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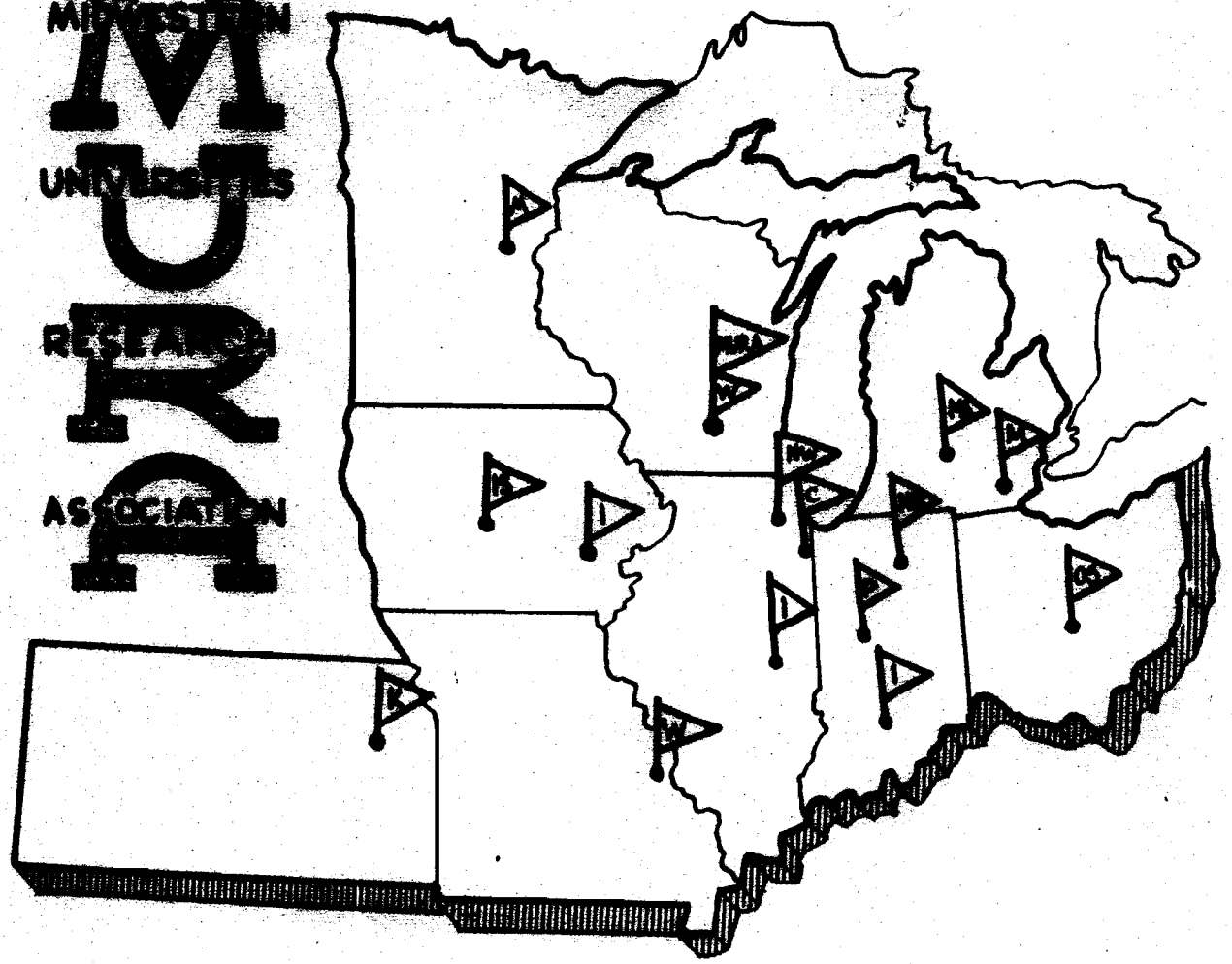

ON HELMHOLTZ'S THEOREM IN FINITE REGIONS

J. Van Bladel

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2203 University Avenue, Madison, Wisconsin

ON HELMHOLTZ'S THEOREM IN FINITE REGIONS

J. Van Bladel
Electrical Engineering Department, University of Wisconsin
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ABSTRACT

The splitting of a continuous vector function into a gradient plus a curl is possible in an infinite number of ways. The present paper is basically concerned with two particular uniquely determined splittings called electric and magnetic. Eigenvector expansions suitable for these splittings are examined, and the application of these expansions to the calculation of the various terms of the splitting is considered.

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I. INTRODUCTION

A certain number of papers[1, 2, 3] have recently been published on the possibility of representing a solenoidal vector by the curl of a vector potential in multiply-bounded and multiply-connected regions, and on the related problem of finding a complete set of eigenvectors for such regions. The present work endeavors to examine these problems in the light of Helmholtz's theorem, i.e. of the splitting of a vector into a curl and a gradient. It culminates in a Table which, notwithstanding its seemingly trivial character, is thought to be of potential interest to physicists. The mathematical background to the present study was provided by Weyl shortly after the turn of the century.

Let $\bar{a}$ be a piecewise continuous vector function. It is sought to investigate whether, and how, $\bar{a}$ can be split into

$$\bar{a} = \text{grad}\ \varphi + \text{curl}\ \bar{w} \tag{1}$$

in a finite regular region V, which can be either simply-bounded and simply-connected as in Fig. 1a, simply-connected and multiply-bounded as in Fig. 1b (and for the sake of clarity doubly-bounded regions only will be considered), or simply-bounded and multiply-connected as in Fig. 1c. There again doubly-connected regions, of the type encountered in toroidal particle accelerators,

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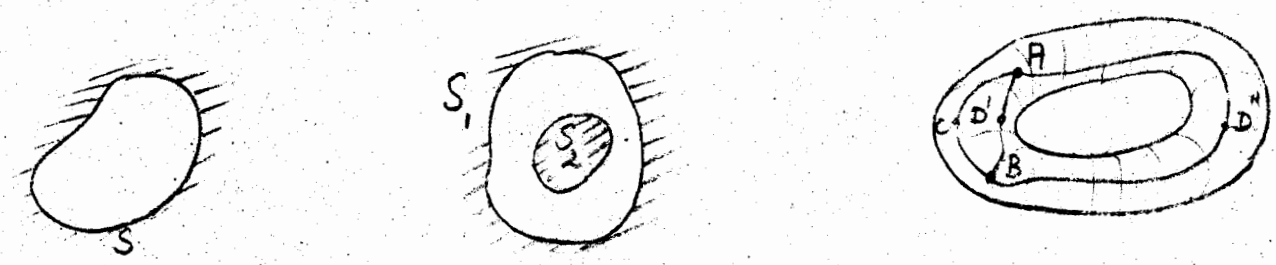
[1] T. Teichmann and E. P. Wigner. "Electromagnetic Field Expansion in Loss-Free Cavities Excited Through Holes." J. Appl. Phys. 24, 262-267, 1953.

[2] A. F. Stevenson. "Note on the Existence and Determination of a Vector Potential." Quart. Appl. Math. XII, 194-198, 1954.

[3] K. Kurokawa. "The Expansions of Electromagnetic Fields in Cavities." IRE Trans. Vol. MTT-6, 178-187, 1958.



Region I (a) Region II (b) Region III (c)

Figure 1

will be examined at the exclusion of more complicated structures. Generalizations are obvious. The gradient and curl terms of the splitting will be called "longitudinal" and "transverse" terms respectively.

First of all, it is apparent that the problem can be solved in at least one way, for the equation

$$\nabla^2 \bar{u} = \text{grad div } \bar{u} - \text{curl curl } \bar{u} = \bar{a} \tag{2}$$

admits at least one solution in the regions considered above. As a matter of fact, it admits an infinity of solutions, and one suspects right away that the splitting must be possible in an infinite number of fashions. This is confirmed by the remark that vectors do exist which can be written either as a gradient or a curl. Addition of such a vector to the longitudinal term, and subtraction from the transverse term, yields a new splitting. It will consequently be possible to impose certain conditions on Ω or $\bar{W}$ (for example that $\bar{W}$ be perpendicular to the boundary surface) to restrict the generality of the splitting, and make it unambiguous. The "electric," "magnetic," and "mixed" splittings, to be discussed hereafter, result from well-chosen additional conditions and

are of particular interest for solenoidal and irrotational vectors. Their use is fruitful in the analysis of electromagnetic fields in finite regions. Other splittings,[4] not to be considered here, are of interest for other branches of physics.

