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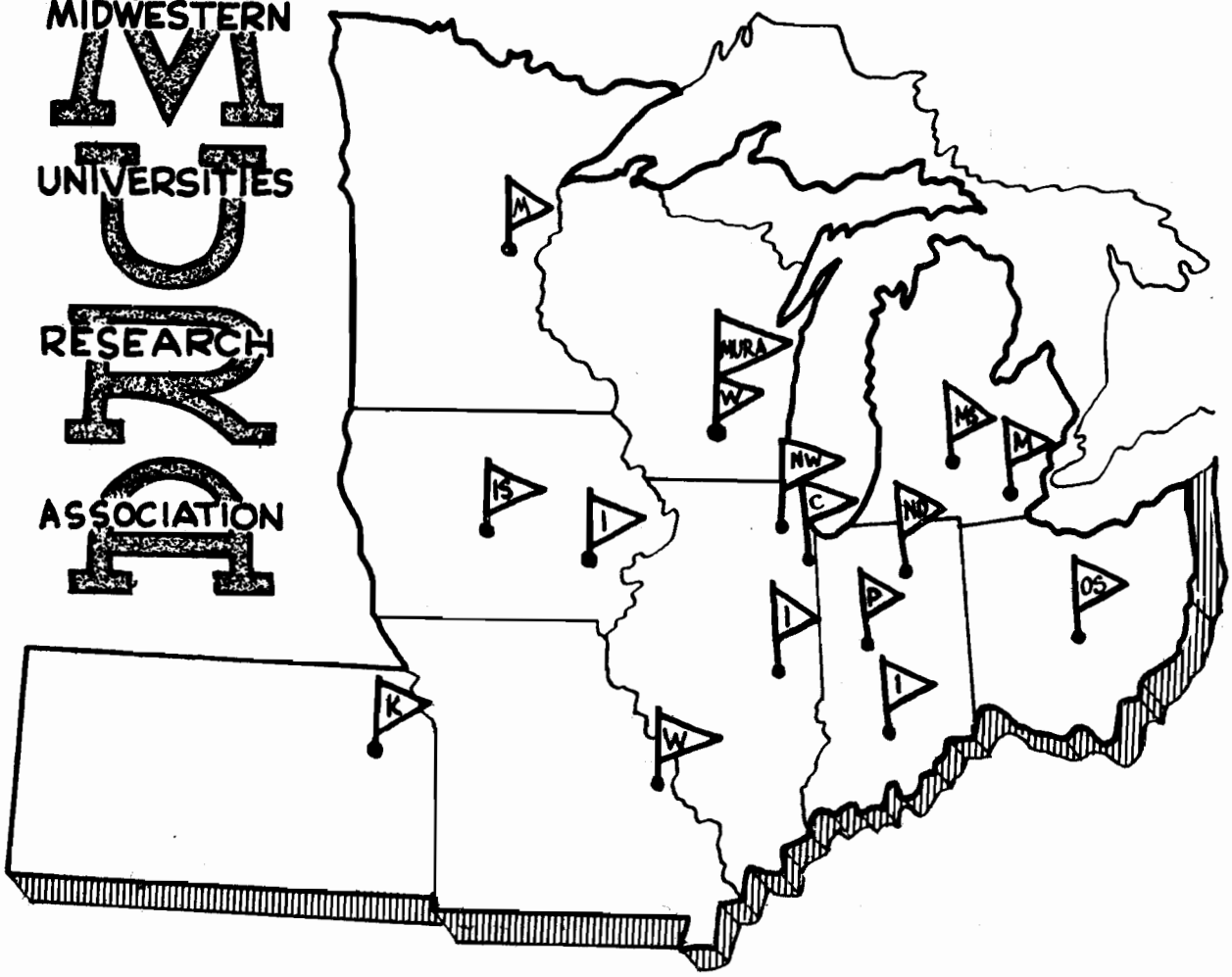

REMARKS CONCERNING THE INTERFERENCE OF HIGHER-ORDER FOURIER COEFFICIENTS IN THE RESULTS OBTAINED WITH THE FORANAL PROGRAM

