 Mev easily satisfies this criterion.

For the second example, with $\Delta E = 0$, the characteristic time T is only 370 microseconds but the criterion for stability (Eq. (42)) demands that ΔE be only greater than 10 kilovolts. Once again, the normal beam with an energy spread of 3 Mev will be stable.

Finally we calculate the rate of instability associated with a triangular distribution, namely Eq. (45), and find an extremely small number so that there need be no concern from this instability.

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FOOTNOTES

[1]D. W. Kerst, Phys. Rev. 60, 47 (1941).

[2]C. E. Nielsen and A. M. Sessler, MURA-413 (Rev. of Sci. Inst., Feb. 1959).

[3]G. J. Budker, Proceedings of the CERN Symposium 1956, Vol. 1, 68.

J. Enoch, MURA-311

As well as many other papers both published and unpublished.

[4]O. Buneman, Phys. Rev. Letters 1, 8 (1958).

[5]The mathematical argument of this section is here presented in a form due to Dr. R. L. Mills. We wish to thank him for allowing us to use his formulation.

[6]Actually, of course, Eq. (19) is only valid when φ and $\dot{\varphi}$ are canonical coordinates and momenta. However, the variable conjugate to φ, namely W (see reference 8) is simply proportional to $\dot{\varphi}$ so that as in most applications of the Boltzmann equation, we may use coordinates and velocities rather than coordinates and momenta.

[7]R. L. Mills and A. M. Sessler, MURA-433. This is equivalent to ignoring short-range collisions, and only considering the interactions between particles represented by average electric fields. See MURA-433 for a more extended discussion of the range of validity of this approximation.

[8]K. R. Symon and A. M. Sessler, Proceedings of the CERN Symposium 1956, Vol. 1, 44. (MURA-106).

[9]Equation (25) may most easily be obtained by transforming to a frame of reference rotating with the average velocity of the particles of ψ_0. Note that in this transformation $\mathcal{E}$ is tangential and hence unaltered. In this

rotating frame of reference we may use the methods of reference 2 if the electric field associated with ψ_1 satisfies the boundary condition of being purely normal at conductors. This is not true for the electric field associated with ψ_0, in this rotating frame, but ψ_0 clearly does not contribute to $\mathcal{E}$. It is true for the electric field associated with ψ_1 as can be seen by the following argument: In the laboratory frame ψ_1 has A. C. components so that if the conducting walls are of high conductivity, the magnetic field due to ψ_1 is purely tangential at the conducting walls. Consequently in the rotating frame of reference, the pure electric field associated with ψ_1 has normal components only. The authors wish to thank Dr. L. Jackson Laslett for discussions leading to this footnote.

[10]A. Vlasov, Journ. of Phys. 9, 25 (1945).

[11]L. Landau, Journ. of Phys. 10, 25 (1946).

N. G. Van Kampen, Physica 21, 949 (1955).

N. G. Van Kampen, Physica 23, 641 (1957).

F. Berz, Proc. Phys. Soc. (London) B69, 939 (1956).

R. Q. Twiss, Phys. Rev. 88, 1392 (1952).

[12]The authors wish to thank Dr. Symon for permission to use, in this section, some of his results.

[13]Actually the difference between the various cases studied in Section IV can be understood rather more completely than is indicated here. This will be discussed in a future report by Dr. Symon, while Twiss(reference 11) has many illuminating comments.

[14]MURA Proposal for a High Energy Accelerator, 1958.

[15]F. T. Cole, E. Day, R. O. Haxby and A. M. Sessler, MURA-375.