II. THE ELECTRIC SPLITTING

A. Preliminary Considerations on Irrotational Vectors ($\text{curl}\,\bar{a} = 0$).

An application of Stoke's theorem shows that, in Regions I and II, $\int_A^B \bar{a} \cdot d\bar{l}$ is independent of the path joining A to B provided this path is entirely contained in the region throughout which $\bar{a}$ is irrotational. It is consequently possible to express $\bar{a}$ as grad Ω, where Ω is defined by the relation $\Omega(B) = \Omega(A) + \int_A^B \bar{a} \cdot d\bar{l}$. Things are somewhat different in Region III. The contour integral of $\bar{a}$ still vanishes along a curve such as A C B D', that does not "encircle" the hole (i. e. that can be reduced to a point without leaving Region III), but takes, in general, a non-zero value T for a contour such as A C B D'' that encircles the hole once. This value T, called the period, is independent of the encircling contour. There now exists a multivalued function Ω such that $\bar{a} = \text{grad}\,\Omega$. The values of this function at any given point differ by an integral number of times T.

B. Existence and Uniqueness.

The electric splitting is generated by the solution of the problem

$$\nabla^2 \bar{f} = \text{grad div}\,\bar{f} - \text{curl curl}\,\bar{f} = \bar{a} \quad \text{with} \quad \left.\begin{array}{l} \text{div}\,\bar{f} = 0 \\ \bar{u}_n \times \bar{f} = 0 \end{array}\right\} \text{on } S \tag{3}$$

[4] H. Weyl. "Das Asymptotische Verteilungsgesetz der Eigenschwingungen Eines Beliebig Gestalteten Elastischen Korpers." Rendic. Circ. Mat. Palermo, 39, 1-50. 1915.

where $\bar{u}_n$ is the unit vector directed along the outside normal to S. The uniqueness of the solution should be investigated first. If there were two different solutions $\bar{f}_1$ and $\bar{f}_2$, their difference $\bar{f}$ would satisfy $\nabla^2 \bar{f} = 0$. Application of the formula

$$\iiint_V \left[\bar{v}.\nabla^2\bar{w} + \operatorname{div}\bar{v}.\operatorname{div}\bar{w} + \operatorname{curl}\bar{v}.\operatorname{curl}\bar{w}\right] dV = \iint_S \left[(\bar{u}_n.\bar{v})\operatorname{div}\bar{w} + (\bar{v}\times\operatorname{curl}\bar{w}).\bar{u}_n\right] dS \quad (4)$$

to $\bar{v} = \bar{w} = \bar{f}$ shows that $\operatorname{div}\bar{f} = \operatorname{curl}\bar{f} = 0$ throughout the region. Such a vector will be called "source-free." It can be represented as grad φ, where φ is a harmonic function which, because of the perpendicularity of $\bar{f}$ to S, takes a constant value along each of the closed surfaces of which the boundary consists. In simply-connected regions, φ is obviously single-valued. In Region III, φ is single-valued too, because the period T is equal to zero. This is easily shown by particularizing the contour over which $\int \bar{f}.d\bar{l}$ is evaluated to lie on the surface S, where $\bar{f}$ is perpendicular to $d\bar{l}$. It is now possible to show that $\bar{f}$ must be zero in Regions I and III, by applying the relation

$$\iiint_V \left[A\nabla^2 B + \operatorname{grad} A.\operatorname{grad} B\right] dV = \iint_S A\frac{\partial B}{\partial n}\, dS \quad (5)$$

valid for single-valued functions, to $A = B = \varphi - \varphi_s$, where φ_s is the constant value which φ takes on the boundary surface. In Region II however, there are non-zero solutions for $\bar{f}$, all multiples of each other, and derived from potentials φ which take different constant values on S_1 and S_2. Let $\bar{f}_0 = \operatorname{grad}\varphi_0$ be the "normalized" solution, determined by the condition

that $\iiint_V |\bar{f}_0|^2 dV = 1$. The following statement can now be made: If $\bar{f}$ exists, it is unique, except in Region II where it is determined with the exception of a multiple of $\bar{f}_0$. In the latter region, the various determinations lead to identical longitudinal and transverse terms, because $\operatorname{div} \bar{f}_0 = 0$ and $\operatorname{curl} \bar{f}_0 = 0$. One concludes that there is at most one way to split a vector electrically.

For a proof of the <u>existence</u> of a solution, one has to turn to a paper by Weyl,[5] where the existence of a Green's dyadic $\mathcal{G}_e(\bar{r}, \bar{r}')$, such that

$$\bar{f}(\bar{r}) = \iiint_V \mathcal{G}_e(\bar{r}, \bar{r}') \cdot \bar{a}(\bar{r}')\, dV' \tag{6}$$

is established for a simply-bounded region. This relation implies that the Laplacian of the second member, taken with respect to the $\bar{r}$ coordinates, reproduces $\bar{a}(\bar{r})$. In other words, the solution of (6), considered as an integral equation in $\bar{a}$ with given second member $\bar{f}$, is $\bar{a} = \nabla^2 \bar{f}$, provided $\bar{f}$ does not have any tangential component on S, and possesses a piecewise continuous Laplacian. In particular, the homogeneous integral equation derived from (6) by setting the second member $\bar{f}$ equal to zero, has $\bar{a} = 0$ as its only solution. This important property will be used later in the argument.

In the course of proving the existence of $\mathcal{G}_e$, Weyl establishes the following properties of the dyadic,

(a) it is symmetrical, i.e. $\mathcal{G}_{ik}(\bar{r}, \bar{r}') = \mathcal{G}_{ki}(\bar{r}', \bar{r})$

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[5] H. Weyl. "Uber die Randwertaufgabe der Strahlungstheorie und Asymptotische Spektralgesetze." Journ. Reine. Angew. Math., <u>143</u>, 177-202, 1913.

(b) it behaves, for $\bar{r} \to \bar{r}'$, in such a manner that

$$\lim_{\bar{r} \to \bar{r}'} \mathcal{G}_{ik}(\bar{r}, \bar{r}') \leq \frac{\text{constant}}{|\bar{r} - \bar{r}'|}$$

When the region is multiply-bounded, no unique solution $\bar{f}$ exists and, consequently, no Green's dyadic in the sense of Eq. (6). There is a way out of this difficulty, however, if one decides to split $\bar{f}$ and $\bar{a}$ into two components:

$$\bar{f} = \bar{f}_0 \iiint_V \bar{f} \cdot \bar{f}_0 \, dV + \bar{f}_c$$

$$\bar{a} = \bar{f}_0 \iiint_V \bar{a} \cdot \bar{f}_0 \, dV + \bar{a}_c$$

All the possible solutions $\bar{f}$ have the same "core" part $\bar{f}_c$, which is, consequently, uniquely determined. The two parts of the splitting can be thought of as the functional projections of $\bar{f}$ (and $\bar{a}$) on $\bar{f}_0$, and on the subspace $\bar{f}_c$ orthogonal to $\bar{f}_0$ respectively. Notice that both projections are orthogonal to each other, in the sense that $\iiint \bar{f}_c \cdot \bar{f}_0 \, dV$

The problem

$$\nabla^2 \bar{f}_c = \bar{a}_c \tag{7}$$

can be proved to have a unique solution, which Weyl shows to be obtainable in the form

$$\bar{f}_c(\bar{r}) = \iiint \mathcal{G}_c'(\bar{r}, \bar{r}') \cdot \bar{a}_c(\bar{r}') \, dV' \tag{8}$$

where $\mathcal{G}_c'$ is an electric Green's dyadic in the extended sense. This dyadic

has the same symmetry property, and the same behavior for $\bar{r} \to \infty$ as the usual Green's dyadic $\mathcal{G}_e$. It now appears that the electric splitting is always possible in the form

$$\bar{a} = \bar{f}_0 \iiint_V \bar{a}\, f\, dV + \text{grad div}\, \bar{f} - \text{curl curl}\, \bar{f} \tag{9}$$

where the term in $\bar{f}_0$ exists in Region II only, and where the three parts of the splitting are uniquely determined. The name "longitudinal" will henceforth be reserved for the second term grad div $\bar{f}$.

C. Further Properties. Application to Solenoidal and Irrotational Vectors.

1. The longitudinal term can be obtained by solving a potential problem. Taking the divergence of both terms of (9) yields, indeed:*

$$\nabla^2 (\text{div}\, \bar{f}) = \text{div}\, \bar{a} \qquad \text{with} \quad \text{div}\, \bar{f} = 0 \quad \text{on } S$$

The longitudinal term turns out to be the electric field created in region V by a charge density $\varepsilon_0\, \text{div}\, \bar{a}$, the walls being metallized and grounded.

2. The three terms of the splitting are functionally orthogonal to each other. The orthogonality of $\bar{f}_0$ and grad div $\bar{f}$ can be established by applying the equation

$$\iiint_V \bar{b} \cdot \text{grad}\, \varphi\, dV = \iint_S \varphi\, (\bar{u}_n \cdot \bar{b})\, dS - \iiint_V \varphi\, \text{div}\, \bar{b}\, dV \tag{10}$$

to $\bar{b} = \bar{f}_0$, $\varphi = \text{div}\, \bar{v}$.