L. Jackson Laslett

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2203 University Avenue, Madison, Wisconsin

REMARKS CONCERNING THE INTERFERENCE OF HIGHER-ORDER FOURIER COEFFICIENTS IN THE RESULTS OBTAINED WITH THE FORANAL PROGRAM

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ABSTRACT

It is pointed out that the Fourier coefficients evaluated by the FORANAL program (MURA Program 52) are in error by an amount determined by the higher-order Fourier coefficients (of order greater than n/2). This error, which arises from the summation of discrete data in the computation of the Fourier coefficients, is briefly analyzed and examples given.

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**Department of Physics and Institute for Atomic Research, Iowa State College, Ames, Iowa.

I. INTRODUCTION

The FORANAL program[1] was prepared for the purpose of computing Fourier coefficients to represent a set of evenly-spaced discrete data which are presumed to be periodic in the interval taken. If the true function which the data represent is

$$q(t) = g_o + \sum_{\ell=1}^{\infty} g_\ell \cos 2\pi\ell t/T + \sum_{\ell=1}^{\infty} f_\ell \sin 2\pi\ell t/T ,$$

the program works with the n discrete values of q at the points $t = kT/n$:

$$q_k = g_o + \sum_{\ell=1}^{\infty} g_\ell \cos 2\pi k\ell/n + \sum_{\ell=1}^{\infty} f_\ell \sin 2\pi k\ell/n ,$$

$$k = 0, 1, 2, \ldots n-1.$$

The program (in the unnormalized case) then evaluates the quantities

$$(g_o)_{\text{Computer}} \equiv \frac{1}{n} \sum_{k=0}^{n-1} q_k$$

$$(g_m)_{\text{Computer}} \equiv \frac{2}{n} \sum_{k=0}^{n-1} q_k \cos 2\pi km/n$$

$$(f_m)_{\text{Computer}} \equiv \frac{2}{n} \sum_{k=0}^{n-1} q_k \sin 2\pi km/n,$$

with m not exceeding n/2.

From the expression for q_k it will be recognized that replacement of ℓ by $Nn + \ell$ (where N is an integer) will not modify any of the sine or cosine factors entering into this expression; similarly replacement of ℓ by $Nn - \ell$ also will not modify these factors, save for a reversal of sign of the sine factors associated with the coefficients f_ℓ. Thus the quantities q_k could, in fact, have been generated by a variety of sets of Fourier coefficients and, conversely, it

may not be surprising to find that the computational algorisms which are employed give coefficients which differ from the desired coefficients of q(t) by an amount which depends on the higher-order Fourier coefficients of this function.

II. ANALYSIS

By application of the FORANAL algorisms to the quantities q_k as given above, one immediately finds

$$(g_o)_{\text{Computer}} \equiv \frac{1}{n} \sum_{k=0}^{n-1} q_k$$

$$= g_o + g_n + g_{2n} + \dots = g_o + \sum_{N=1}^{\infty} g_{Nn} ,$$

$$(g_m)_{\text{Computer}} \equiv \frac{2}{n} \sum_{k=0}^{n-1} q_k \cos 2\pi km/n$$

$m \neq o$
$(o < m \leq n/2)$

$$= \frac{2}{n} \sum_{\ell=1}^{\infty} g_\ell \sum_{k=0}^{n-1} \cos 2\pi k\ell/n \cos 2\pi km/n$$

$$= \frac{1}{n} \sum_{\ell=1}^{\infty} g_\ell \sum_{k=0}^{n-1} \left[\cos 2\pi k(\ell-m)/n + \cos 2\pi k(\ell+m)/n\right]$$

$$= g_m + \sum_{N=1}^{\infty} \left[g_{Nn+m} + g_{Nn-m}\right] ,$$

and similarly

$$(f_m)_{\text{Computer}} = f_m + \sum_{N=1}^{\infty} \left[f_{Nn+m} - f_{Nn-m}\right] .$$

III. EXAMPLES

Example I: We consider here the function q(t) to be "rectangular":

$q(t) = \pi/\sqrt{2}$ for $-\pi/4 < t < 3\pi/4$, mod. 2π;
$q(t) = 0$ for $3\pi/4 < t < 7\pi/4$, mod. 2π; and
$q(t)$ = average value $= \pi/(2\sqrt{2})$ at the points of discontinuity.