The orthogonality of $\bar{f}_0$ and grad div $\bar{f}$ to - curl curl $\bar{f}$ results from applying

$$\iiint_V \bar{b} \cdot \text{curl}\, \bar{a}\, dV = \iint_S \bar{u}_n \cdot (\bar{a} \times \bar{b})\, dS + \iiint_V \bar{a} \cdot \text{curl}\, \bar{b}\, dV \tag{11}$$

*It is assumed that $\bar{a}$ has piecewise continuous first derivatives.

to $\bar{a} = -\operatorname{curl} \bar{f}$ and to $\bar{b} = \bar{f}_0$ and grad $\operatorname{div} \bar{f}$ in succession.

3. When $\bar{a}$ is solenoidal, i.e. $\operatorname{div} \bar{a} = 0$, the longitudinal term vanishes and the splitting takes the form

$$\bar{a} = \bar{f}_0 \iiint \bar{a} \cdot \bar{f}_0 \, dV - \operatorname{curl} \operatorname{curl} \bar{f} \tag{12}$$

The solenoidal vector $\bar{a}$ cannot, in general, be derived from a vector potential in a multiply-bounded region of Type II. To justify this assertion, consider first the fact that $\bar{f}_0$ cannot be derived from a vector potential. If, indeed, $\bar{f}_0$ were equal to curl $\bar{M}$, an application of the formula

$$\iiint_V \left[\operatorname{curl} \bar{a} \cdot \operatorname{curl} \bar{b} - \bar{a} \cdot \operatorname{curl} \operatorname{curl} \bar{b}\right] dV = \iint_{S_1+S_2} \bar{u}_n \cdot (\bar{a} \times \operatorname{curl} \bar{b}) \, dS \tag{13}$$

to $\bar{a} = \bar{b} = \bar{M}$ would yield $\iiint_V |\operatorname{curl} \bar{M}|^2 dV = 0$ i.e. $\bar{f}_0$ equal to zero, an obvious contradiction. Clearly then, $\bar{a}$ derives from a vector potential if, and only if, its functional projection on $\bar{f}_0$ is zero, i.e. $\iiint_V \bar{a} \cdot \bar{f}_0 \, dV$ vanishes. But this implies that

$$\iiint_V \bar{f}_0 \cdot \bar{a} \, dV = \iiint_V \operatorname{grad} \varphi_0 \cdot \bar{a} \, dV = \iint_{S_1+S_2} \varphi_0 (\bar{a} \cdot \bar{u}_n) \, dS - \iiint_V \varphi_0 \operatorname{div} \bar{a} \, dV \tag{14}$$

$$= \varphi_{01} \iint_{S_1} (\bar{a} \cdot \bar{u}_{n1}) \, dS_1 + \varphi_{02} \iint_{S_2} (\bar{a} \cdot \bar{u}_{n2}) \, dS_2 = 0$$

where φ_{01} and φ_{02} are the two (unequal) values of φ_0 on S_1 and S_2. On the other hand, an application of Gauss' theorem to the solenoidal vector $\bar{a}$ yields

$$\iiint_V \operatorname{div} \bar{a} \, dV = 0 = \iint_{S_1} (\bar{a} \cdot \bar{u}_{n1}) \, dS_1 + \iint_{S_2} (\bar{a} \cdot \bar{u}_{n2}) \, dS_2 \tag{15}$$

A comparison of Eqs. (14) and (15) shows that a solenoidal vector in a doubly-bounded region derives from a vector potential if, and only if, its flux through each of the two boundary surfaces is equal to zero. This result has previously been obtained by Stevenson[2] by somewhat different methods.

4. When $\bar{a}$ is irrotational, and possesses a tangential component along S, all terms of the electric splitting are present. Both $\bar{f}_0$ and the longitudinal part being perpendicular to S, it is the transverse term which will be responsible for the tangential component. The main function of the transverse term is, so to speak, to account for that component, because - curl curl $\bar{f}$ vanishes when $\bar{a}$ is perpendicular to S. This can be proved by applying (13) to $\bar{a} = \bar{b} = \text{curl}\,\bar{f}$

$$\iiint_V [\text{curl curl}\,\bar{f}]^2 dV = \iiint_V \text{curl}\,\bar{f} \cdot \text{curl curl curl}\,\bar{f}\, dV + \iint_S \bar{u}_n \cdot [\text{curl}\,\bar{f} \times \text{curl curl}\,\bar{f}]\, dS$$

The second member vanishes because curl curl curl $\bar{f}$ = - curl $\bar{a}$, is equal to zero, by hypothesis, and because curl curl $\bar{f}$ is perpendicular to S. This implies the vanishing of the first member, and of curl curl $\bar{f}$ in consequence.

The irrotational character of $\bar{a}$ allows representation of that vector as the gradient of a scalar potential. A look at Eq. (9) shows that the transverse term can, consequently, also be derived from a scalar potential and, furthermore, one that is harmonic throughout V . A consequence of this fact, which rests on properties to be enunciated later, is that the transverse term cannot be purely tangential in a simply-connected region. It then contributes part of the normal component of $\bar{a}$ at the surface, the other part being furnished by the $\bar{f}_0$ and longitudinal terms.

5. The electric splitting can be put in the form

$$\bar{a} = \bar{f}_0 \iiint \bar{a} \cdot \bar{f}_0 \, dV + \operatorname{grad} \varphi + \operatorname{curl} \bar{W} \tag{16}$$

where φ vanishes on the boundary. Conversely, if a splitting such as (16) has been achieved, it must be the electric splitting. Taking the divergence of both terms shows, indeed, that $\nabla^2 \varphi = \operatorname{div} \bar{a}$. This equation, together with the condition for φ at the boundary, is also satisfied by $\operatorname{div} \bar{f}$. The solution of this potential problem being unique, φ <u>must</u> be $\operatorname{div} \bar{f}$, and the splitting (16) is unique. Notice that $\bar{W}$ is a tangential vector, being the curl of $\bar{f}$, a vector perpendicular to the boundary.

D. Eigenvector Expansions Suitable for the Electric Splitting.

In the search for a complete set of eigenvectors, some fundamental properties of integral operators will first be reviewed.[6] Let $G(x,x')$ be a symmetrical kernel (i.e. such that $G(x,x') = G(x',x)$), and let G be square integrable in the plane domain $a \leq x \leq b$, $a \leq x \leq b$. Then there exists a sequence of orthogonal eigenfunctions $\varphi_m(x)$, and corresponding non-zero eigenvalues μ_m, such that

$$\int_a^b G(x,x') \varphi_m(x') \, dx' + \mu_m \varphi_m(x) = 0$$

where the φ_m belong to the space of square integrable functions. Any square integrable function ψ in (a,b) admits the expansion

$$\psi(x) = \sum_m \varphi_m(x) \int_a^b \varphi_m(x') \psi(x') \, dx' + \theta(x)$$

[6] F. Riesz et B. S. Nagy. "Lecons d' Analyse Fonctionnelle." 3d Edition, p. 239. Gauthier-Villars, Paris, 1955.

where the convergence is in the mean, and the rest term $\theta(x)$ is a solution of

$$\int_a^b G(x,x')\,\theta(x')\,dx' = 0$$

When the latter integral equation admits zero as its only solution, no rest term is necessary, and the set of the φ_m is complete.

The same sort of expansion theorem can be established[7, 8] in three-dimensional space, where the elements φ_m and ψ are replaced by square integrable three-dimensional vectors, and $G(x,x')$ is replaced by a symmetric dyadic $\mathcal{G}$ (i.e. such that $\mathcal{G}_{xy}(\bar{r},\bar{r}') = \mathcal{G}_{yx}(\bar{r}',\bar{r})$) having the property that $\iiint_V [\bar{a}(\bar{r})\cdot\mathcal{G}(\bar{r},\bar{r}')]^2\,dV'$ is a finite and continuous function of $\bar{r}$ whenever $\bar{a}$ is continuous. The two dyadics $\mathcal{G}_c$ and $\mathcal{G}_c'$ have sufficiently mild discontinuities at $\bar{r}=\bar{r}'$ to satisfy these requirements. Consequently, in Regions I and III, the eigenvectors $\bar{F}_m$ of

$$\iiint_V \mathcal{G}_c(\bar{r},\bar{r}')\cdot\bar{F}_m(\bar{r}')\,dV' + \mu_m\,\bar{F}_m(\bar{r}) = 0 \tag{17}$$

form a complete set. The homogeneous integral equation (6) has indeed been shown to have zero as its only solution. It will be noticed, by taking the Laplacian of both sides, that

$$\bar{F}_m(\bar{r}) + \mu_m \nabla^2 \bar{F}_m(\bar{r}) = 0$$

On the other hand, the nature of $\mathcal{G}_c$ is such that the vector represented by the integral in (17) satisfies $\bar{n}\times\bar{F}_m = \operatorname{div}\bar{F}_m = 0$ on S. The $\bar{F}_m$ are

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[7] C. E. Weatherburn. "Vector Integral Equations of the First Kind." Quart. J. Pure Appl. Math. 46, 334-356, 1915.

[8] C. E. Weatherburn. "Vector Integral Equations and Gibbs' Dyadics." Trans. Cambr. Philos. Soc., Vol. XXII, No. VIII, 133-158, 1916.

consequently also the eigenvectors of the differential operator ∇^2 corresponding to these boundary conditions, the eigenvalues being now $\lambda_m = \frac{1}{\mu_m}$.

It is shown in Appendix I that these eigenvectors fall into two categories:

(a) solenoidal eigenvectors $\bar{e}_m$, solutions of

$$-\operatorname{curl}\operatorname{curl}\bar{e}_m + \lambda'_m \bar{e}_m = 0 \qquad \bar{n}\times\bar{e}_m = 0 \quad \text{on } S \tag{18}$$

(b) irrotational eigenvectors $\bar{f}_m = \operatorname{grad}\varphi_m$, obtained from the eigenfunctions φ_m of

$$\nabla^2\varphi_m + \lambda''_m \varphi_m = 0 \tag{19}$$

These eigenvectors are all orthogonal to each other, and it will be assumed that they have been normalized.

In Region II, similarly, the eigenvectors $\bar{F}'_m$ of $\iiint_V g'_e(\bar{r},\bar{r}')\,dV'$ form a complete set with respect to the subspace of vectors that are orthogonal to $\bar{f}_0$. This restriction is made necessary by the fact that $\nabla^2 \iiint_V g'_e(\bar{r},\bar{r}')\cdot\bar{a}(\bar{r})\,dV'$ is equal to $\bar{a}_c(\bar{r})$. In consequence, if the integral $\iiint_V g'_e(\bar{r},\bar{r}')\cdot\bar{a}(\bar{r})\,dV'$ vanishes for all $\bar{r}$, we can conclude that the "core" part $\bar{a}_c$ of $\bar{a}$ vanishes, but not that $\bar{a}$ itself is equal to zero. In fact, any multiple of $\bar{f}_0$ is a solution of the homogeneous integral equation. It then follows that the "core" part of any square integrable vector can be expanded in the $\bar{F}'_m$. In formula:

$$\bar{a}(\bar{r}) = \bar{f}_0(\bar{r})\iiint_V \bar{a}(\bar{r}')\cdot\bar{f}_0(\bar{r}')\,dV' + \sum_m \bar{F}'_m(\bar{r})\iiint_V \overline{F'_m}(\bar{r}')\cdot\bar{a}(\bar{r}')\,dV' \tag{20}$$

It is found, by arguments similar to those used for $\bar{F}_m$, that the $\bar{F}'_m$ belong to the subspace of vectors orthogonal to $\bar{f}_0$, and that they are the eigenvectors of ∇^2 belonging to eigenvalues different from zero. It is a simple matter, indeed, to establish the orthogonality of the latter eigenvectors to $\bar{f}_0$ by applying the formula

$$\iiint_V \left[\bar{a}\cdot\nabla^2\bar{b} - \bar{b}\cdot\nabla^2\bar{a}\right] dV = \iint_S \left[(\bar{a}\cdot\bar{u}_n)\,\mathrm{div}\,\bar{b} - (\bar{b}\cdot\bar{u}_n)\,\mathrm{div}\,\bar{a} + (\bar{u}_n\times\bar{a})\cdot\mathrm{curl}\,\bar{b} - (\bar{u}_n\times\bar{b})\cdot\mathrm{curl}\,\bar{a}\right] dS \qquad (21)$$

to $\bar{a} = \bar{f}_0$, $\bar{b} = \bar{F}'_m$. It is apparent from (20), by the way, that $\bar{f}_0$ can be included in the set of eigenvectors to make it complete with respect to the space of square integrable vectors. The $\bar{F}'_m$ can also be subdivided into categories (18) and (19) already introduced for simply-bounded regions. It is now possible to expand each term of the electric splitting (16) in the $\bar{f}_0$, the $\bar{e}_m$ and $\bar{f}_m$. The coefficients of the expansion can be found by repeated use of formulas (10), (11), and (13). The resulting formula,* of great importance for practical applications, is:

$$\bar{a}(\bar{r}) = \underbrace{\bar{f}_0\left[\iint_{S+S_2}(\bar{u}_n\cdot\bar{a})\,\varphi_0\,dS - \iiint_V \varphi_0\,\mathrm{div}\,\bar{a}\,dV\right]}_{\text{Term present in Region II only}} + \underbrace{\sum_m \bar{f}_m\left[-\iiint_V \varphi_m\,\mathrm{div}\,\bar{a}\,dV\right]}_{\text{grad div } \bar{f}} + \underbrace{\sum_m \frac{\bar{e}_m}{\lambda'_m}\left[\iiint_V \mathrm{curl}\,\bar{e}_m\cdot\mathrm{curl}\,\bar{a}\,dV - \iint_S (\bar{u}_n\times\bar{a})\cdot\mathrm{curl}\,\bar{e}_m\,dS\right]}_{-\mathrm{curl}\,\mathrm{curl}\,\bar{f}} \qquad (22)$$

All the results relative to solenoidal and irrotational vectors can be obtained directly from this formula.

A short additional remark will be of interest here. Formula (22) shows that, in a region of Type I, a sourceless vector is expandable in the solenoidal

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* This formula is valid only when the first order partial derivatives of $\bar{a}$ are sufficiently regular for Gauss' and Stokes' theorems to be applicable.

vectors $\bar{e}_m$, but does not have any functional projection on the $\bar{f}_m = \text{grad}\,\varphi_m$. On the other hand, a sourceless vector can certainly be represented as the gradient of a scalar potential φ. The latter can be expanded in the complete set φ_m, and one obtains a representation of $\bar{a}$ as $\text{grad}\left[c_1\varphi_1 + \cdots + c_n\varphi_n + \cdots\right]$. Here then is an interesting example of a situation where the gradient of a sum is not the sum of the gradients, i.e. $\bar{a}$ is not $c_1\,\text{grad}\,\varphi_1 + \cdots + c_n\,\text{grad}\,\varphi_n + \cdots$ throughout the region, a fact which is due to the non-continuous character of differential operators, and which parallels the well-known property that a Fourier series cannot, in general, be differentiated term by term. To convince himself of the validity of these statements, the reader might want to direct his attention to Appendix II, where a particular sourceless vector is expanded in terms of the $\bar{e}_m$, and where it is checked that the series, indeed, converges to the original vector. The convergence is in the mean, implying that the series converges everywhere but in a set of zero measure, which in this case is made out of the boundary points. Notice, indeed, that all $\bar{e}_m$, and consequently their sum, are perpendicular to the boundary, while the original vector is not. As a result, the expansion converges "uniformly" to the normal component of a, but the convergence to the tangential component presents a Gibbs phenomenon, as suggested by Fig. 2, and of a nature similar to the better known Gibbs phenomenon associated with the convergence of the series $\sum_n a_n \sin nx$ around $x = 0$

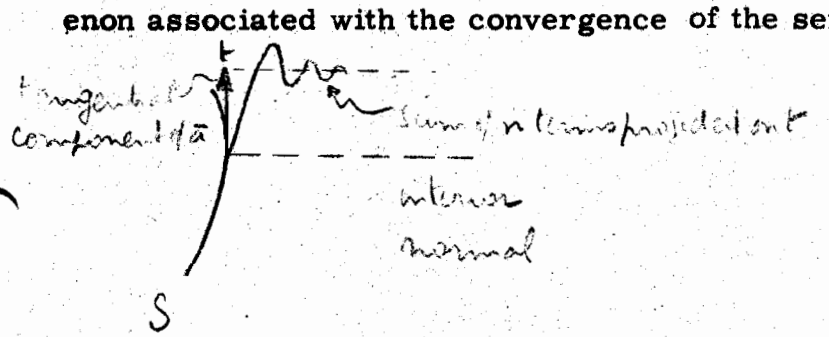


Figure 2

III. THE MAGNETIC SPLITTING

A. Existence and Uniqueness.

The magnetic splitting is generated by the solution of the problem

$$\nabla^2 \bar{h} = \text{grad div}\, \bar{h} - \text{curl curl}\, \bar{h} = \bar{a} \quad \text{with} \quad \left.\begin{array}{l} \bar{u}_n \cdot \bar{h} = 0 \\ \bar{u}_n \times \text{curl}\, \bar{h} = 0 \end{array}\right\} \text{on } S \tag{23}$$