The true Fourier coefficients for this function are then found to be

$$g_o = \pi/(2\sqrt{2}) \;;$$

$$g_m = \frac{1}{m} \cdot \left[1,\ 1,\ -1,\ -1,\ 1,\ 1,\ \ldots\right]$$
$$\text{for } m = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots \;;$$

$$f_m = \frac{1}{m} \cdot \left[1,\ -1,\ -1,\ 1,\ 1,\ -1,\ \ldots\right]$$
$$\text{for } m = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots .$$

If, to take a simple case, the discrete quantities q_k were obtained from this function by taking $n = 8$

$$\left[q_o = q_1 = q_2 = \pi/\sqrt{2},\ q_3 = \pi/(2\sqrt{2}),\ q_4 = q_5 = q_6 = 0,\ q_7 = \pi/(2\sqrt{2})\right],$$

the computer would undertake to produce Fourier components through those of fourth order. A direct computation by means of the FORANAL algorisms shows immediately that one would obtain

$$(g_o)_{\text{Computer}} = \pi/(2\sqrt{2})$$
$$(g_1)_{\text{Computer}} = (\pi/8)\left[\sqrt{2} + 1\right]$$
$$(g_3)_{\text{Computer}} = (\pi/8)\left[\sqrt{2} - 1\right]$$
$$(f_1)_{\text{Computer}} = (\pi/8)\left[\sqrt{2} + 1\right]$$
$$(f_3)_{\text{Computer}} = -(\pi/8)\left[\sqrt{2} - 1\right],$$

with the other coefficients (g_2, g_4, f_2, and f_4) computed to be zero - - g_o is computed correctly in this case, but the other coefficients exhibited here will be seen to be in error.

Alternatively, from the arguments given in the preceding section, we could say

$$
\begin{aligned}
(g_1)_{\text{Computer}} &= g_1 + g_7 + g_{15} + g_{23} + \cdots \\
&\qquad + g_9 + g_{17} + g_{25} + \cdots \\
&= 1 - \left(\frac{1}{7} - \frac{1}{9}\right) - \left(\frac{1}{15} - \frac{1}{17}\right) - \left(\frac{1}{23} - \frac{1}{25}\right) - \cdots \\
&= 1 - 2 \sum_{j=1}^{\infty} \frac{1}{(8j-1)(8j+1)} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
(g_3)_{\text{Computer}} &= g_3 + g_5 + g_{13} + g_{21} + \cdots \\
&\qquad + g_{11} + g_{19} + g_{27} + \cdots \\
&= \frac{1}{3} - \left(\frac{1}{5} - \frac{1}{11}\right) - \left(\frac{1}{13} - \frac{1}{19}\right) - \left(\frac{1}{21} - \frac{1}{27}\right) - \cdots \\
&= \frac{1}{3} - 6 \sum_{j=1}^{\infty} \frac{1}{(8j-3)(8j+3)} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
(f_1)_{\text{Computer}} &= f_1 - f_7 - f_{15} - f_{23} - \cdots \\
&\qquad + f_9 + f_{17} + f_{25} + \cdots \\
&= 1 - \left(\frac{1}{7} - \frac{1}{9}\right) - \left(\frac{1}{15} - \frac{1}{17}\right) - \left(\frac{1}{23} - \frac{1}{25}\right) - \cdots \\
&= 1 - 2 \sum_{j=1}^{\infty} \frac{1}{(8j-1)(8j+1)} \qquad \left[\text{as for } (g_1)_{\text{Computer}}\right] ,
\end{aligned}
$$