The investigation of the solution of this problem proceeds along lines which parallel those used for the electric splitting. The uniqueness is proved by examining the solutions of the homogeneous problem $\nabla^2 \bar{h}' = 0$. An application of (4) shows these solutions to be sourceless vectors. They can consequently be put in the form $\bar{h}' = \text{grad}\, \psi$, with ψ harmonic, and $\bar{h}'$ tangent to S (i.e. $\frac{\partial \psi}{\partial n} = 0$ on S). An application of (5) shows that no such vector can exist in Regions I and II. This implies that the only solution to $\nabla^2 \bar{h}' = 0$ is $\bar{h}' = 0$ or, in other words, that an eventual solution of (23) must be unique. In Region III, however, there is an infinite number of non-zero solutions ψ, all multiples of each other. In a region bounded by a surface of revolution, for instance, ψ is a multiple of θ + constant, and $\bar{h}'$ is a multiple of $\frac{\bar{u}_\theta}{r}$. Let $\bar{h}_0 = \text{grad}\, \psi_0$ be the normalized solution. The following statement can now be made: if $\bar{h}$ exists, it is unique, except in Region III where it is determined with the exception of a multiple of $\bar{h}_0$. In the latter region, the various determinations lead to identical longitudinal and transverse terms. Notice

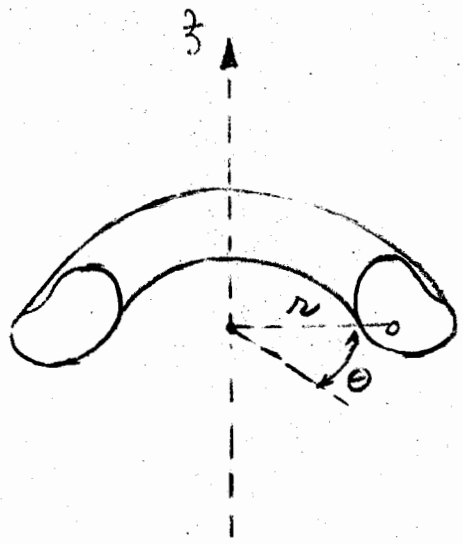


Figure 3

that $\bar{h}_o$, which derives from a multivalued scalar potential ψ_o, can also be derived from a vector potential. An application of the electric splitting to $\bar{h}_o$ shows, indeed, that there is a vector $\bar{f}$ such that $\bar{h}_o = -\text{curl curl}\,\bar{f}$, where $\bar{f}$ is perpendicular to S. In consequence, $\bar{h}_o$ is the curl of a vector tangent to S.

The existence of a solution follows, again, from results obtained by Weyl,[4, 5, 9] who indicates that, in a simply-connected region, a dyadic exists such that

$$\bar{h}(\bar{r}) = \iiint_V \mathcal{G}_H(\bar{r},\bar{r}') \cdot \bar{a}(\bar{r}')\, dV' \tag{24}$$

This dyadic has the symmetry property, and the behavior for $\bar{r} \to \bar{r}'$, which were associated with its electric counterpart. Also, the homogeneous integral equation possesses $\bar{a} = 0$ as its only solution. In a doubly-connected region, a magnetic Green's dyadic in the extended sense can be introduced, such that

$$\bar{h}_c(\bar{r}) = \iiint_V \mathcal{G}'_H(\bar{r},\bar{r}') \cdot \bar{a}_c(\bar{r}')\, dV' \tag{25}$$

where $\bar{a}_c$ indicates the "core" part of a vector, obtained by subtracting from the vector its functional projection on $\bar{h}_o$. The dyadic $\mathcal{G}'_H$ has properties similar to those of $\mathcal{G}'_c$. It is now possible to state that: "the magnetic splitting is always possible in the form

$$\bar{a} = \bar{h}_o \iiint_V \bar{a} \cdot \bar{h}_o\, dV + \text{grad div}\,\bar{h} - \text{curl curl}\,\bar{h} \tag{26}$$

[9] H. Weyl. "Uber das Spektrum der Hohlraumstrahlung." Journ. Reine Angew. Math, 141, 163-181, 1912.

where the term in $\bar{h}_0$ exists in Region III only, and where the three parts of the splitting are uniquely determined." The name "transverse" will from now on be reserved for the third term "- curl curl $\bar{h}$".

B. Further properties. Application to Solenoidal and Irrotational Vectors.

1. The boundary conditions satisfied by $\bar{h}$ imply that $\frac{\partial}{\partial n}(\operatorname{div}\bar{h}) = a_n$ on S. This result is easily obtained by projecting (23) on the normal to S. The vector curl $\bar{h}$ being perpendicular to S, its curl, i.e. curl curl $\bar{h}$, is tangential to S (this property is established by applying Stokes' theorem to any small curve drawn on surface S, along which the cir culation of curl $\bar{h}$ vanishes, so that the flux of curl curl $\bar{h}$ through <u>any</u> portion of S is zero; this implies that curl curl $\bar{h}$ is tangential). The projection of the left member of (23) is $\bar{u}_n \cdot \operatorname{grad} \operatorname{div}\bar{h} = \frac{\partial}{\partial n}(\operatorname{div}\bar{h})$, the projection of the right member a_n, and the two must be equal. The scalar $\operatorname{div}\bar{h}$ can now be obtained by solving the potential problem

$$\nabla^2(\operatorname{div}\bar{h}) = \operatorname{div}\bar{a} \qquad \text{with } \frac{\partial}{\partial n}(\operatorname{div}\bar{h}) = a_n \text{ on } S \tag{27}$$

and the restriction that $\operatorname{div}\bar{h}$ must be single-valued. Once this problem is solved, the longitudinal term follows directly by taking the gradient of $\operatorname{div}\bar{h}$. Notice that this term accounts for the normal component of $\bar{a}$ on S.

2. The three terms of the splitting (26) are functionally orthogonal. This can easily be checked by applying relations (10) and (11).

3. When $\bar{a}$ is solenoidal and tangential to S, the only single-valued solution to the potential problem (27) is $\operatorname{div}\bar{h}$ = constant. The longitudinal term vanishes.

4. When $\bar{a}$ is irrotational, the transverse term - curl curl $\bar{h}$ vanishes. This term can, indeed, be derived from a scalar potential, all other terms of (26) being the gradient of something. So, - curl curl $\bar{h}$ = grad ψ, where ψ is harmonic, and $\frac{\partial \psi}{\partial n} = 0$ on S, the transverse term being tangential to the boundary surface. In a simply-connected region, the only solution satisfying these requirements is the trivial one $\psi = 0$. In doubly-connected Region III, ψ must be a multiple of ψ_0, and -curl curl $\bar{h}$ a multiple of $\bar{h}_0$. But -curl curl $\bar{h}$ must simultaneously be functionally orthogonal to $\bar{h}_0$. The zero vector is the only one to have that property, so that - curl curl $\bar{f}$ vanishes.

5. The magnetic splitting can be put in the form

$$\bar{a} = \bar{h}_0 \iiint \bar{a} \cdot \bar{h}_0 \, dV + \text{grad}\, \psi + \text{curl}\, \bar{v} \tag{28}$$

where $\bar{v}$ is perpendicular to the boundary and ψ is single-valued. Conversely, if a splitting such as (28) has been achieved, it is the magnetic one. Taking the divergence of both members of (28) shows, indeed, that ψ must satisfy

$$\nabla^2 \psi = \text{div}\, \bar{a} \qquad \text{while} \qquad \frac{\partial \psi}{\partial n} = a_n$$

But these are exactly the relations satisfied by div $\bar{h}$. In consequence, ψ and div $\bar{h}$ differ by a constant (the only single-valued solution of the homogeneous problem), and the gradients of these two functions are identical.

C. Eigenvector Expansions Suitable for the Magnetic Splitting.

The considerations valid for the electric splitting can be duplicated here. In simply-connected regions, the eigenvectors relative to the integral

operator $\iiint_V G_H(\bar{r},\bar{r}')\,dV'$ form a complete set. These eigenvectors fall into two categories:

(a) solenoidal eigenvectors $\bar{h}_m$, solutions of

$$-\text{curl curl}\,\bar{h}_m + \nu_m'\,\bar{h}_m = 0 \qquad \bar{u}_n \times \text{curl}\,\bar{h}_m = 0 \text{ on } S \tag{29}$$

The normalized $\bar{h}_m$ is simply related to the normalized $\bar{e}_m$ encountered in the electric problem. The relation is $\bar{h}_m = \frac{1}{\sqrt{\lambda_m'}}\,\text{curl}\,\bar{e}_m$, and both $\bar{e}_m$ and $\bar{h}_m$ have the same eigenvalues $\nu_m' = \lambda_m'$.

(b) irrotational eigenvectors $\bar{g}_m = \text{grad}\,\psi_m$, obtained from the eigenfunctions ψ_m of

$$\nabla^2\psi_m + \nu_m''\,\psi_m = 0 \qquad \frac{\partial\psi_m}{\partial n} = 0 \text{ on } S \tag{30}$$

These eigenvectors are all orthogonal to each other, and it will be assumed that they have been normalized. Their eigenvalues are different from zero. In Region III, the vector $\bar{h}_0$ must be added to obtain a set that is complete with respect to the space of square integrable vectors.

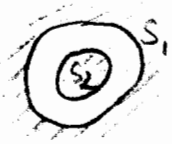
The expansion of the various terms of the magnetic splitting in terms of the $\bar{h}_0$, the $\bar{h}_m$ and the $\bar{g}_m$ is expressed by the important formula:

$$\bar{a} = \underbrace{\bar{h}_0 \iiint_V \bar{a}\cdot\bar{h}_0\,dV}_{\text{Term present in Region III only}} + \underbrace{\sum_m \bar{g}_m\left[\iint_S (\bar{u}_n\cdot\bar{a})\psi_m\,dS - \iiint_V \psi_m\,\text{div}\,\bar{a}\,dV\right]}_{\text{grad div}\,\bar{h}} + \underbrace{\sum_m \frac{\bar{h}_m}{\lambda_m'}\iiint \text{curl}\,\bar{h}_m\cdot\text{curl}\,\bar{a}\,dV}_{-\text{curl curl}\,\bar{h}} \tag{31}$$

The convergence is in the mean, and, at a boundary point, the series converges to the tangential component of $\bar{a}$, all vectors of the expansion being tangential to the boundary surface.

The information obtained from the electric and the magnetic splitting is summarized in the following table. For each category of vectors, the electric splitting is shown on the the first line, the magnetic on the second, and it should be remembered that each of these splittings can be realized in one and only one way. The symbols Ω, φ, ψ denote respectively some scalar function, a function which vanishes at the boundary, and a function which takes some constant value on each surface of which the boundary is comprised. The symbols $\bar{v}$ and $\bar{t}$ denote respectively a vector which is perpendicular to the boundary, and a vector which is tangential to the latter. The notation $\langle \bar{a}, \bar{f}_0 \rangle$ stands for the functional projection of $\bar{a}$ on $\bar{f}_0$, i.e. $\iiint \bar{a} \cdot \bar{f}_0 \, dV$. The Ω, φ and ψ are all single-valued. For irrotational and sourceless vectors in Region III, the projection $\langle \bar{a}, \bar{h}_0 \rangle$ is proportional to the circulation of $\bar{a}$ around the hole.

TABLE

Category	Region I	Region II 	Region III
Any continuous vector $\bar{a}$	grad φ + curl $\bar{t}$ grad Ω + curl $\bar{v}$	grad φ + curl $\bar{t}$ + $\langle \bar{a}, \bar{f}_0 \rangle \bar{f}_0$ grad Ω + curl $\bar{v}$	grad φ + curl $\bar{t}$ grad Ω + curl $\bar{v}$ + $\langle \bar{a}, \bar{h}_0 \rangle \bar{h}_0$
Solenoidal vector ($\mathrm{div}\, \bar{a} = 0$)	curl $\bar{t}$ grad Ω + curl $\bar{v}$ (Ω harmonic)	curl $\bar{t}$ + $\langle \bar{a}, \bar{f}_0 \rangle \bar{f}_0$ grad Ω + curl $\bar{v}$ (Ω harmonic)	curl $\bar{t}$ grad Ω + curl $\bar{v}$ + $\langle \bar{a}, \bar{h}_0 \rangle \bar{h}_0$ (Ω harmonic)
Solenoidal vector tangent to boundary	curl $\bar{t}$ curl $\bar{v}$	curl $\bar{t}$ curl $\bar{v}$	curl $\bar{t}$ curl $\bar{v}$ + $\langle \bar{a}, \bar{h}_0 \rangle \bar{h}_0$
Irrotational vector	grad φ + curl $\bar{t}$ grad Ω	grad φ + curl $\bar{t}$ + $\langle \bar{a}, \bar{f}_0 \rangle \bar{f}_0$ grad Ω	grad φ + curl $\bar{t}$ grad Ω + $\langle \bar{a}, \bar{h}_0 \rangle \bar{h}_0$
Irrotational vector perpendicular to boundary	grad φ grad ψ	grad φ + $\langle \bar{a}, \bar{f}_0 \rangle \bar{f}_0$ grad ψ	grad φ grad ψ
Sourceless vector	curl $\bar{t}$ grad Ω (Ω harmonic)	curl $\bar{t}$ + $\langle \bar{a}, \bar{f}_0 \rangle \bar{f}_0$ grad Ω (Ω harmonic)	curl $\bar{t}$ grad Ω + $\langle \bar{a}, \bar{h}_0 \rangle \bar{h}_0$ (Ω harmonic)

IV. THE MIXED SPLITTING

It consists of splitting a vector as

$$\bar{a} = \text{grad}\,\varphi + \text{curl}\,\bar{v} + \bar{\alpha} \tag{32}$$

where φ vanishes on the boundary, $\bar{v}$ is perpendicular to the latter, and $\bar{\alpha}$ denotes a certain sourceless vector. To obtain this splitting, one can first calculate grad φ, which is a term furnished by the electric splitting, and then apply the magnetic splitting to what is left when grad φ is subtracted from $\bar{a}$. The mixed splitting is unique. The relative importance of the three terms can best be illustrated by writing down their expansions.

$$\bar{a} = \overbrace{\sum_m \bar{f}_m \left[-\iiint \varphi_m \,\text{div}\,\bar{a}\, dV\right]}^{\text{grad}\,\varphi} + \overbrace{\sum_m \frac{\bar{h}_m}{\lambda'_m} \iiint \text{curl}\,\bar{h}_m \cdot \text{curl}\,\bar{a}\, dV}^{\text{curl}\,\bar{v}} + \bar{\alpha} \tag{33}$$

(perpendicular to the boundary) (tangential to the boundary)

which indicates which terms disappear for a solenoidal or an irrotational vector. The term $\bar{\alpha}$, which appears as a rest term in (32), can be expressed as

$$\bar{\alpha} = \sum_m \bar{g}_m \iint_S \left(a_n - \frac{\partial \varphi}{\partial n}\right) \psi_m\, dS + \bar{h}_0 \iiint_V \bar{a} \cdot \bar{h}_0\, dV \tag{34}$$

The term $-\bar{h}_0 \iiint \text{grad}\,\varphi\, \bar{h}_0\, dV$ which the application of (31) introduces in Region III can be discarded. This term vanishes because $\iiint \text{grad}\,\varphi \cdot \text{grad}\,\psi_0\, dV$ is equal to zero, as seen from an application of (5) where A would be the single-valued function φ, and B the harmonic function ψ_0. Notice that φ vanishes for a solenoidal vector, and that the term $\frac{\partial \varphi}{\partial n}$ appearing in (34) can be put equal to zero in those circumstances.

Another expression is possible for $\bar{\alpha}$

$$\bar{\alpha} = \bar{f}_0 \left[\iint (\bar{u}_n \cdot \bar{a}) \varphi_0 \, dS - \iiint \varphi_0 \, \mathrm{div}\, \bar{a} \, dV \right] - \sum_m \frac{\bar{e}_m}{\lambda'_m} \iint_S \left[\bar{u}_n \times (\bar{a} - \mathrm{curl}\, \bar{v}) \right] \cdot \mathrm{curl}\, \bar{e}_m \, dS \tag{35}$$

Notice that $\bar{v}$ vanishes for an irrotational vector. It is sometimes desirable to calculate φ directly, instead of finding its gradient by a formula such as (33). The steps are rather trivial, and involve expansion in the normalized* eigenfunctions φ'_m of (19). The final result is:

$$\varphi = -\sum_m \frac{\varphi'_m}{\lambda''_m} \iiint \varphi'_m \, \mathrm{div}\, \bar{a} \, dV = -\sum_m \varphi_m \iiint \varphi_m \, \mathrm{div}\, \bar{a} \, dV \tag{36}$$

where φ_m is the scalar potential used above for the normalized vectors $\bar{f}_m = \mathrm{grad}\, \varphi_m$. Similarly, it is interesting to have an expansion for $\bar{v}$ in terms of the eigenvectors of the electric splitting. Notice that $\bar{v}$ stands for - curl $\bar{f}$, and is consequently solenoidal. This implies that the $\bar{f}_m$ do not appear in the expansion. Notice also that $\bar{v}$ being the curl of minus $\bar{f}$, does not have any functional projection on $\bar{f}_0$, because $\bar{f}_0$ cannot be represented as a curl. In consequence, the terms in $\bar{e}_m$ are the only ones to survive in the expansion of $\bar{v}$, which then turns out to be

$$\bar{v} = \sum_m \frac{\bar{e}_m}{\lambda'_m} \iiint \bar{e}_m \cdot \mathrm{curl}\, \bar{a} \, dV \tag{37}$$

---

*Notice that the φ_m appearing in formulas (22) and (33) are not normalized. It is their gradient $\bar{f}_m$ which is normalized, and an application of (5) shows that $\iiint \varphi_m^2 \, dV = \frac{1}{\lambda'_m}$. The functions φ'_m are, consequently, identical with $\sqrt{\lambda'_m}\, \varphi_m$

Obviously, any gradient perpendicular to the boundary can be added to (37), and still furnish an acceptable $\bar{v}$, i.e. yield an unmodified transverse term curl $\bar{v}$. The $\bar{v}$ given by (37) is the particular choice associated with the magnetic splitting.