$$
\begin{aligned}
(f_3)_{\text{Computer}} &= f_3 - f_5 - f_{13} - f_{21} - \cdots \\
&\qquad + f_{11} + f_{19} + f_{27} + \cdots \\
&= -\frac{1}{3} + \left(\frac{1}{5} - \frac{1}{11}\right) + \left(\frac{1}{13} - \frac{1}{19}\right) + \left(\frac{1}{21} - \frac{1}{27}\right) + \cdots \\
&= -\left[\frac{1}{3} - 6 \sum_{j=1}^{\infty} \frac{1}{(8j-3)(8j+3)}\right] \qquad \left[\text{as for } -(g_3)_{\text{Computer}}\right] .
\end{aligned}
$$

The numberical values are summarized in the table below:[2]

Coefficient	True Value	Calculated by Algorism	Calculated[2] from Higher-order Fourier Coefficients	From FORANAL Run[3]
g_0	1.110 720 735	1.110 720 735	1.110 720 735	1.110 720 734
g_1	1.	0.948 059 449	0.948 06	0.948 059 4
g_3	0.333 333 333	0.162 661 286	0.162 65	0.162 661 3
f_1	1.	0.948 059 449	0.948 06	0.948 059 4
f_3	-0.333 333 333	-0.162 661 286	-0.162 65	-0.162 661 3

Example II: As a second, very simple, example we may consider the function q(t) to be given by

$$q(t) = \cos 4t + \sin 4t,$$

for which the true Fourier coefficients are, of course, $g_4 = f_4 = 1$, with all others zero. If we obtain the q_k by use of $n = 4$, however, the discrete values employed in the computation are all equal to unity and one clearly must obtain

$$(g_o)_{\text{Computer}} = \frac{1}{n} \sum q_k$$

$$= \frac{1}{4} \left[1 + 1 + 1 + 1\right] = 1,$$

with all other computed coefficients (through those of second-order) zero.

In terms of the higher-order Fourier coefficients, one may write

$$(g_o)_{\text{Computer}} = g_o + \sum_{N=1}^{\infty} g_{Nn}$$

$$= g_o + g_4 + g_8 + \ldots = 1, \qquad \text{since } g_4 = 1,$$

as found above.

IV. CONCLUSION

The arguments made in this report merely serve to point out the need to use caution in interpreting the results of such numerical computations based on a limited number of points, as is of course well-known. The report may serve to

emphasize once again, however, the desirability of employing a sufficient number of points that, not only are the Fourier Coefficients of interest well-represented by the input data, but also that significant errors are not introduced into these coefficients by virtue of sizable coefficients of order greater than n/2.[4]

In a recent application[5] of the FORANAL program, the conclusions reached in the present report have permitted a useful comparison of $(g_m)_{Computer}$ with the value of $g_m + g_{N-m}$ expected on the basis of a tentative analytic theory intended to describe the input data (m near n/2).

V. REFERENCE AND NOTES

1. J. N. Snyder, "FORANAL" (Program 52), A Program for Fourier Analysis", MURA - 228 Internal (I. B. M. Program), February-March, 1957.

2. As a straight-forward check, numerical evaluation of sums such as $\sum_{j=1}^{\infty} \frac{1}{(8j-1)(8j+1)}$ was accomplished by summing the first few individual terms and replacing the remainder by an integral.

3. FORANAL run 15000. The following 8 input values have been reproduced on the output: .222 144 14688, .222 144 146 88, .222 144 146 88, .111 072 073 45, 0, 0, 0, .111 072 073 45. These are to be regarded as 1/10 the input values to which reference was made in the text and the Fourier components accordingly are interpreted as 10 times the values printed.

4. With a large number of input data the limitation of the original program to output coefficients of order not to exceed 24 may not be an essential limitation, however. We are indebted to J. McNall, of the MURA Computer Division, for attention to the possibility of relaxing this limitation.

5. L. J. Laslett, work in progress concerning eigen-functions characterizing particle orbits in an A-G accelerator, perturbed so that the operating point lies at the edge of a stop-band.