APPENDIX I

In this appendix, the properties of the eigenvectors of the Laplacian operator are examined for two different sets of boundary conditions. Only those eigenvectors relative to eigenvalues different from zero are considered.

(a) Consider first the eigenvectors $\overline{F}_m$ of

$$\text{grad div}\,\overline{F}_m - \text{curl curl}\,\overline{F}_m + \lambda_m \overline{F}_m = 0 \quad \text{with} \left\{ \begin{array}{l} \overline{u}_n \cdot \overline{F}_m = 0 \\ \text{div}\,\overline{F}_m = 0 \end{array} \right\} \text{on } S \tag{38}$$

The eigenvector $\overline{F}_m$ can be split electrically as grad α_m + curl $\overline{W}_m$. No term in $\overline{f}_0$ is necessary in Region II, because $\overline{F}_m$ and $\overline{f}_0$ are functionally orthogonal to each other. This results from the formula

$$\iiint_V \overline{f}_0 \cdot \nabla^2 \overline{F}_m \, dV - \lambda_m \iiint_V \overline{f}_0 \cdot \overline{F}_m \, dV = \iiint_V \overline{F}_m \cdot \nabla^2 \overline{f}_0 \, dV + \iint_S [(\overline{u}_n \cdot \overline{f}_0)\,\text{div}\,\overline{F}_m - (\overline{u}_n \cdot \overline{F}_m)\,\text{div}\,\overline{f}_0$$
$$+ (\overline{u}_n \times \overline{f}_0) \cdot \text{curl}\,\overline{F}_m - (\overline{u}_n \times \overline{F}_m) \cdot \text{curl}\,\overline{f}_0]\, dS$$

All integrals of the last member vanish, which implies that $\iiint \overline{f}_0 \cdot \overline{F}_m \, dV = 0$. Introducing grad α_m + curl $\overline{W}_m$ into (38), one arrives at the formula

$$\text{grad}\,[\nabla^2 \alpha_m + \lambda_m \alpha_m] = \text{curl}\,[\text{curl curl}\,\overline{W}_m - \lambda_m \overline{W}_m] \tag{39}$$

Calling $\overline{g}$ the common value of the two members, one sees right away that $\overline{g}$ must be sourceless and perpendicular to the boundary surface. On this surface, indeed, α_m is zero, and $\text{div}\,\overline{F}_m = \nabla^2 \alpha_m$ vanishes too. The gradient of $\nabla^2 \alpha_m + \lambda_m \alpha_m$ is consequently perpendicular to that surface. In simply-bounded regions, there is no non-zero sourceless vector perpendicular to the boundary, and both members of (39) must vanish. This implies that

$\nabla^2 \alpha_m + \lambda_m \alpha_m$ is constant throughout the volume, and that the constant is zero since this is the value $\nabla^2 \alpha_m + \lambda_m \alpha_m$ takes on the boundary. We write

$$\nabla^2 \alpha_m + \lambda_m \alpha_m = 0 \qquad \alpha_m = 0 \quad \text{on } S \tag{40}$$

But this implies also that the transverse part curl $\overline{W}_m$, which will now be written as $\overline{e}_m$, satisfies

$$-\text{curl curl } \overline{e}_m + \lambda_m \overline{e}_m = 0 \qquad \overline{u}_n \times \overline{e}_m = 0 \quad \text{on } S \tag{41}$$

One then arrives at the conclusion that to each eigenvalue λ_m there correspond two possible sorts of eigenvectors viz: grad α_m, where α_m satisfies (40), and (or) $\overline{e}_m$, where $\overline{e}_m$ satisfies (41). Conversely any eigenvector of (41), and the gradient of any eigenfunction of (40), are eigenvectors of the original problem (38). There is, consequently, complete identity between the latter and the ensemble of the grad α_m supplemented by the ensemble of the $\overline{e}_m$. The same is true for the eigenvalues λ_m, which are made up of the eigenvalues λ_m'' of (40) supplemented by the eigenvalues λ_m' of (41), some of the eigenvalues being possibly common to both classes, as evidenced by the example treated in Appendix II.

In doubly-bounded regions, the same sequence of arguments can be used. Vector $\overline{g}$ must be a multiple of $\overline{f}_0$, being sourceless and perpendicular to the boundary. But (39) indicates that $\overline{g}$ can be put in the form of the curl of a vector potential. This is impossible for $\overline{f}_0$ and its non-zero multiples, so $\overline{g}$ must vanish.

(b) Consider now the eigenvectors $\overline{H}_m$ of

$$\text{grad div } \overline{H}_m - \text{curl curl } \overline{H}_m + \nu_m \overline{H}_m = 0 \quad \text{with} \quad \left\{ \begin{array}{l} \overline{u}_n \times \text{curl } \overline{H}_m = 0 \\ \overline{u}_n \cdot \overline{H}_m = 0 \end{array} \right\} \text{ on } S \tag{42}$$

The same sort of argument as in part (a) can be applied here. The eigenvector $\overline{H}_m$ can be split magnetically as grad β_m + curl $\overline{v}_m$, with no term in $\overline{h}_0$ needed in doubly-connected regions. Introducing in (42), one obtains

$$\text{grad} \left[\nabla^2 \beta_m + \nu_m \beta_m \right] = \text{curl} \left[\text{curl curl } \overline{v}_m - \nu_m \overline{v}_m \right] \tag{43}$$

The common value of both members is a sourceless vector. This vector is tangent to the boundary surface, because grad β_m is tangent, being the difference of the two tangential vectors $\overline{H}_m$ and curl $\overline{v}_m$, and because grad div $\overline{H}_m$ = grad $\nabla^2 \beta_m$ is tangent, being the difference of the two tangential vectors $-\nu_m \overline{H}_m$ and curl curl $\overline{H}_m$. In consequence, both members vanish in a simply-connected region, where a sourceless tangential vector cannot exist. This implies that $\nabla^2 \beta_m + \nu_m \beta_m$ is equal to a constant c_m throughout the volume, or that $\nabla^2 \left(\beta_m - \frac{c_m}{\nu_m} \right) + \nu_m \left(\beta_m - \frac{c_m}{\nu_m} \right)$ vanishes throughout the volume. In other words, the longitudinal part can be put in the form grad ψ_m where

$$\nabla^2 \psi_m + \nu_m \psi_m = 0 \qquad \frac{\partial \psi_m}{\partial n} = 0 \text{ on } S \tag{44}$$

The transverse part $\overline{h}_m$ = curl $\overline{v}_m$ satisfies

$$-\text{curl curl } \overline{h}_m + \nu_m \overline{h}_m = 0 \qquad \overline{u}_n \times \text{curl } \overline{h}_m = 0 \text{ on } S \tag{45}$$

Conversely, each grad ψ_m from (44), and each $\bar{h}_m$ from (45), is an eigenvector of the original problem (42). Notice, by taking the curl of (45), that curl $\bar{h}_m$ satisfies the eigenvector equation defining $\bar{e}_m$, i.e. (41). Conversely, curl $\bar{e}_m$ satisfies the eigenvector equation (45) defining the $\bar{h}_m$. This is seen by taking the curl of (41), and noticing that

$\bar{a}_m \times \bar{e}_m = \bar{\nabla} \times \mathrm{curl}\left(\frac{\mathrm{curl}\,\bar{e}_m}{\lambda_m}\right) = 0$ on S. <u>The solenoidal eigenvectors of both problems are multiples of the curl of each other</u>.

In a doubly-connected region, both members of (43) are multiples of $\bar{h}_0$, or vanish. But β_m, and $\nabla^2\beta_m + \nu_m\beta_m$ are single-valued functions, a property of the magnetic splitting. The integral of their gradient around the hole vanishes. The integral of $\bar{h}_0$, however, is different from zero, and the only way to resolve this inconsistency is for grad $\left[\nabla^2\beta_m + \nu_m\beta_m\right]$ to vanish. From there on, the steps follow as for simply-connected regions.

APPENDIX II

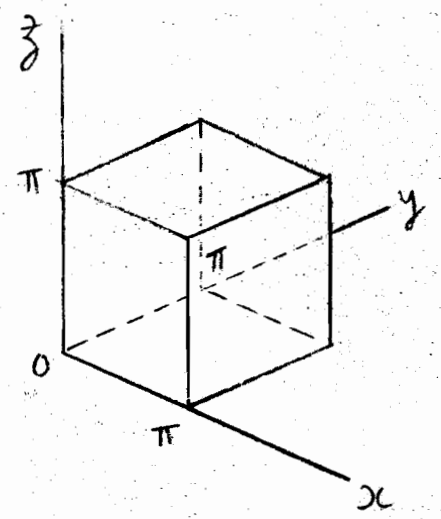


Figure 4

The vector function

$$\bar{a} = \text{grad}(xy) = y\,\bar{u}_x + x\,\bar{u}_y$$

has zero divergence and curl. This is true, in particular, throughout the cube of side π indicated in Fig. 4. It is desired to expand $\bar{a}$ in terms of the normalized electric eigenvectors of the cube. The latter are, as usual, of two types.[10]

1. Irrotational eigenvectors.

The functions φ'_m are $\sqrt{\frac{8}{\pi^3}}\,\sin lx \sin ny \cdot \sin pz$ and the corresponding normalized eigenvectors

$$\bar{f}_m = \sqrt{\frac{8}{\pi^3(l^2+n^2+p^2)}}\left[l\cos lx \cdot \sin ny \cdot \sin pz\,\bar{u}_x + n\sin lx \cos ny \cdot \sin pz\,\bar{u}_y + p\sin lx \sin ny \cos pz\,\bar{u}_z\right]$$

where m stands for the triple l, n, p, the latter numbers being integers different from zero. The corresponding eigenvalues λ''_m are $l^2+n^2+p^2$, the lowest value being 3.

2. Solenoidal eigenvectors.

They can be split into two categories

(a) $$\bar{e}'_m = \sqrt{\frac{8}{\pi^3(\sum_l n^2 + \sum_n l^2)}}\left[-n\cos lx \sin ny \sin pz\,\bar{u}_x + l\sin lx \cos ny \sin pz\,\bar{u}_y\right]$$

[10] P. M. Morse and H. Feshbach. "Methods of Theoretical Physics." p. 1849 McGraw-Hill Company, New York, 1953.

where either l or n is allowed to be zero, and where ε_i is two for $i=0$, one for $i \neq 0$. The eigenvalues λ'_m are, again, $l^2+n^2+p^2$, the lowest value being 2.

$$\text{(b)}\quad \bar{e}'_m = \sqrt{\frac{8}{\pi^3(l^2+n^2)[p^2+\varepsilon_l(l^2+n^2)]}}\Big[pl\cos lx\,\sin ny\,\sin pz\,\bar{u}_x + pn\sin lx\cos ny\,\sin pz\,\bar{u}_y - (l^2+n^2)\sin lx\,\sin ny\,\cos pz\,\bar{u}_z\Big]$$

where p is allowed to vanish, and where λ'_m is, again, $l^2+n^2+p^2$, with minimum value 2. The expansion of $\bar{a}$, which contains only the $\bar{e}'_m$ terms, yields, after calculations

$$\bar{a} = \sum_{n=1,2,3}^{\infty}\sum_{p=1,3,5}^{\infty} \frac{8}{\pi}\cdot\frac{(-1)^{n-1}}{np}\sin ny\,\sin pz\,\bar{u}_x + \sum_{l=1,2,3}^{\infty}\sum_{p=1,3,5}^{\infty}\frac{8}{\pi}\frac{(-1)^{l-1}}{lp}\sin lx\,\sin pz\,\bar{u}_y$$

It is a simple matter to check that the two Fourier expansions converge to y and x respectively. In the plane $x=0$, $\bar{a}$ reduces to $y\,\bar{u}_x$, and is perpendicular to the boundary. The series converges to $y\,\bar{u}_x$ everywhere on that plane, except at those points where the normal is not defined, i.e. along the edges. In the plane $z=0$, the vector $\bar{a}$ is purely tangential, and, as expected, the series converges to the normal component, i.e. vanishes. The convergence to the tangential component is non-uniform, and presents a Gibbs phenomenon, of the type associated with $\sum_p a_p \sin pz$ at $z=0$, when the expansion coefficients a_p correspond to a function which does not vanish at $z=0$.

A last remark: the vector $\bar{f}$, solution of $\nabla^2\bar{f} = y\,\bar{u}_x + x\,\bar{u}_y = \bar{a}$ with $\bar{u}_n\cdot\bar{f}=0$ and $\operatorname{div}\bar{f}=0$ on S, can be expressed as

$$\bar{f} = -\sum_{n=1,2,3}^{\infty}\sum_{p=1,3}^{\infty}\frac{8}{\pi}\frac{(-1)^{n-1}}{np}\frac{\sin ny\,\sin pz}{n^2+p^2}\bar{u}_x - \sum_{l=1,2,3}^{\infty}\sum_{p=1,3,5}^{\infty}\frac{8}{\pi}\frac{(-1)^{l-1}}{lp}\frac{\sin lx\,\sin pz}{l^2+p^2}\bar{u}_y$$

The first double sum can easily be checked to be the Fourier expansion of f_x, i.e. the solution of $\nabla^2 f_x = y$ that vanishes on the rectangular contour.

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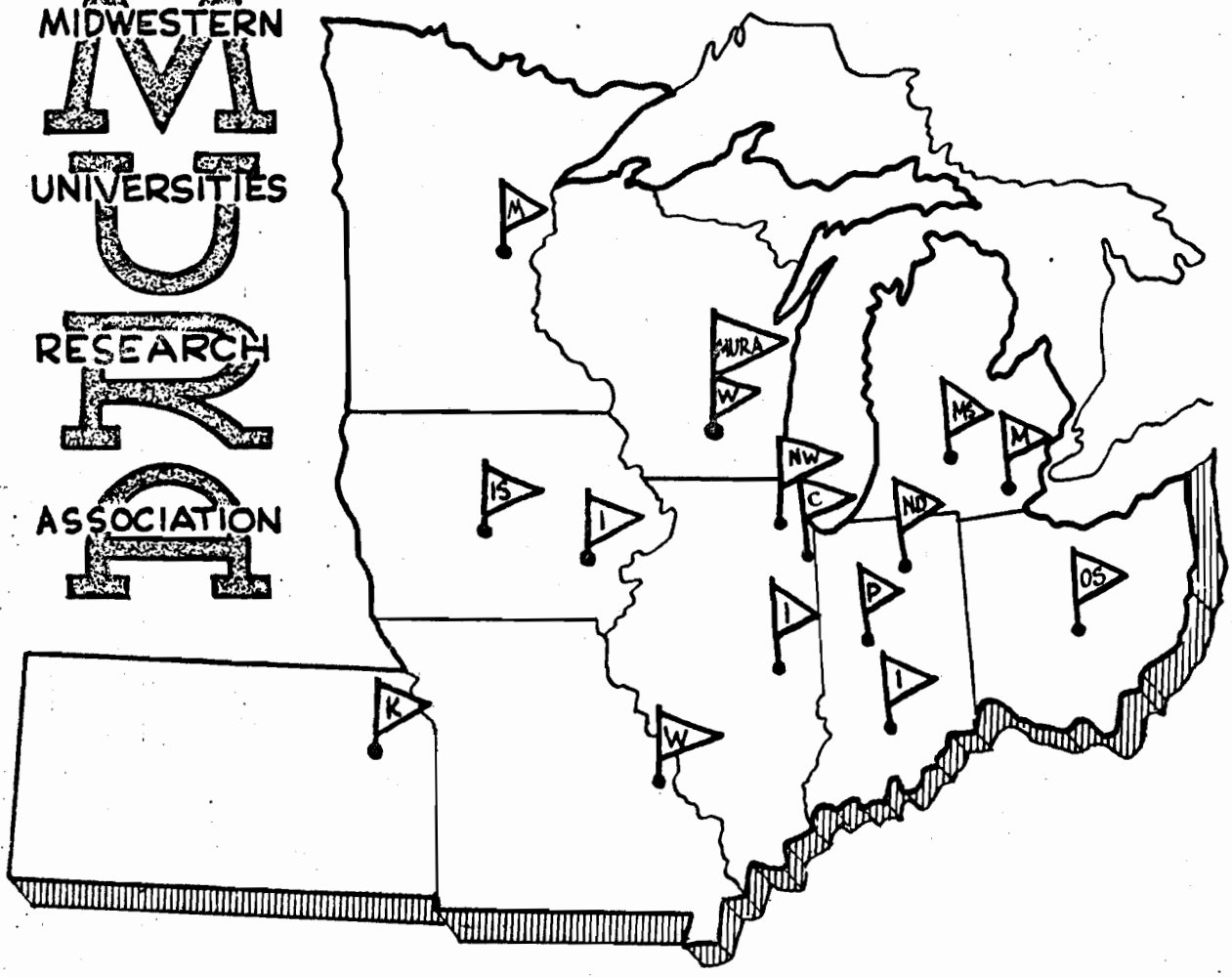

CONCERNING THE $\nu/N \longrightarrow 1/3$ RESONANCE, IV

THE LIMITING-AMPLITUDE SOLUTION OF THE EQUATION

$$\frac{d^2u}{d\phi^2} + (a + b\cos 2\phi)\,u + \frac{B_1}{2}(\sin 2\phi)\,u^2 = 0$$

L. Jackson Laslett

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